II. The Hodge decomposition & residue theory

A. Sobolev spaces

Let $T = \left(\mathbb{R}/2\pi\mathbb{Z}\right)^n$ (coords. $x_1, \ldots, x_n$)

If you had to extend $C^s(T)$ to get a Hilbert space, you might try completing it in the norm

(A.1) $$\sum_{|J| \leq s} \int_T |D^J(\cdot)|^2 \, d\underline{x}$$

$D^J = \frac{1}{i^{|J|}} (\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$ $\qquad$ $d\underline{x} = \frac{1}{(2\pi)^n} dx_1 \wedge \cdots \wedge dx_n$

which is the idea behind the spaces $\{\mathcal{H}_s\}$ we now introduce.

<u>Definition 1</u>: Writing

$$\mathcal{F} := \left\{ \text{formal power series } \underbrace{\sum_{\underline{\xi} \in \mathbb{Z}^n} \mu_{\underline{\xi}} e^{i\langle \underline{\xi}, \underline{x} \rangle}}_{= \mu} \;\middle|\; \mu_{\underline{\xi}} \in \mathbb{C} \right\},$$

Eucl. inner prod.

the s^{th} <u>Sobolev space</u> is

$$\mathcal{H}_s := \left\{ \mu \in \mathcal{F} \;\middle|\; \|\mu\|_s^2 := \sum_{\underline{\xi} \in \mathbb{Z}^n} (1 + \|\underline{\xi}\|^2)^s |\mu_{\underline{\xi}}|^2 < \infty \right\}.$$

s^{th} Sobolev norm

<u>Lemma 1</u>: "$(\mu, \nu)_s := \sum_{\underline{\xi} \in \mathbb{Z}^n} (1 + \|\underline{\xi}\|^2)^s \mu_{\underline{\xi}} \overline{\nu_{\underline{\xi}}}$" makes $\mathcal{H}_s$ into a Hilbert (= complete inner product) space.

<u>Proof</u>:
- Cauchy-Schwarz $\Rightarrow |(\mu, \nu)_s| \leq \|\mu\|_s \|\nu\|_s < \infty$ (so $(\,,\,)_s$ well-defined)
- $(\,,\,)_s$ clearly then a pos.-definite Hermitian inner product
- Completeness argument same as for ℓ_2 (easy). $\square$

Noting the inclusions

$$\cdots \supset \mathcal{H}_{-1} \supset \underset{\left(\overset{\shortparallel}{\ell_2}\right)}{\mathcal{H}_0} \supset \mathcal{H}_1 \supset \cdots$$

we set $\mathcal{H}_\infty := \bigcap_s \mathcal{H}_s$, $\mathcal{H}_{-\infty} = \bigcup_s \mathcal{H}_s$.

Let $\varphi \in C^0(\mathbb{T})$. Define the Fourier coefficients of φ by

$$\varphi_{\underline{\xi}} := \int_{\mathbb{T}} \varphi(\underline{x})\, e^{-i\langle \underline{\xi}, \underline{x}\rangle}\, d\underline{x} .$$

Lemma 2 (Fourier / parseval): (a) $\sum_{\xi \in \mathbb{Z}} \varphi_{\underline{\xi}}\, e^{i\langle \underline{\xi}, \underline{x}\rangle}$ converges absolutely + uniformly to φ

(b) $\int_{\mathbb{T}} |\varphi|^2 d\underline{x} = \sum_{\underline{\xi}} |\varphi_{\underline{\xi}}|^2$.

Proof: Set $\Phi_N = \sum_{\|\xi\| \le N} \varphi_{\underline{\xi}}\, e^{i\langle \underline{\xi}, \underline{x}\rangle}$. Dropping "$\mathbb{T}$", "$d\underline{x}$" we have

$$0 \le \int |\varphi - \Phi_N|^2 = \int |\varphi|^2 - \underbrace{\left(\int \varphi \overline{\Phi}_N + \int \overline{\varphi}\, \Phi_N\right)}_{\substack{2\sum_{\|\xi\|\le N} |\varphi_{\underline{\xi}}|^2 \\ \text{by defn. of } \varphi_{\xi}}} + \underbrace{\int |\Phi_N|^2}_{\sum_{\|\xi\|\le N} |\varphi_\xi|^2}$$

[henceforth write $\sum$ for $\sum_{\xi \in \mathbb{Z}^n}$, $\sum_{\le N}$ for $\sum_{\|\xi\| \le N}$]

$$\Rightarrow \int |\varphi|^2 \ge \sum_{\le N} |\varphi_\xi|^2 \underset{N \to \infty}{\Longrightarrow} \infty > \int |\varphi|^2 \ge \sum |\varphi_{\underline{\xi}}|^2 .$$

Suppose $\lambda = \sum_{\le N} \lambda_{\underline{\xi}}\, e^{i\langle \underline{\xi}, \underline{x}\rangle}$. Then

$$\text{(A.2)}\quad \left\{\begin{aligned} 0 \le \int |\varphi - \lambda|^2 &= \int |\varphi|^2 - \int \varphi \overline{\lambda} - \int \overline{\varphi}\lambda + \int |\lambda|^2 \\ &\underset{\left(\substack{\text{defn}\\ \text{of } \varphi_\xi}\right)}{=} \int |\varphi|^2 - \sum_{\le N} \varphi_{\underline{\xi}} \overline{\lambda}_{\underline{\xi}} - \sum_{\le N} \overline{\varphi}_\xi \lambda_\xi + \sum_{\le N} |\lambda_\xi|^2 \\ &= \int |\varphi|^2 - \sum_{\le N} |\varphi_\xi|^2 + \sum_{\le N} |\varphi_\xi - \lambda_\xi|^2 \quad \text{is minimized} \end{aligned}\right.$$

when $\lambda = \Phi_N$. But Stone – Weierstraß $\Longrightarrow$
(+ fact that the $\{\mu \cdot e^{i\langle \xi, x\rangle}\}$ separate points)

given $\epsilon > 0$, $\exists\, N \gg 0$ & λ as above s.t. $\|\varphi - \lambda\|_\infty < \sqrt{\epsilon} \Longrightarrow$

$$0 \le \underbrace{\int |\varphi - \Phi_N|^2}_{\overset{(A.2)}{=}\ \int |\varphi|^2 - \sum_{\le N} |\varphi_\xi|^2} < \epsilon$$

$\square$

Now lemma 2(b) $\Longrightarrow$

<u>Corollary 1</u>: $(C^0(T), \|\cdot\|_{L^2}) \hookrightarrow (\mathcal{H}_0, \|\cdot\|_0)$

Given $\varphi, \psi \in C^s(T)$ and $|J| \le s$, $\int$ by parts $\Longrightarrow$

$$(A.3) \qquad \int_T (D^J \varphi)\, \bar{\psi}\, d\underline{x} = \int_T \varphi\, \overline{D^J \psi}\, d\underline{x} \Longrightarrow$$

$$(D^J\varphi)_\xi = \int_T (D^J \varphi)\, e^{-i\langle \underline{\xi}, \underline{x}\rangle} d\underline{x} \underset{(A.3)}{=} \int_T \varphi\, \overline{\underline{\xi}^J e^{i\langle \underline{\xi}, \underline{x}\rangle}}\, d\underline{x}$$

$$= \underline{\xi}^J \int_T \varphi\, e^{-i\langle \underline{\xi}, \underline{x}\rangle} d\underline{x} = \underline{\xi}^J \varphi_{\underline{\xi}} \quad \overset{\text{parseval}}{\Longrightarrow}$$

<u>Corollary 2</u>: $\|D^J \varphi\|_0^2 = \sum_{\underline{\xi} \in \mathbb{Z}^n} |\underline{\xi}^J|^2 |\varphi_{\underline{\xi}}|^2 \quad (\forall\, \varphi \in C^s(T), |J| \le s)$

i.e. L^2
(using lemma 2(b))

Now
$$\sum_{|J| \le s} |\underline{\xi}^J|^2 \le (1 + \|\underline{\xi}\|^2)^s \le K_s \sum_{|J| \le s} |\underline{\xi}^J|^2 \quad (\forall\, \underline{\xi})$$
(>0)

$$\Longrightarrow \sum_{|J| \le s} \sum_{\underline{\xi} \in \mathbb{Z}^n} |\underline{\xi}^J|^2 |\varphi_{\underline{\xi}}|^2 \le \sum_{\underline{\xi}} (1 + \|\underline{\xi}\|^2)^s |\varphi_{\underline{\xi}}|^2 \le K_s \sum_{|J| \le s} \sum_{\underline{\xi}} |\underline{\xi}^J|^2 |\varphi_{\underline{\xi}}|^2$$

$$\underset{(\text{Cor. 2})}{=} K_s \sum_{|J| \le s} \|D^J \varphi\|_0^2 < \infty$$

$\Longrightarrow$

Corollary 3: (a) Fourier series induce an embedding $C^s(T) \hookrightarrow \mathcal{H}_s$

(b) $\|\cdot\|_s$ is equivalent to $\underbrace{\sum_{|J|\le s} \|D^J(\cdot)\|_0^2}$ on $C^s(T)$.

(c) $\mathcal{H}_s$ is the completion of $C^\infty(T)$ in this latter norm.

Pf of (c): by defn. of $\mathcal{H}_s$, the $\mu_\xi e^{i\langle \xi, x\rangle}$'s are dense; and these belong to $C^\infty(T)$. Apply (b). $\square$

Lemma 3 (Sobolev): $\mathcal{H}_{s+\lfloor n/2\rfloor+1} \subset C^s(T)$

[Remark 1: This partial converse (to Cor. 3(a)) says that for L^2 control over enough formal derivatives of $\mu \in \mathcal{T}$, it is the F.S. of (& converges A.+U. to) a C^s fcn.]

Proof: Let $\mu = \sum \mu_\xi e^{i\langle\xi, x\rangle} \in \mathcal{H}_{\lfloor n/2\rfloor+1}$, with partial sums $s_N \in C^0(T)$.

$$\sum_{\le N} |\mu_\xi| = \sum_{\le N} |\mu_\xi| (1+\|\xi\|^2)^{\frac{\lfloor n/2\rfloor+1}{2}} (1+\|\xi\|^2)^{-\frac{\lfloor n/2\rfloor+1}{2}}$$

$$\underset{\text{Cauchy-Schwarz}}{\le} \left(\sum (1+\|\xi\|^2)^{\lfloor n/2\rfloor+1} |\mu_\xi|^2\right)^{1/2} \left(\sum (1+\|\xi\|^2)^{-(\lfloor n/2\rfloor+1)}\right)^{1/2}$$

$$\le \underbrace{\|\mu\|_{\lfloor n/2\rfloor+1}}_{<\infty \text{ by defn.}} \cdot \underbrace{\left(\sum_{\|\xi\|\le N} \frac{1}{(1+\|\xi\|^2)^{\lfloor n/2\rfloor+1}}\right)^{1/2}}_{\text{bounded as } N\to\infty:\ \sim \frac{N^{n-1}}{(1+N^2)^{\lfloor n/2\rfloor+1}} \sim N^{-2 \text{ or } -3}}$$

so the s_N are uniformly Cauchy, done for $s=0$.

For the inductive step, assume (for simplicity) $n=1$, so $\langle\xi, x\rangle = \xi x$. Wts: $\mathcal{H}_{s+1} \subset C^s(T) \Rightarrow \mathcal{H}_{s+2} \subset C^{s+1}(T)$.

Given $\mu \in \mathcal{H}_{s+2}$,

$$\sum_\xi (1+\|\xi\|^2)^{s+1}\, \xi^2 |\mu_\xi|^2 \le \sum_\xi (1+\|\xi\|^2)^{s+2} |\mu_\xi|^2 = \|\mu_{s+2}\|^2 < \infty$$

$$\Rightarrow \nu := \underbrace{\sum_{\xi\in\mathbb{Z}} i\xi \mu_\xi e^{i\xi x}}_{} \in \mathcal{H}_{s+1}$$

of induction $\Rightarrow$ converges uniformly to C^s fcn. $(=\nu)$

$$\Rightarrow \int_0^x \nu(t)\,dt = \sum \mu_\xi e^{i\xi x} - \mu(0) = \mu(x) - \mu(0)$$

$\underset{\text{FTC}}{\Rightarrow}$ $\mu' = \nu$, so $\mu \in C^{s+1}$. $\square$

<u>Corollary 4</u>: $\mathcal{H}_\infty = C^\infty(T)$

<u>Pf</u>: Cor 3(c) of Lemma 3 ("sandwich" argument) $\square$

<u>Lemma 4 (Rellich)</u>: For $s > r$, the inclusion $\mathcal{H}_s \overset{j}{\subset} \mathcal{H}_r$ is a compact operator.

[<u>Remark 2</u>: this means that j sends bounded sequences into <u>bounded sequences w./Cauchy subsequences</u> — i.e. image of ball has compact closure.]

<u>Proof</u>: Given $\{\mu^{(k)}\} \subset \mathcal{H}_s$ sequence w./ $\|\mu^{(k)}\|_s^2 < C$ $(\forall k)$,

$$\sum (1+\|\xi\|^2)^r |\mu_\xi^{(k)}|^2 \le \sum (1+\|\xi\|^2)^s |\mu_\xi^{(k)}|^2 < C \quad (\forall k) \Rightarrow$$

$\{(1+\|\xi\|^2)^{r/2} \mu_\xi^{(k)}\}_k$ bounded (for each fixed $\xi \in \mathbb{Z}^n$) $\underset{\text{replace } k \text{ by subsequence}}{\Longrightarrow}$

" Cauchy " $\Longrightarrow$

$\Longrightarrow$

given $\epsilon > 0$, can choose R & m s.t.

$$\frac{4C}{(1+\|\xi\|^2)^{s-r}} < \frac{\epsilon}{2} \quad \text{for } \|\xi\| > R \quad \underline{\text{AND}} \quad \sum_{\|\xi\|\le R} (1+\|\xi\|^2)^r \left|\mu_\xi^{(k)} - \mu_\xi^{(l)}\right|^2 < \frac{\epsilon}{2}$$

for $k, l \geq m$.

$$\Longrightarrow \|\mu^{(k)} - \mu^{(l)}\|_r^2 = \sum_{\|\xi\|\le R} (1+\|\xi\|^2)^r \left|\mu_\xi^{(k)} - \mu_\xi^{(l)}\right|^2 + \sum_{\|\xi\|>R} \frac{(1+\|\xi\|^2)^s}{(1+\|\xi\|^2)^{s-r}} \left|\mu_\xi^{(k)} - \mu_\xi^{(l)}\right|^2$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{8C} \sum_{\|\xi\|>R} (1+\|\xi\|^2)^s \left\{ |\mu_\xi^{(k)}|^2 + 2|\mu_\xi^{(k)}||\mu_\xi^{(l)}| + |\mu_\xi^{(l)}|^2 \right\}$$

$$\underset{\text{Cauchy-Schwarz}}{\le} \frac{\epsilon}{2} + \frac{\epsilon}{8C}\left(2\|\mu^{(k)}\|_s^2 + 2\|\mu^{(l)}\|_s^2\right) < \epsilon .$$

$\square$

<u>Corollary 5 ("Rellich Selection Theorem")</u>: If a sequence of functions (here, in $C^\infty(T)$) <u>together with their 1st derivatives</u> is bounded in norm (i.e. bdd. seq. in $\mathcal{H}_1$), then a subsequence convergent in L^2 can be selected.

Now let $E \to M$ be a Hermitian (C^∞, complex) vector bundle over a Riemannian (C^∞, real) n-manifold

set $T^\vee_{M,\mathbb{C}} := T^\vee_M \otimes_{\mathbb{R}} \mathbb{C}$

$dV :=$ generator of $H^n_{dR}(M, \mathbb{R})$.

$$\left.\begin{aligned} \nabla = D_E &: C^\infty(M, E) \to C^\infty(M, T^\vee_{M,\mathbb{C}} \otimes E) \\ D_{T^\vee} &: C^\infty(M, T^\vee_{M,\mathbb{C}}) \to C^\infty(M, (T^\vee_{M,\mathbb{C}})^{\otimes 2}) \end{aligned}\right\} \text{connections}$$

$$\nabla = D_{T^\vee} \otimes 1 + 1 \otimes D_E : C^\infty(M, T^\vee_{M,\mathbb{C}} \otimes E) \to C^\infty(M, (T^\vee_{M,\mathbb{C}})^{\otimes 2} \otimes E)$$

$$\nabla^k = \underbrace{\nabla \circ \cdots \circ \nabla}_{k \text{ times}} : C^\infty(M, E) \to C^\infty(M, (T^\vee_{M,\mathbb{C}})^{\otimes k} \otimes E)$$

(Note that each $(T^\vee_{M,\mathbb{C}})^{\otimes k} \otimes E$ has a metric "$(g^\vee)^{\otimes k} \otimes h_E$".)

Consider the restriction of a section $\sigma \in C^\infty(M, E)$ to a coordinate neighborhood $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$, op. $\cup$ $W_{\alpha j}$ rel. compact (i.e. compact closures), so that $E|_{W_{\alpha j}} = \mathbb{C}^r \times W_{\alpha j}$ and we can regard $(\sigma|_{W_{\alpha j}} \cdot \eta_{\alpha j})$ (partition of 1 fcn) as a function on T (by fitting $W_{\alpha j}$ inside a fund. domain for T). Fourier transforming, applying the s-norm*, and summing over the α_j's produces a Sobolev norm on $C^\infty(M, E)$. Essentially by Cor. 3(b), the following norm is equivalent:

<u>Definition 2</u>: <u>$\mathcal{H}_s(M, E)$ is the completion of $C^\infty(M, E)$ in the norm</u>

$$\|\sigma\|_s^2 := \sum_{0 \le k \le s} \int_M \|\nabla^k \sigma\|^2 \, dV .$$

(norm induced by metrics)

It is a Hilbert space under

$$\langle \sigma, \nu \rangle_s := \sum_{0 \le k \le s} \int_M \langle \nabla^k \sigma, \nabla^k \nu \rangle \, dV ,$$

---

* that the function is vector-valued is not a problem: replace $|\eta_\xi|$ by $\|\eta_\xi\|$, etc.

and the connection to the torus Sobolev spaces given by the equivalent partition-of-1 norm gives immediately:

$\boxed{\text{Sobolev Embedding Lemma:}}$ $\mathcal{H}_{s+[n/2]+1}(M,E) \subset C^s(M,E)$ and

$$\bigcap_s \mathcal{H}_s(M,E) = C^\infty(M,E).$$

$\boxed{\text{Rellich Lemma:}}$ $s > r \Rightarrow \mathcal{H}_s(M,E) \hookrightarrow \mathcal{H}_r(M,E)$ is compact.

There are two more main Lemmas involved in the proof of the Hodge theorem. To state them, we'll have to introduce elliptic partial differential operators.