B. Elliptic differential operators

Let $\begin{cases} E \xrightarrow{\pi} M \\ E' \xrightarrow{\pi'} M \end{cases}$ be rank $\begin{cases} m \\ m' \end{cases}$ $\mathbb{C}$-vector bundles over a C^∞ n-manifold

with trivializations $(\pi^{(\prime)})^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^{m^{(\prime)}}$ (coords. $(x_1, \dots, x_n; u_1^{(\prime)}, \dots, u_{m^{(\prime)}}^{(\prime)})$).

Consider a map $\mathcal{Q} : C^\infty(E) \to C^\infty(E')$ of sheaves locally ($\forall \alpha$) of the form

$$(\mathcal{Q}_{U_\alpha})(\underline{u}(\underline{x})) = \sum_{|J| \leq d} A_J(\underline{x}) \overset{\text{matrix mult.}}{\cdot} (D^J \underline{u}(\underline{x})),$$

where $A_J(\underline{x})$ are $m' \times m$-valued C^∞ functions.

<u>Definition 1</u>: (a) The resulting $P := \mathcal{Q}_M : C^\infty(M, E) \to C^\infty(M, E')$ is called a (linear partial) <u>differential operator of degree d from E to E'</u>.

(b) Its (principal) <u>symbol</u> $\sigma_P \in \Gamma(M, \mathrm{Hom}(E, E') \otimes S^d T_M)$ is given locally by

$$\sigma_P(\underline{x}) = \sum_{|J| = d} A_J(\underline{x})\, D^J .$$

$\square$

<u>Remark 1</u>: (i) Writing $\sigma_P(\underline{x}, \omega) = \sum_{|J|=d} A_J(\underline{x}) D_J(\omega)$ for the (no differentiation here) associated degree-d homogeneous polynomial map from $T_M^\vee$ to $\mathrm{Hom}(E, E')$, for any $f \in C^\infty(M)$

$$e^{-tf(x)} P e^{tf(x)} = t^d \sigma_P(\underline{x}, df) + O(t^{d-1}).$$

This shows that σ_P is well-defined.

(ii) $\sigma_{P \circ Q} = \sigma_P \cdot \sigma_Q$

Definition 2 : P is elliptic $\Leftrightarrow$ $\sigma_P(\underline{x}, \omega) : E_x \to E'_x$ is injective $\forall\, x \in M$ & $\xi \in T^{*}_{M,x} \setminus \{0\}$.

Returning to the n-torus T (which is where we shall work today), the cotangent bundle is just $T \times \mathbb{R}^n$ (coords. $(x_1, \ldots, x_n; \omega_1, \ldots, \omega_n)$).
So if P is a (degree d) differential operator from $\underbrace{\mathbb{C}^m(\times T)}_{\text{trivial bundle}}$ to itself, then σ_P is a function on $T \times \mathbb{R}^n$ smooth/periodic in $\underline{x}$ and homogeneous of degree d in $\{\omega_j\}$.

We continue the line of thought from pp. 109 - 114 but with vector valued functions and Fourier coefficients : so

$\mathcal{H}_\infty = C^\infty(T, \mathbb{C}^m)$ (rather than $C^\infty(T)$) and so forth.

Begin by noting that we can extend D^J from $\mathcal{H}_\infty$ to $\mathcal{F}$ by setting $(D^J \mu)_{\underline{\xi}} := \underline{\xi}^{\underline{J}} \mu_{\underline{\xi}}$; then $\underline{\xi}^{2\underline{J}} \leq (1 + |\underline{\xi}|^2)^{|J|} \Longrightarrow$
$\|D^J \mu\|_{s-|J|} \leq c \|\mu\|_s \;\; \forall\, \mu \in \mathcal{H}_s$. How to extend differential operators in general? Start with multiplication by (coefficient) functions : for $f \in C^\infty(T)$ and $s \geq 0$, by equivalence of the "two" Sobolev norms, we have (for all $\varphi \in \mathcal{H}_\infty$)

(general symbol for constants we don't care about)

$$\|f\varphi\|_s \leq \# \sum_{|J| \leq s} \|D^J f \varphi\|_{L^2}$$

$$\leq \# \Bigg(\sum_{|J| \leq s} \|f D^J \varphi\|_{L^2} + \overbrace{\sum_{|J| \leq s} \|\underbrace{(D^J f - f D^J)}_{\text{degree} \leq s-1} \varphi\|_{L^2}}^{\leq \# \sum_{|J| \leq s-1} \|D^J \varphi\|} \Bigg) \Longrightarrow$$

(B.1)* $\|f\varphi\|_s \leq C\|f\|_\infty \|\varphi\|_s + C'\|\varphi\|_{s-1}$ $\Longrightarrow$

(B.2) $\|f\varphi\|_s \leq C''\|\varphi\|_s$ $\xRightarrow[\text{density of } \mathcal{H}_\infty]{}$ multiplication by f extends to bounded operator on $\mathcal{H}_s$ (and (B.1-2) apply to $\varphi \in \mathcal{H}_s$).

[Note: these hold for $s<0$ too by similar argument.]

So for degree d operators:

bound on highest order coeff. functions in P — i.e. the ones in σ_P

(B.3) $\|P\varphi\|_{s-d} \overset{(B.1)}{\leq} CK\|\varphi\|_s + C'\|\varphi\|_{s-1}$ $\Longrightarrow$

(B.4) $\|P\varphi\|_{s-d} \leq C''\|\varphi\|_s$ $\xRightarrow[\text{density of } \mathcal{H}_\infty]{}$ $\exists$ (by continuity) bounded extensions $\underline{\overline{P : \mathcal{H}_s \to \mathcal{H}_{s-d}}}$ ($\forall s \in \mathbb{Z}$).

Let $\begin{cases} f \in C^\infty(T) \\ \mu, \nu \in \mathcal{H}_\infty \end{cases}$; then

(B.5) $\langle f\mu, \nu\rangle_0 = \langle \mu, \bar{f}\nu\rangle_0$ (extends to $\mu, \nu \in \mathcal{H}_0$)

Introduce $K^t :=$ operator on $\mathcal{H}_\infty$ defined by

$$(K^t\varphi)_{\underline{\xi}} := \left(1+|\underline{\xi}|^2\right)^t \varphi_{\underline{\xi}} \quad \text{OR} \quad K^t\varphi = \Big(1 - \sum_j \frac{\partial^2}{\partial x_j^2}\Big)^t \varphi,$$

[clearly sends $\mathcal{H}_\infty \to \mathcal{H}_\infty$] equivalently, for $t>0$ [$t>0$ only]

which extends to an isometry $K^t : \mathcal{H}_s \to \mathcal{H}_{s-2t}$. (Obvious.)

Say $s<0$. Then $\langle f\mu, \nu\rangle_s = \langle f K^{-s}K^s\mu, K^s\nu\rangle_0$

$= \langle K^s\mu, K^{-s}\bar{f}K^s\nu\rangle_0$

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* important note: in all inequalities, constants are independent of φ

$$= \langle \mu, \bar f \nu \rangle_s + \langle K^s \mu, \underbrace{(K^{-s} f - \bar f K^{-s})}_{=:L \text{ (deg. } -2s-1 \text{ operator)}} K^s \nu \rangle_0$$

$$\leq \|K^s\mu\|_{-s} \|L K^s \nu\|_s$$

$$\overset{(B.4)}{\leq} \|K^s \mu\|_{-s} \|K^s \nu\|_{-s-1}$$

$$\leq \|\mu\|_s \|\nu\|_{s-1}$$

A similar argument for $s>0$ + symmetry argument $\Rightarrow$

(B.6) $$\left|\langle f\mu, \nu\rangle_s - \langle \mu, \bar f \nu\rangle_s\right| \leq C\left(\|\mu\|_s \|\nu\|_{s-1} + \|\mu\|_{s-1}\|\nu\|_s\right).$$ [extends to $\mu, \nu \in \mathcal{H}_s$]

Given P, & f ℝ-valued,

(B.7) $$\left|\langle P(f^2\mu), P\mu\rangle_s - \|P(f\mu)\|_s^2\right| \leq C\|\mu\|_{s+d}\|\mu\|_{s+d-1}$$ $(\forall \mu \in \mathcal{H}_{s+d})$

suff. to prove for $\mu \in \mathcal{H}_\infty$

since
$$\text{LHS} \leq \left|\langle fP(f\mu), P\mu\rangle_s - \langle P(f\mu), fP\mu\rangle_s\right| + \left|\langle P(f\mu), \underbrace{(fP - Pf)}_{\deg.\ d-1}\mu\rangle_s\right| + \left|\langle \underbrace{(Pf - fP)}_{\deg\ d-1}(f\mu), P\mu\rangle_s\right|$$

Using (B.2) & (B.4) together with
- (B.6) for the 1st term
- Cauchy-Schwarz for the latter two

$\Rightarrow\ \leq \text{RHS}$

Henceforth we assume P is elliptic (of deg. d).

Theorem 1 (Gårding inequality): Given $s \in \mathbb{Z}$, $\exists\ c > 0$ s.t.

(B.8) $$\|\mu\|_{s+d} \leq C\left(\|P\mu\|_s + \|\mu\|_s\right) \quad \forall \mu \in \mathcal{H}_{s+d}.$$

Proof: It will suffice to prove this for $\mu = \varphi \in \mathcal{H}_\infty$.

$\boxed{\text{Step 1}}$: Let P_0 be an elliptic operator with constant coeffs. and all terms of degree d.

main point

ellipticity of P_0 $\Rightarrow$ $\displaystyle\min_{\substack{|\xi|=1=|u| \\ \xi\in\mathbb{C}^n,\, u\in\mathbb{C}^m}} |\sigma_{P_0}(\xi)u|^2 = c$ (†)

$$\Rightarrow |\sigma_{P_0}(\xi)u|^2 \geq c|\xi|^{2d}|u|^2 \quad \forall \begin{cases}\xi\in\mathbb{C}^n \\ u\in\mathbb{C}^m\end{cases}$$

$$\Rightarrow \|P_0\varphi\|_s^2 = \sum_{\xi\in\mathbb{Z}^n} |\sigma_{P_0}(\xi)\varphi_\xi|^2 (1+|\xi|^2)^s$$

$$\geq \# \sum_{\xi\in\mathbb{Z}^n} |\xi|^{2d}|\varphi_\xi|^2(1+|\xi|^2)^s$$

ξth Fourier coeff. of $P_0\varphi$

$$\Rightarrow (\|P_0\varphi\|_s + \|\varphi\|_s)^2 \geq \|P_0\varphi\|_s^2 + \|\varphi\|_s^2$$

$$\geq \sum_\xi |\varphi_\xi|^2(1+|\xi|^2)^s(1+\#|\xi|^{2d})$$

$$\geq \# \sum_\xi |\varphi_\xi|^2(1+|\xi|^2)^{s+d}$$

$$= \# \|\varphi\|_{s+d}.$$

$\boxed{\text{Step 2}}$: P general, $q\in\mathbb{R}^n$

$P_0 :=$ highest order part of P at q, extended "by translation" (to have constant coeffs.)

Now Step 1 $\Rightarrow$

$$(B.9) \quad \|\varphi\|_{s+d} \leq k(\|P_0\varphi\|_s + \|\varphi\|_s)$$

$$\leq k(\|(P_0-P)\varphi\|_s + \|P\varphi\|_s + \|\varphi\|_s).$$

Take $\epsilon\in(0,\frac{1}{2ck})$ and choose $q\in U$ s.t. (P_0-P)'s coeff. functions are $<\epsilon$ there;

the c from (B.3)

(†) this also suggests the term "elliptic"!

& let $\tilde{P}$ be a parabolic operator agreeing with $P_0 - P$ on U and with $\sigma_{\tilde{P}}$'s coeff. funs. $< \epsilon$ everywhere. Then for φ supported in U,

$$\|\varphi\|_{s+d} \underset{(B.9)}{\leq} k\left(\|P\varphi\|_s + \|\tilde{P}\varphi\|_s + \|\varphi\|_s\right)$$

$$\underset{\substack{(B.3) \\ (\text{\& noting meaning of } \kappa)}}{\leq} \tfrac{1}{2}\|\varphi\|_{s+d} + \#\left(\|P\varphi\|_s + \|\varphi\|_{s+d-1} + \|\varphi\|_s\right)$$

$$\leq \tfrac{3}{4}\|\varphi\|_{s+d} + \#\left(\|P\varphi\|_s + \|\varphi\|_s\right)$$

where at the end we have applied the Peter–Paul inequality

[Given $\underbrace{t' < t < t''}_{\in \mathbb{Z}}$ & $\epsilon > 0$, $\exists\, C(\epsilon) > 0$ s.t. $\|\mu\|_t^2 \leq \epsilon \|\mu\|_{t''}^2 + C(\epsilon)\|\mu\|_{t'}^2$ for all $\mu \in \mathcal{H}_{t''}$. (Exercise: show $C(\epsilon) := \epsilon^{(t'-t)/(t''-t)}$ does the job.)]

with $t' = s$, $t = s+d-1$, $t'' = s+d$, $\epsilon = \frac{1}{4}$.

$\boxed{\text{Step 3}}$ Consider a finite open cover of T by these U's & let η_j be a partition of 1. For $\varphi \in \mathcal{H}_\infty$,

$$\|\varphi\|_{s+d}^2 = \langle \varphi, \varphi\rangle_{s+d} = \langle \textstyle\sum \eta_j \varphi, \varphi\rangle_{s+d} \underset{\substack{(B.6) \\ +(B.2)}}{=} \sum \langle \eta_j^{1/2}\varphi, \eta_j^{1/2}\varphi\rangle + \#\|\varphi\|_{s+d}\|\varphi\|_{s+d-1}$$

$$\underset{\text{Step 2}}{\leq} \#\left(\sum_j \|P(\eta_j^{1/2}\varphi)\|_s^2 + \|\varphi\|_s^2 + \|\varphi\|_{s+d}\|\varphi\|_{s+d-1}\right)$$

$$\underset{\substack{(B.7) \\ \text{with } f = \eta_j^{1/2}}}{\leq} \#\left(\underbrace{\sum_j \langle P(\eta_j\varphi), P\varphi\rangle_s}_{\|P\varphi\|_s^2} + \|\varphi\|_s^2 + \|\varphi\|_{s+d}\|\varphi\|_{s+d-1}\right)$$

$$\leq \tfrac{1}{2}\|\varphi\|_{s+d}^2 + \#\left(\|P\varphi\|_s^2 + \|\varphi\|_s^2 + \|\varphi\|_{s+d-1}^2\right)$$

$$\underset{\text{Peter–Paul}}{\leq} \tfrac{3}{4}\|\varphi\|_{s+d}^2 + \#\cdot\left(\|P\varphi\|_s^2 + \|\varphi\|_s^2\right).$$

$\square$

Let $\begin{cases} \varphi \in \mathcal{H}_\infty \\ \underline{h} \in \mathbb{R}^n \setminus \{0\} \end{cases}$; Then $\varphi(\underline{x}) \mapsto \varphi(\underline{x}+\underline{h})$ sends $\underbrace{\varphi_{\underline{\xi}} \mapsto e^{i\underline{h}\cdot\underline{\xi}}\varphi_{\underline{\xi}}}$

extends to $\widetilde{\mathcal{D}}$: call this T_h
— clearly an isometry on each $\mathcal{H}_s$.

Set $\mu^h := \dfrac{T_h(\mu)-\mu}{|\underline{h}|}$; then $\left|\dfrac{e^{ih\cdot\xi}-1}{|\underline{h}|}\right|^2 \leq 1+|\underline{\xi}|^2 \Rightarrow \|\mu^h\|_s \leq \|\mu\|_{s+1}$.

<u>Lemma 1</u>: Suppose $\exists\, k>0$ s.t. $\|\mu^h\|_s \leq k \quad \forall\, \underline{h} \in \mathbb{R}^n\setminus\{0\}$. Then $\mu \in \mathcal{H}_{s+1}$

(roughly "you can integrate" μ^h)

<u>Proof</u>: Let $\mu_{\leq N}$ denote the truncation (partial sum), $\{\hat{\underline{e}}_j\}$ o.n. basis of $\mathbb{R}^n$,

$\underline{h} = t\hat{\underline{e}}_j$

$$\left|\frac{e^{i\underline{h}\cdot\underline{\xi}}-1}{|\underline{h}|}\right|^2 = \left|\frac{e^{it\xi_j}-1}{t}\right|^2 \xrightarrow[t\to 0]{} |\xi_j|^2$$

By assumption, $\displaystyle\sum_{|\underline{\xi}|\leq N} (1+|\underline{\xi}|^2)^s |\mu_{\underline{\xi}}|^2 \left|\frac{e^{i\underline{h}\cdot\underline{\xi}}-1}{|\underline{h}|}\right|^2 \leq k^2$

$$\Big\downarrow\ t\to 0$$

$$\sum_{|\underline{\xi}|\leq N} (1+|\underline{\xi}|^2)^s |\mu_{\underline{\xi}}|^2 |\xi_j|^2 \leq k^2$$

$$\Longrightarrow \|\mu_{\leq N}\|_{s+1} \leq nk^2 + \|\mu\|_s^2$$

$$\underset{N\to\infty}{\Longrightarrow} \|\mu\|_{s+1} < \infty .$$

$\square$

<u>Theorem 2 (Periodic Regularity)</u>: Given $\mu \in \mathcal{H}_{-\infty}$, $\nu \in \mathcal{H}_s$, s.t. $P\mu = \nu$, we have $\mu \in \mathcal{H}_{s+d}$.

<u>Proof</u>: Suppose $\mu \in \mathcal{H}_s$, $\nu = L\mu \in \mathcal{H}_{s-d+1}$. <u>Wts</u>: $\mu \in \mathcal{H}_{s+1}$.

Applying $(\cdot)^h$ to coeff. functions of P defines "P^h".

Given $\varphi \in C^\infty$, $P(\varphi^h) = (P\varphi)^h - P^h(T_h\varphi)$ (trivial – write it out)

$\Longrightarrow$ (density of $\mathcal{H}_\infty$ + continuity of extension of P) works for $\mathcal{H}_{-\infty}$. By Gårding*, and writing $\#'$ for constants indep. of h,

$$\|\mu^h\|_s \leq \#'\left(\|P\mu^h\|_{s-d} + \|\mu^h\|_{s-d}\right)$$

$$\leq \#'\left(\|(P\mu)^h\|_{s-d} + \|P^h(T_h\mu)\|_{s-d} + \|\mu^h\|_{s-d}\right)$$

(coeffs. of P^h uniformly bounded + (B.4))

$$\leq \#'\left(\|(P\mu)^h\|_{s-d} + \|T_h\mu\|_s + \|\mu^h\|_{s-d}\right)$$

$$\leq \#'\left(\|P\mu\|_{s-d+1} + \|\mu\|_s\right)$$

$\left\{\begin{array}{l} \cdot\ T_h \text{ isometry} \\ \cdot\ \|\mu\|_s \geq \|\mu\|_{s-d} \\ \cdot\ \|(\cdot)^h\|_t \leq \|(\cdot)\|_{t+1} \end{array}\right.$

$\underset{\text{Lemma 1}}{\Longrightarrow} \mu \in \mathcal{H}_{s+1}$.

□

Now for any (periodic) differential operator $P = \sum_{|J|\leq d} \overbrace{A_J(\underline{x})}^{m\times m \text{ matrices}} D^J$ on $\mathbb{T}$, we have the

Definition 3: The *formal adjoint* of P is $P^* := \sum_{|J|\leq d} D^J\, {}^t\overline{A_J(\underline{x})}$.

Ex/ (a) Using $\int$ by parts, $\langle P\varphi, \psi\rangle = \langle \varphi, P^*\psi\rangle$ $\forall \varphi, \psi \in \mathcal{H}_\infty$

(b) Taking $\varphi_j \xrightarrow[\|\cdot\|_s]{\in \mathcal{H}_\infty} \varphi$, show that this extends to the setting $\varphi \in \mathcal{H}_s$ ($\psi \in \mathcal{H}_\infty$)

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* this is where ellipticity enters into this proof!

Consider the setting

$$(B.10)\quad \begin{cases} \bullet\ P \text{ diff. op. on } T \quad (\text{degree } d) \\ \bullet\ f \in \mathcal{H}_\infty \\ \bullet\ \underset{\text{open}}{V} \subset \bar{V} \subset \left\{ \begin{matrix} \text{fund. domain} \\ \text{for } T \end{matrix} \right\} (\subset \mathbb{R}^n) \end{cases}$$

If $\varphi \in \mathcal{H}_\infty$ satisfies $P\varphi = f$ on $V (\subset T)$, then

$$\langle f, \psi \rangle = \langle P\varphi, \psi \rangle = \langle \varphi, P^* \psi \rangle \quad \forall\ \psi \in \mathcal{H}_\infty^c(V) \left(:= \underset{\nwarrow \text{compact support}}{C_c^\infty}(V, \mathbb{C}^m) \right)$$

<u>Definition 4</u>: A <u>weak solution</u> of $P(\cdot) = f$ on V is a B.L.F. (bounded* linear functional) $l : \mathcal{H}_\infty^c(V) \to \mathbb{C}$ satisfying

$$l(\underbrace{P^*\psi}_{\text{note that this is still in } \mathcal{H}_\infty^c(V)}) = \langle f, \psi \rangle \quad (\forall\ \psi \in \mathcal{H}_\infty^c(V)).$$

<u>Proposition 1 (Local regularity)</u>: Assume P, f, V, l are as in (B.10) and Definition 4, and that P is elliptic. Then for each $p \in V$ $\exists$ neighborhood W_p and $\varphi_p \in \mathcal{H}_\infty$ s.t. $l(z) = \langle \varphi_p, z \rangle$ $\forall z \in \mathcal{H}_\infty^c(W_p)$
($\Rightarrow P\varphi_p = f$ on V).

<u>Remark 2</u>: Here's a simple example of how strong this is:

Ex / ($n=2$, think $\mathbb{R}^2 = \mathbb{C}$) $P = \partial/\partial\bar{z}$ is elliptic //

Consequence: Any distributional, or L^2, or C^s solution of $\partial/\partial\bar{z}(\cdot) = 0$ is actually <u>smooth</u>. So holo. $\Rightarrow$ smooth, but also: holo. distributions are genuine holo. functions.

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* i.e. $\exists\ c > 0$ s.t. $\forall\ z \in \mathcal{H}_\infty^c(V)$, $|l(z)| \le c \|z\|_0$.

To prove Prop. 1 we'll need 2 basic functional analysis results:

<u>Riesz Representation Theorem</u>: Every B.L.F. l on a Hilbert space $\mathbb{H}$ is of the form $l(\cdot) = \langle \mu, \cdot \rangle$ for some $\mu \in \mathbb{H}$.

<u>Hahn-Banach Theorem</u>: Every B.L.F. l on a linear subspace of a normed vector space $\mathbb{H}$ extends to a B.L.F. on $\mathbb{H}$.

<u>Proof of Prop. 1</u>: H-B with $U = \mathcal{H}^c_\infty(V)$, $\mathbb{H} = \mathcal{H}_0 \, (= L^2) \Rightarrow$ $\exists$ bounded extension $\tilde{l} : \mathcal{H}_0 \to \mathbb{C}$. Now Riesz $\Rightarrow \exists \mu \in \mathcal{H}_0$ s.t. $\tilde{l}(t) = \langle \mu, t \rangle_0$ $(\forall t \in \mathcal{H}_0)$.

Consider a countable sequence of open neighborhoods of p:

$$V \supset (\overline{V_0} \supset) \, V_0 \supset (\overline{V_1} \supset) \, V_1 \supset (\overline{V_2} \supset) \, V_2 \supset \dots \supset (\overline{V_\infty} \supset) V_\infty \supset (\overline{W_p} \supset) W_p$$

together with fcns. $\begin{cases} \eta_j \in C_c^\infty(V_{j-1}, [0,1]) \text{ s.t. } \eta_j|_{V_j} \equiv 1 \\ \eta \in C_c^\infty(V_\infty, [0,1]) \text{ s.t. } \eta|_{W_p} \equiv 1. \end{cases}$

Then*

$$\mu_1 := \eta_1 \mu \in \mathcal{H}_0$$

has

$$\text{(B.11)} \qquad P\mu_1 = \eta_1 \underbrace{P\mu}_{\in \mathcal{H}_{-d}} + \underbrace{\overbrace{(P\eta_1 - \eta_1 P)}^{\text{deg. } d-1}\mu}_{\in \mathcal{H}_{1-d}}.$$

Given $\varphi \in \mathcal{H}_\infty$,

$$\langle \eta_1 P\mu - \eta_1 f, \varphi \rangle = \langle \eta_1 P\mu, \varphi \rangle_0 - \langle \eta_1 f, \varphi \rangle_0$$

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* recall that we extended multiplication by C^∞ functions, as well as C^∞ diff. ops., to Sobolev spaces above.

$$\left(\text{by Exercise (a) above}\right) = \langle \mu, P^* \eta_1 \varphi \rangle_0 - \langle f, \eta_1 \beta \rangle_0$$

$$= \tilde{l}(P^* \eta_1 \varphi) - \tilde{l}(P^* \eta_1 \varphi)$$

$$= 0$$

$\overset{(*^1)}{\Longrightarrow}$ $\eta_1 P \mu = \eta_1 f \ (\in \mathcal{H}_\infty)$

$\Longrightarrow$ RHS (B.11) $\in \mathcal{H}_{1-d}$

$\underset{\text{(periodic) reg. thm.}}{\Longrightarrow}$ $\mu_1 \in \mathcal{H}_{1 \, (=(1-d)+d)}$

Next,

$\mu_2 := \eta_2 \mu$

has

$$P\mu_2 = \eta_2 P \mu + \overbrace{(P\eta_2 - \eta_2 P)}^{\deg d-1} \mu$$

$$= \eta_2 f + (P\eta_2 - \eta_2 P) \underset{\mathcal{H}_1}{\underset{\cap}{\mu_1}} \quad \text{since } \mu \ \& \ \mu_1 \text{ are equal on } V_1 \ \overset{(*^2)}{}$$

$$\in \mathcal{H}_{2-d \, (=1-(d-1))}$$

$\underset{\text{(periodic) reg. thm.}}{\Longrightarrow}$ $\mu_2 \in \mathcal{H}_2$.

Continuing in this fashion, we find $\mu_k := \eta_k \mu \in \mathcal{H}_k$ $(\forall k \in \mathbb{N})$

$\Longrightarrow$ $\varphi_p := \eta \mu = \eta \eta_k \mu \in \mathcal{H}_\infty$.

$(*^1)$ A little argument here is needed: if $\varepsilon \in \mathcal{H}_s$, with $\langle \varepsilon, \varphi \rangle_0 = 0$ $(\forall \varphi \in \mathcal{H}_\infty)$,

then taking $\varphi_j \overset{\in \mathcal{H}_\infty}{\underset{\|\cdot\|_{-s}}{\longrightarrow}} K^s \varepsilon$ we have

$$\langle \varepsilon, \varepsilon \rangle_s \underset{\substack{\uparrow \\ K^s \text{ isometry}}}{=} \langle K^s \varepsilon, K^s \varepsilon \rangle_{-s} = \lim_j \langle K^s \varepsilon, \varphi_j \rangle_{-s} = \lim_j \cancel{\langle \varepsilon, \underbrace{K^s \varphi_j}_{\in \mathcal{H}_\infty} \rangle} \to 0 = 0$$

$\Longrightarrow \varepsilon = 0$.

$(*^2)$ Suppose ν' & $\nu'' \in \mathcal{H}_s$ are <u>equal on U</u> $\left(\overset{\text{def.}}{\Longleftrightarrow} \langle \nu' - \nu'', \varphi \rangle_0 = 0 \ \forall \varphi \in \mathcal{H}^c_\infty(U)\right)$

If some $\hat{P}$ has compact supp. in U, then

$$\langle \hat{P}(\nu' - \nu''), \psi \rangle_0 = \langle \nu' - \nu'', \underbrace{\hat{P}^* \psi}_{\in \mathcal{H}^c_\infty(U)} \rangle_0 = 0 \quad \forall \psi \in \mathcal{H}_\infty \overset{\text{by } (*^1)}{\Longrightarrow} \hat{P}\nu' = \hat{P}\nu''.$$

But then, for $z \in \mathcal{H}^c_\infty(W_p)$,

$$l(z) = \tilde{l}(z) = \langle \mu, z\rangle_0 = \langle \mu, \eta z\rangle_0 = \langle \eta\mu, z\rangle_0 = \langle \varphi_p, z\rangle. \qquad \square$$

Now let $\mathcal{D}$ be a degree-d elliptic differential operator from E to E' (over M), assumed holomorphic & Hermitian with $m = m'$ and let $\sigma' \in C^\infty(M, E')$. Taking a finite cover $\{U_\alpha\}$ of M with $U_\alpha \subset \overline{U_\alpha} \subset \{\text{fund. domain for } T\} \subset \mathbb{R}^n$, we have (using the Hermitian metrics!)

- $\langle \omega, \nu\rangle_E = \sum_\alpha h_\alpha(\omega_\alpha, \underset{\uparrow}{\eta_\alpha} \nu_\alpha) = \sum_\alpha \langle \omega_\alpha, \overset{\text{Hermitian mx}}{H_\alpha} \eta_\alpha \nu_\alpha\rangle$

  Π of 1

  modify on complement of $\mathrm{supp}(\eta_\alpha)$, regard as $\in \mathcal{H}_\infty$ (i.e. periodic)

- $\langle \mathcal{D}\omega, \nu'\rangle_{E'} = \sum_\alpha \langle D_\alpha \omega_\alpha, H'_\alpha \eta_\alpha \nu'_\alpha\rangle_0 \underset{\int \text{ by parts}}{=} \sum_\alpha \langle \omega_\alpha, H_\alpha \eta_\alpha \underbrace{\{\eta_\alpha^{-1} H_\alpha^{-1} (D_\alpha)^* H'_\alpha \eta_\alpha\}}_{=: (\mathcal{D}^*)_\alpha} \nu'_\alpha\rangle_0$

  as defined in Defn. 3

  $=: \langle \omega, \mathcal{D}^* \nu'\rangle_E$ defines a formal adjoint.

  (Note that since $m = m'$, $\mathcal{D}^*$ is also elliptic.)

- We can define weak sol'n of $\mathcal{D}(\cdot) = \sigma'$ as a B.L.F.

  $$L : C^\infty(M, E) \to \mathbb{C} \text{ s.t. } L(\mathcal{D}^* \nu') = \langle \sigma', \nu'\rangle_{E'} \quad (\forall \nu').$$

Theorem 3 (Global regularity): Given a weak solution L of $\mathcal{D}(\cdot) = \sigma'$, $\exists\, \beta \in C^\infty(M, E)$ s.t. $L(\sigma) = \langle \beta, \sigma\rangle$ $\forall \sigma \in C^\infty(M, E)$; hence, $\mathcal{D}\beta = \sigma'$.

Proof: It will suffice to show that for any $p \in M$, $\exists \begin{cases} W_p \text{ s.t.} \\ \beta_p \in C^\infty(W_p, E) \end{cases}$ on $(\sigma \in)\, C^\infty_c(W_p, E)$, $L(\sigma) = \langle \beta_p, \sigma\rangle_E$. For given two such "solutions"

"local problem"

β_p & β_q, for $\sigma \in C_c^\infty(W_p \cap W_q, E)$ $\langle \beta_p, \sigma \rangle = L(\sigma) = \langle \beta_q, \sigma \rangle$

$\Rightarrow \langle \beta_p - \beta_q, \sigma \rangle = 0 \ (\forall \sigma) \Rightarrow \beta_p = \beta_q$ on $W_p \cap W_q$. But then for

general $L(\sigma) = \sum_p L(\eta_p \sigma) = \sum_p \langle \beta_p, \eta_p \sigma \rangle = \langle \underbrace{\sum \eta_p \beta_p}_{=:\beta}, \sigma \rangle$,

η_p: Π of 1 associated to some finite subcover $\{W_p\}$

and $\langle \sigma', \nu' \rangle_{E'} = L(\mathcal{D}^* \nu') = \langle \beta, \mathcal{D}^* \nu' \rangle_E = \langle \mathcal{D}\beta, \nu' \rangle_{E'}$, $(\forall \nu')$

$\Rightarrow \mathcal{D}\beta = \sigma'$!

To solve the local problem, pick $p \in W_p \subset \overline{W_p} \subset U_\alpha \subset T$, and define

- an elliptic diff. operator $\tilde{\mathcal{D}}_p \begin{cases} = \mathcal{D}_\alpha & \text{on } W_p \\ \equiv \mathcal{D}_{\alpha,p} & \text{on } T \backslash U_\alpha \end{cases}$ (this will be elliptic for W_p suff. small)
- $f_p \in \mathcal{H}_\infty$ extending $\sigma'|_{W_p}$
- a B.L.F. on $\mathcal{H}_\infty^c(W_p)$ by $l_p(z) := L(\underbrace{H_\alpha^{-1} z}_{})$

extend by 0 to $C^\infty(M, E)$

[Ex/ l_p is bounded since L is and the matrix norms $\|H_\alpha'^{-1}\|$ are (on $\overline{W_p}$)]

But then $l_p((\tilde{\mathcal{D}}_p)^* \underset{\mathcal{H}_\infty^c(W_p)}{\underset{\cap}{\psi}}) = L(H_\alpha^{-1} (\mathcal{D}_\alpha)^* \psi) = L(\mathcal{D}^* H_\alpha'^{-1} \psi)$

$(\mathcal{D}^*)_\alpha = H_\alpha^{-1} (\mathcal{D}_\alpha)^* H_\alpha'$ (near p)

$$= \langle \sigma', H_\alpha'^{-1} \psi \rangle_{E'} = \langle f_p, \psi \rangle_0$$

and local regularity $\Longrightarrow$

$$l_p(z) = \langle \beta_p, z \rangle_0 \Longrightarrow$$

on $(\sigma \in) C^\infty(W_p, E)$, $L(\sigma) = l_p(H_\alpha \sigma) = \langle \beta_p, H_\alpha \sigma \rangle_0 = \langle \beta_p, \sigma \rangle_E$,

as desired. $\square$

Proposition 2 (Global Gårding): Given a sequence $\{\omega_n\} \subset C^\infty(M,E)$ and $C>0$ s.t. $\|\omega_n\|_E, \|D\omega_n\|_{E'} \le C$ $(\forall n)$, then $\exists$ $\|\cdot\|_E$-Cauchy subsequence $\{\omega_{n_k}\}$.

Proof: It's sufficient to show $\{\eta_\alpha \omega_n\} =: \{\varphi_n\} \subset \mathcal{H}_\infty$ has one, in $\|\cdot\|_0$ (diagonal subsequence argument will give a "global" Cauchy ssq.).

Now Rellich $\Rightarrow$ suffices to show $\|\varphi_n\|_1$ bounded.

Replacing ω_n by a periodic ($\in \mathcal{H}_\infty$) function $\tilde{\omega}_n$ (agreeing w/ original ω_n on supp(η_α)), D by periodic P (similarly to last proof), we have $\|\varphi_n\|_1 =$

$$\|\eta\tilde{\omega}_n\|_1 \underset{\text{Gårding ineq.}}{\le} \#\left(\|P\eta\tilde{\omega}_n\|_{1-d} + \|\eta\tilde{\omega}_n\|_{1-d}\right)$$

insert $\tau \begin{cases} \equiv 1 & \text{on } supp(\eta) \\ \equiv 0 & \text{off } U_\alpha \end{cases}$

$$\le \#\Big(\underbrace{\|\eta P\tilde{\omega}_n\|_{1-d}}_{} + \underbrace{\|\overbrace{(P\eta - \eta P)}^{\deg d-1}\tilde{\omega}_n\|_{1-d}}_{} + \underbrace{\|\eta\tilde{\omega}_n\|_{1-d}}_{}\Big)$$

$$\begin{array}{lll}
\le \|\eta P\overset{(\sim)}{\omega}_n\|_0 & \le \#\|\tau\overset{(\sim)}{\omega}_n\|_0 & \le \|\eta\overset{(\sim)}{\omega}_n\|_0 \\
\le \#\|\eta D\omega_n\|_{E'} & \le \#\|\tau\omega_n\|_E & \le \#\|\eta\omega_n\|_E \\
\le \#\|D\omega_n\|_{E'} & \le \#\|\omega_n\|_E & \le \#\|\omega_n\|_E
\end{array}$$

where none of the constants depend on n. $\square$

We are now ready for the two profound results:

$\boxed{\text{Finiteness Theorem:}}$ $\mathrm{Ker}(D) \subset C^\infty(M,E)$ is finite-dimensional.

Proof: Suppose not; then $\ker(D) \supseteq$ infinite orthonormal seq. $\{\omega_n\}$ $\underset{\text{prop. 2}}{\Longrightarrow}$ $\exists$ $\{\omega_{n_k}\}$ Cauchy. But this is a contradiction, because that $\|\omega_{n_k} - \omega_{n_\ell}\| = \sqrt{2}$. $\square$

Remark 3: Obviously the theorem applies also to $\mathcal{D}^*$.

If $E = E'$, then

$$\dim(\ker \mathcal{D}) - \dim(\ker \mathcal{D}^*)$$

defines the analytical index of $\mathcal{D}$. There is a topological index defined using the Chern classes of E and the symbol $\sigma_{\mathcal{D}}$, and their equality is the content of the Atiyah – Singer index theorem.

$\boxed{\text{Decomposition Theorem}}$: $C^\infty(M,E) = \mathcal{D}^*(C^\infty(M,E')) \oplus \ker(\mathcal{D})$.

Remark 4: swapping $\mathcal{D} \leftrightarrow \mathcal{D}^*$, we see that $\ker \mathcal{D}^* \cong \operatorname{coker} \mathcal{D}$.

For the proof, we will need

Lemma 2: $\exists\, C > 0$ s.t. $\|\beta\|_E \le C \|\mathcal{D}\beta\|_{E'}$ $\forall \beta \in (\ker \mathcal{D})^\perp$.

Pf: Otherwise $\exists$ $\boxed{\{\beta_j\} \subset (\ker \mathcal{D})^\perp \text{ with } \begin{cases} \|\beta_j\|_E = 1 \\ \|\mathcal{D}\beta_j\|_{E'} \to 0 \end{cases}}$ $(*)$

$\Longrightarrow$ $\exists$ Cauchy subsequence (still call β_j)

$\overset{\text{Prop. 2}}{\Longrightarrow}$ $\langle \beta_j, \psi \rangle_E$ Cauchy $\forall \psi \in C^\infty(M,E)$

$\Rightarrow$ can define a B.L.F. on $C^\infty(M,E)$ by $\hat{\ell}(\psi) := \lim_{j\to\infty} \langle \beta_j, \psi\rangle_E$, and

$$\hat{\ell}(\mathcal{D}^*\varphi) = \lim_{j\to\infty} \langle \beta_j, \mathcal{D}^*\varphi\rangle_E = \lim_{j\to\infty} \langle \mathcal{D}\beta_j, \varphi\rangle_E = 0$$

$\Rightarrow$ $\hat{\ell}$ weak solution of $\mathcal{D}(\cdot) = 0$

$\underset{\text{Global Reg.}}{\Longrightarrow}$ $\exists\, \beta \in C^\infty(M,E)$ s.t. $\hat{\ell}(\psi) = \langle \beta, \psi\rangle$ $(\forall \psi)$

$\Longrightarrow$ $\beta_j \to \beta$

$\underset{(*)}{\Longrightarrow}$ $\|\beta\|_E = 1$ and $\beta \in (\ker(\mathcal{D}))^\perp$

But $\hat{\ell}(\psi) = \langle \beta, \psi \rangle$ also $\Rightarrow$ $\mathcal{D}\beta = 0$. $\#$ $\square$

Proof of D.T.: Let $\{\omega_1, \dots, \omega_\ell\} \subset \ker \mathcal{D}$ be an o.n. basis (under $\|\cdot\|_E$).

For any $\alpha \in C^\infty(M,E)$,

$$\alpha - \underbrace{\sum \langle \alpha, \omega_i \rangle \omega_i}_{=:\pi(\alpha)} \in (\ker \mathcal{D})^\perp$$

$$\Longrightarrow C^\infty(M,E) = (\ker \mathcal{D}) \oplus (\ker \mathcal{D})^\perp.$$

We must show $(\ker \mathcal{D})^\perp = \mathcal{D}^*(C^\infty(M,E'))$.

($\supseteq$): $\omega \in C^\infty(M,E')$ and $\alpha \in \ker \mathcal{D} \Longrightarrow \langle \mathcal{D}^*\omega, \alpha \rangle = \langle \omega, \underbrace{\mathcal{D}\alpha}_{0} \rangle = 0$.

($\subseteq$): Given $\alpha \in (\ker(\mathcal{D}))^\perp$, define a B.L.F. on $\mathcal{D}(C^\infty(M,E))$ by

(B.12) $$\ell(\mathcal{D}\varphi) := \langle \alpha, \varphi \rangle \qquad (\forall \varphi \in C^\infty(M,E)).$$

Check

well-defined? $\mathcal{D}\varphi_1 = \mathcal{D}\varphi_2 \Rightarrow \varphi_1 - \varphi_2 \in \ker \mathcal{D} \Rightarrow \langle \alpha, \varphi_1 - \varphi_2 \rangle = 0$.

bounded? let $\varphi \in C^\infty(M,E)$, $\psi := \varphi - \pi(\varphi) \in (\ker \mathcal{D})^\perp$. Then

$$|\ell(\mathcal{D}\varphi)| = |\ell(\mathcal{D}\psi)| = |\langle \alpha, \psi \rangle| \le \|\alpha\| \|\psi\| \underset{\text{Lemma 2}}{\le} c\|\alpha\| \|\mathcal{D}\psi\|$$
$$= c\|\alpha\| \, \|\mathcal{D}\varphi\|.$$

Now Hahn–Banach $\Rightarrow$ ℓ extends to a B.L.F. on $C^\infty(M,E')$

$\underset{\text{(B.12)}}{\Longrightarrow}$ $\tilde{\ell}$ is a weak solution of $\mathcal{D}^*(\cdot) = \alpha$

$\underset{\text{global regularity}}{\Longrightarrow}$ $\exists\, \omega \in C^\infty(M,E')$ s.t. $\mathcal{D}^*\omega = \alpha$. $\square$

Corollary 1: If $E = E'$ and $\mathcal{D} = \mathcal{D}^*$ (is formally self-adjoint), then $\boxed{C^\infty(M,E) = \operatorname{Im} \mathcal{D} \oplus \ker \mathcal{D}}$.