C. Harmonic representatives

Hodge star

Let V = finite-dim'l (n) $\mathbb{R}$-vector space with $\begin{cases} \text{inner product } \langle , \rangle \\ \text{o.n. basis } \{e_1, \dots, e_n\} \\ \text{choice of orientation "} e_1 \wedge \dots \wedge e_n > 0 \text{"} \end{cases}$

Extend $\langle , \rangle$ to $\Lambda^{\cdot} V := \sum_{k \geq 0} \Lambda^k V$ by $\begin{cases} \langle \Lambda^k V, \Lambda^l V \rangle = 0 \text{ if } l \neq k \\ \langle v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k \rangle := \det\{\langle v_i, w_j \rangle\} \\ \text{extend linearly} \end{cases}$

<u>Definition 1</u>: The <u>Hodge star</u> is the linear transformation

$$\begin{array}{ccc} * : \Lambda^{\cdot} V & \xrightarrow{\cong} & \Lambda^{\cdot} V \\ \cup & & \cup \\ \Lambda^k V & \xrightarrow{\cong} & \Lambda^{n-k} V \end{array} \quad \text{defined by} \quad \begin{cases} *(1) = e_1 \wedge \dots \wedge e_n \\ *(e_1 \wedge \dots \wedge e_n) = 1 \\ *(e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(k)}) = (-1)^{\operatorname{sgn}(\sigma)} e_{\sigma(k+1)} \wedge \dots \wedge e_{\sigma(n)} \\ \quad \forall \{ \sigma \in S_n \\ \quad\ \ \{ k \\ \text{linear extension} \end{cases}$$

<u>Proposition 1</u>:

(a) $**|_{\Lambda^k V} = (-1)^{k(n-k)}$

(b) $*$ independent of choice of o.n. basis (depends only on $\langle , \rangle$)

(c) $\omega, \eta \in \Lambda^k V \implies \begin{cases} \langle \omega, \eta \rangle = *(\omega \wedge * \eta) \quad (\in \mathbb{R}) \\ \langle * \omega, * \eta \rangle = \langle \omega, \eta \rangle \quad (* \text{ is an isometry}) \end{cases}$

<u>Sketch of Pf</u>: (a) $**(e_1 \wedge \dots \wedge e_k) = *(e_{k+1} \wedge \dots \wedge e_n) = (-1)^{\operatorname{sgn}\{\{1,\dots,k\},\{k+1,\dots,n\}\}} e_1 \wedge \dots \wedge e_k$

(b) $\det(\text{transition matrix}) = 1$ (same volume of parallelepiped)

(c) use fact that $e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(k)}$ are the o.n. elts of $\Lambda^k V$. $\square$

We also get an inner product ($\&$ Hodge $*$) on $\Lambda^{\cdot} V^{\vee}$:

$\langle , \rangle$ induces map $\begin{array}{c} V \xrightarrow{\xi} V^{\vee} \\ v \longmapsto \langle v, - \rangle \end{array}$ hence $V^{\vee} \otimes V^{\vee} \xrightarrow{(\xi^{-1})^{\otimes 2}} V \otimes V \xrightarrow{\langle , \rangle} \mathbb{R}$

Let (M,g) be a compact oriented Riemannian manifold, so that the above discussion applies to $T_{M,p}$ & $T^\vee_{M,p}$ for each $p \in M$.
The resulting (fiberwise) Hodge star induces C^∞ bundle morphisms

$$*: \Lambda^k T_M^\vee \longrightarrow \Lambda^{n-k} T_M^\vee,$$

which we may also extend $\mathbb{C}$-linearly to

$$*: \underbrace{\Lambda^k T_M^\vee \otimes \mathbb{C}}_{=: E^k_M} \longrightarrow \Lambda^{n-k} T_M^\vee \otimes \mathbb{C}$$

and (taking sections)

$$*: A^k(M) \to A^{n-k}(M).$$

We also get Hermitian metrics on the E^k_M

$$\langle \alpha, \beta \rangle_{E^k_{M,p}} := *(\alpha \wedge * \bar{\beta}) \underset{\text{Prop. 1(e)}}{=\!=} \overset{\text{extended to } \Lambda T^\vee_{M,p}}{g}(\alpha, \bar{\beta}) \qquad (\alpha, \beta \in E^k_{M,p}).$$

Integrating this, we have

Definition 2: The Hodge inner product on $\sum_{k \geq 0} A^k(M)$ is

$$\langle \omega, \eta \rangle := \int_M \omega \wedge * \bar{\eta} \quad \left(= \int_M g(\omega, \bar{\eta})\, dV \right),$$ w./corr. norm $\|\omega\|$.

The Laplacian

Now consider

$$A^{n-k} \dashrightarrow A^{n-k+1}$$

$$\delta := \begin{cases} (-1)^{n(k+1)+1} * d * : A^k(M) \to A^{k-1}(M) \\ (0 \quad \text{on } A^0(M)) \end{cases},$$

and note $* \delta = (-1)^{n(k+1)+1} (-1)^{(n-k+1)(k-1)} d* = (-1)^k d*$.

Proposition 2: $\langle d\omega, \eta \rangle = \langle \omega, \delta \eta \rangle$.

Proof: wlog wma $\begin{cases} \omega \in A^{k-1}(M) \\ \eta \in A^k(M) \end{cases} \implies d(\omega \wedge * \bar{\eta}) = d\omega \wedge * \bar{\eta} + (-1)^{k-1} \omega \wedge d * \bar{\eta}$

$$= d\omega \wedge * \bar{\eta} - \omega \wedge * \delta \bar{\eta}$$

$$\underset{\text{Stokes}}{\implies} 0 = \int_M d\omega \wedge * \bar{\eta} - \int_M \omega \wedge * \delta \bar{\eta}$$

$$= \langle d\omega, \eta \rangle - \langle \omega, \delta \eta \rangle. \qquad \square$$

Definition 3: The (d-) Laplacian on M is

$$\Delta_{(d)} := \delta d + d\delta : \begin{cases} E^k_M \to E^k_M \\ A^k(M) \to A^k(M) \end{cases}$$

Ex/
(C.1) $\boxed{*\Delta = \Delta *}$

Example 1: $(M = \mathbb{R}^2, k=0)$

fcn. on $\mathbb{R}^2$

$$-\Delta f = \underbrace{*d*df}_{f_x dx + f_y dy} + \cancel{d*d\underbrace{*f}_{f\,dx\wedge dy}}_{0}$$
$$= *d(f_x dy - f_y dx)$$
$$= *((f_{xx} + f_{yy})dx \wedge dy)$$
$$= f_{xx} + f_{yy}.$$ □

Corollary 1 (of prop 2): $\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle$

Proposition 3: For $\alpha \in A^k(M)$, $\Delta\alpha = 0 \iff d\alpha = 0 = \delta\alpha$

Proof: $(\Rightarrow)$: $0 = \langle \Delta\alpha, \alpha \rangle = \langle d\delta\alpha, \alpha \rangle + \langle \delta d\alpha, \alpha \rangle = \|\delta\alpha\|^2 + \|d\alpha\|^2$. □

Note that this makes constants the only (global) harmonic functions on M. Speaking of which, we have the

Definition 4: The harmonic k-forms are $\mathcal{H}^k(M) := \ker(\Delta) \subset A^k(M)$.

Why would we care about these? Historically, the Hodge theorem was motivated by the idea that one should look for representatives of cohomology classes with least norm.

Proposition 4: A given $\eta \in \ker(d) \subset A^k(M)$ is of least Hodge norm in its class $[\eta] = \eta + dA^{k-1}(M) \in H^k(M, \mathbb{C}) \iff \eta \in \mathcal{H}^k(M)$ [equiv. $\delta\eta = 0$].

Proof: $(\Longleftarrow)$:
$$\begin{aligned}\|\eta + d\omega\|^2 &= \langle \eta + d\omega, \eta + d\omega\rangle \\ &= \|\eta\|^2 + \|d\omega\|^2 + 2\,\mathrm{Re}\,\underbrace{\langle \eta, d\omega\rangle}_{\langle \cancel{\delta\eta}, \omega\rangle = 0} \\ &= \|\eta\|^2 + \|d\omega\|^2 \\ &\geq \|\eta\|^2 \quad \text{with equality} \Leftrightarrow d\omega = 0.\end{aligned}$$

Hence η is the unique class of minimal norm.

$(\Longrightarrow)$: $\|\eta\|$ minimal $\Rightarrow$ for every ω (e.g. $\delta\eta$),
$$\begin{aligned}0 &= \left(\frac{d}{dt}\|\eta + t\,d\omega\|^2\right)\Big|_{t=0} \\ &= \left(\frac{d}{dt}\langle \eta + t\,d\omega, \eta + t\,d\omega\rangle\right)_{t=0} \\ &= 2\,\mathrm{Re}\langle \eta, d\omega\rangle \\ &= 2\,\mathrm{Re}\langle \delta\eta, \omega\rangle\end{aligned}$$
$\Rightarrow \delta\eta = 0$. $\square$

The hard question is whether every class *has* a harmonic representative at all.

Let U, V, W = finite-dim'l inner-product spaces

$$U \underset{A^*}{\overset{A}{\rightleftarrows}} V \underset{B^*}{\overset{B}{\rightleftarrows}} W \qquad \text{adjoints}$$

Note that A^* is injective on $\mathrm{im}(A)$ since $\langle A^*Au, u\rangle = \|Au\|^2$.

Lemma 1: $B^*B + AA^* \in \mathrm{Aut}(V)$

Proof: Given $v \neq 0$, $\langle \underbrace{(B^*B + AA^*)v}, v\rangle = \|Bv\|^2 + \|A^*v\|^2$

If $Bv \neq 0$ then this $\neq 0$, done.

If $Bv = 0$ then $v \in \mathrm{im}(A)$, & so $A^*v \neq 0$, done. $\square$

Set $\begin{cases} U = E^{k-1}_{M,p},\ V = E^k_{M,p},\ W = E^{k+1}_{M,p} \\ \text{equipped with Hodge } \langle\,,\rangle \end{cases}$ and A, B given by $\xi \wedge (\cdot) =: \mu_\xi$ for $\xi \in T^\vee_{M,p}$

Ex/ Check the sequence is indeed exact in this case.

Lemma 2: $A^* = (-1)^{n(k-1)} * \mu_\xi *$, $B^* = (-1)^{nk} * \mu_\xi *$.

Proof: $\langle \mu_\xi u, v \rangle = *((\xi \wedge u) \wedge * v) = (-1)^{k-1} * (u \wedge \underbrace{\xi \wedge * v}_{\deg\ n-k+1})$

$= (-1)^{k-1} (-1)^{(n-k+1)(k-1)} * (u \wedge ** \mu_\xi * v) = \langle u, (-1)^{n(k-1)} * \mu_\xi * v \rangle$ □

Corollary 2: $(-1)^{nk} * \mu_\xi * \mu_\xi + (-1)^{n(k-1)} \mu_\xi * \mu_\xi * \in Aut(E^k_{M,p}) \quad \forall \xi \in T^\vee_{M,p}$

Proposition 5: Δ is elliptic [as a diff. op. $A^k(M) \to A^k(M)$, for any k]

Proof: Wts for each $p \in M$ and any $\xi \in T^\vee_{M,p} \setminus \{0\}$ and $v_p \in E^k_{M,p} \setminus \{0\}$, that

$$(C.2) \qquad \sigma_\Delta(p,\xi)(v_p) = \Delta(\varphi^2 v)\,|_p$$

(σ_Δ homog. of deg. 2 in ξ, linear in v) (any C^∞ fcn. with $d\varphi(p) = \xi$) (any local C^∞ section extending v_p)

is nonzero. But RHS (C.2) $= \{(-1)^{n(k+1)+1} d*d* + (-1)^{nk+1} *d*d\}(\varphi^2 v)\,|_p$

$= -2\{(-1)^{nk} * \mu_\xi * \mu_\xi + (-1)^{n(k-1)} \mu_\xi * \mu_\xi *\}(v_p)$

$\neq 0$ by Cor. 2. □

where e.g.
$(d*d*)(\varphi^2 v))|_p =$
$(d*d\varphi^2 * v)_p =$
$(2 d * \varphi d\varphi * v)_p =$
$(2(d\varphi) \wedge *(d\varphi) * v)_p =$
$2 \mu_\xi * \mu_\xi * v_p$.

The Hodge Theorems

Theorem 1: For each k, $\dim \mathcal{H}^k(M) < \infty$ and

$$A^k(M) = \Delta(A^k(M)) \overset{\perp}{\oplus} \mathcal{H}^k(M)$$
$$= d\delta(A^k(M)) \overset{\perp}{\oplus} \delta d(A^k(M)) \overset{\perp}{\oplus} \mathcal{H}^k(M)$$
$$= d(A^{k-1}(M)) \overset{\perp}{\oplus} \delta(A^{k+1}(M)) \overset{\perp}{\oplus} \mathcal{H}^k(M)$$

where "$\perp$" is w.r.t. Hodge $\langle , \rangle$.

<u>Proof</u>: Since $\begin{cases}\Delta \text{ is an elliptic diff. op.,} \\ \mathcal{H}^k(M) = \ker(\Delta)\end{cases}$ the Finiteness + Decomposition Thms. apply and give the "$<\infty$" and the first "$=$". Next, we have

$$\Delta(A^k(M)) \subseteq d\delta(A^k(M)) + \delta d(A^k(M))$$
$$\subseteq d(A^{k-1}(M)) + \delta(A^{k+1}(M))$$

making these inclusions equalities

By Prop. 2-3, the last 2 spaces are $\perp$ to $\mathcal{H}^k(M)$. So it remains to check that the sums are orthogonal ($\Rightarrow$ direct). This again follows from Prop. 2 since in general $\langle \delta\alpha, d\beta\rangle = \langle \alpha, \underbrace{dd\beta}_{0}\rangle = 0$. □

<u>Corollary 3</u>: $\Delta(\cdot) = \beta \,(\in A^k(M))$ soluble $\Leftrightarrow \beta \perp \mathcal{H}^k(M)$.
(The solution is unique mod $\mathcal{H}^k(M)$.)

Recall from the proof of the Decomposition Thm. the orthogonal projection which reads in this case

$$H : A^k(M) \twoheadrightarrow \mathcal{H}^k(M).$$

<u>Definition 5</u>: Taking $G(\alpha) :=$ unique solution in $(\mathcal{H}^k(M))^\perp$ of

$$\Delta(\cdot) = \alpha - H(\alpha) \qquad \text{(cf. Cor. 3 above)}$$

defines the <u>Green's operator</u>

$$G : A^k(M) \to (\mathcal{H}^k(M))^\perp.$$

Clearly, it satisfies

$$\text{(C.3)} \qquad \boxed{\mathbb{1} = H + \Delta G}$$

Remark that neither H nor G is defined locally: if you start with a form ω (say, d-closed) <u>compactly supported in some $V \subset M$</u>, $H(\omega)$ and $G(\omega)$ will not have this property.

Proposition 6: (a) G is bounded + self-adjoint

(b) G takes bounded sequences to sequences w/ Cauchy subseq's

(c) G commutes with d, δ, and Δ

Proof: (a) Lemma B.2 $\Rightarrow$ $\exists\, C > 0$ s.t. $\|\beta\| \le C\|\Delta\beta\|$ $(\forall \beta \in (\mathcal{H}^h(M))^\perp)$

(C.4) $$\Longrightarrow \|G(\alpha)\| \le C\|\Delta G(\alpha)\| \le C\sqrt{\|\Delta G(\alpha)\|^2 + \|H(\alpha)\|^2}$$

$$= C\|\alpha\|$$

$\uparrow$

$\alpha = H(\alpha) + \Delta(G(\alpha))$

$\perp$

$$\langle G(\alpha), \beta\rangle = \langle G(\alpha), H(\beta) + \Delta G(\beta)\rangle$$
$$= \langle G(\alpha), \Delta G(\beta)\rangle$$
$$= \langle \Delta G(\alpha), G(\beta)\rangle$$
$$= \langle \alpha - H(\alpha), G(\beta)\rangle$$
$$= \langle \alpha, G(\beta)\rangle.$$

(b) Given $\|\alpha_\ell\| \le K$, (C.4) $\Rightarrow$ $\|G(\alpha_\ell)\|$, $\|\Delta G(\alpha_\ell)\|$ bounded.

Done by global Gårding.

(c) <u>wts</u>: G commutes w/ any linear operator $T: A^p(M) \to A^q(M)$ s.t. $[T, \Delta] = 0$.

First note $\begin{cases} T(\mathcal{H}^p(M)) \subset \mathcal{H}^q(M) & \text{(easy)} \\ T((\mathcal{H}^p(M))^\perp) \subset (\mathcal{H}^q(M))^\perp & (\beta \in (\mathcal{H}^p)^\perp \Rightarrow \beta = \Delta G(\omega) \Rightarrow \langle T\beta, \mathcal{H}^q\rangle = 0) \end{cases}$

But then T "preserves $\mathcal{H}$- and $\mathcal{H}^\perp$-components $\Longrightarrow$

$$0 = T(1-H) - (1-H)T$$
$$= T\Delta G - \Delta G T$$
$$= \Delta(TG - GT)$$

range $\perp$ to $\ker(\Delta)$

$$\Longrightarrow 0 = TG - GT.$$

$\square$

The next 3 results hold for an arbitrary compact, orientable, C^∞ manifold M, since such an M can always be equipped with a Riemannian metric.

<u>Theorem 2</u>: Every dR cohomology class contains a unique harmonic representative; that is, $\mathcal{H}^k(M) \xrightarrow{\cong} H^k(M,\mathbb{C})$.

<u>Proof</u>: need to show $\exists$ of rep. (then done by Prop. 4).

Given any $\alpha \in \ker(d) \subset A^k(M)$,

$$\alpha \underset{(C.3)}{=} H\alpha + d\delta G\alpha + \delta d G\alpha$$

$$\underset{\text{Prop. 6(c)}}{=} H\alpha + d\delta G\alpha + \delta G \cancel{d\alpha}^{\,0}$$

$$= H\alpha + d(\delta G\alpha).$$

$$\Longrightarrow [\alpha] = [H(\alpha)] \in H^k(M,\mathbb{C}).$$

$\square$

<u>Corollary 4</u>: $\dim(H^k(M)) < \infty$.

<u>Corollary 5 (Poincaré duality)</u>: The pairing

$$H^{n-k}(M,\mathbb{C}) \times H^k(M,\mathbb{C}) \to \mathbb{C}$$

induced by $([\alpha],[\beta]) \longmapsto \int_M \alpha\wedge\beta$, is perfect.

<u>Proof</u>: Given $[\alpha]$ (w.t.s. $\exists$ class with which it pairs nontrivially), let α_0 = <u>harmonic</u> representative (of $[\alpha]$). By (C.1),

$$0 = \underbrace{*\Delta\bar{\alpha}_0}_{\overline{\Delta\alpha_0}} = \Delta(*\bar{\alpha}_0) \Longrightarrow *\bar{\alpha}_0 \text{ harmonic a fortiori } d\text{-closed}.$$

Now $\int_M \alpha_0 \wedge * \overline{\alpha_0} = \|\alpha_0\|^2 \neq 0$

$\Longrightarrow [*\overline{\alpha}_0]$ pairs nontrivially with $[\alpha]$. ∎

Remark 1:

(i) All of the above works $/\mathbb{R}$ rather than $/\mathbb{C}$.

(ii) Another approach to the proof of the Hodge Theorems 1 & 2 uses the Spectral theorem (e.g. to construct G). The eigenvalues of Δ go to ∞, but are (w/ the exception of 0 itself) bounded away from 0.

(iii) We could have done Thm. 2/Cor. 4 w/o constructing G — cf. Voisin or the related proof for Δ_E below

(iv) Henceforth, for consistency, we drop δ for the adjoint of d and just write d^*. Note that when n is even, $\begin{cases} ** = (-1)^k \\ d^* = -*d* . \end{cases}$

∂, $\bar\partial$, and $\bar\partial_E$

Given $V = \mathbb{C}\langle \{e_j\}_{j=1}^n \rangle$ with Hermitian metric $h = g - i\omega$ (o.n. in)

$W = V^{\mathbb{R}} = \mathbb{R}\langle \{e_j\}, \{J(e_j)\} \rangle$ w/ J-invariant metric g (o.n. in)

$W_{\mathbb{C}} = \mathbb{C}\left\langle \left\{ \underset{\overset{\shortparallel}{e_j^+}}{\frac{e_j + iJ(e_j)}{\sqrt{2}}} \right\}, \left\{ \underset{\overset{\shortparallel}{e_j^-}}{\frac{e_j - iJ(e_j)}{\sqrt{2}}} \right\} \right\rangle$ with g extended to Hermitian metric (o.n. in)

(recall the independence Prop 1(b))

$\Downarrow$

$1 = \langle e_I^+ \wedge e_J^-, e_I^+ \wedge e_J^- \rangle = *\left(e_I^+ \wedge e_J^- \wedge *\left(\overline{e_I^+ \wedge e_J^-}\right)\right) \Longrightarrow *\left(\overline{e_I^+ \wedge e_J^-}\right) = e^+_{I^c} \wedge e^-_{J^c}$ ($I^c = \{1,\dots,n\} \setminus I$)

$\Longrightarrow *\left(e_I^+ \wedge e_J^-\right) = \overline{e^+_{I^c} \wedge e^-_{J^c}} = e^+_{J^c} \wedge e^-_{I^c}$

$*$ $\mathbb{C}$-linear

hence $\Lambda^{p,q} W_{\mathbb{C}} \xrightarrow{*} \Lambda^{n-q,n-p} W_{\mathbb{C}}$.

i.e. Hermitian n-manifold

Let M = $\mathbb{C}$-n-manifold with Hermitian metric $h = g - i\omega$ (142)

$\uparrow$ ($M^{\mathbb{R}}$ = underlying real $2n$-manifold)

g: J-invariant Riemannian metric on $M^{\mathbb{R}}$

$\Rightarrow \; * : A^k \to A^{2n-k}$ sends $A^{p,q} \to A^{n-q,n-p}$

and $\bar{*}(\cdot) := *\overline{(\cdot)}$ sends $A^{p,q} \to A^{n-p,n-q}$

E = holomorphic Hermitian vector bundle

Hermitian: inner product induces $E \xrightarrow{\varphi} E^\vee$ (antilinear)

hence have $\bar{*}_E : A^k(E) \to A^k(E^\vee)$

sending $A^{p,q}(E) \to A^{n-p,n-q}(E^\vee)$

$\omega \otimes \varepsilon \longmapsto *\bar{\omega} \otimes \varphi(\varepsilon)$

We shall restrict attention to

$$\bar{\partial}_E : A^{0,q}(E) \to A^{0,q+1}(E),$$

so

$$\bar{*}_E : A^{0,k}(E) \to A^{n,n-k}(E^\vee) \cong A^{0,n-k}(\underbrace{K_M}_{= \Omega^n_M} \otimes E^\vee)$$

will suffice.

We will need the adjoint operators

Lemma 3: (i) $\left\{ \begin{array}{ll} \partial^* = -*\bar{\partial}* & : A^{p,q} \to A^{p-1,q} \\ \bar{\partial}^* = -*\partial* & : A^{p,q} \to A^{p,q-1} \end{array} \right\}$ under $\langle \cdot,\cdot \rangle = \int_M (\cdot) \wedge *\overline{(\cdot)}$

(ii) $\bar{\partial}_E^* = (-1)^q \bar{*}_E^{-1} \bar{\partial}_{K_M \otimes E^\vee} \bar{*}_E : A^{0,q}(E) \to A^{0,q-1}(E)$

under $\langle \cdot,\cdot \rangle_E \overset{(\dagger)}{=} \int_M (\cdot) \wedge \bar{*}_E(\cdot)$ (includes action of $E^\vee$ on E.)

Proof (for $\bar{\partial}$, Ex/rest): $\left. \begin{array}{l} \omega \in A^{k-1}(M) \\ \eta \in A^k(M) \end{array} \right\} \Longrightarrow$

$$\bar{\partial}(\overbrace{\omega \wedge *\bar{\eta}}^{2n-1\text{ form}}) = \bar{\partial}\omega \wedge *\bar{\eta} + (-1)^{k-1} \omega \wedge \underbrace{\bar{\partial}*\bar{\eta}}_{(-1)^{k-1}**}$$

$$= \bar{\partial}\omega \wedge *\bar{\eta} + \omega \wedge *\overline{*\partial*\eta}$$

($\dagger$) note that we are wedging $A^{0,q}(E)$ with $A^{n,n-q}(E^\vee)$ — thus gives (letting $E^\vee$ & E pair off) something in $A^{n,n}$ (hence natural to integrate $/M$).

For degree $(2n-1)$ forms $\alpha^{(n,n-1)}+\beta^{(n-1,n)}$, $\bar\partial(\alpha+\beta)=\bar\partial\alpha = d\alpha \Rightarrow$

$\int_M \bar\partial(\alpha+\beta) = \int_M d\alpha \underset{\text{Stokes}}{=} 0$. So $0=\int_M \bar\partial(\omega\wedge *\bar\eta) = \langle\bar\partial\omega,\eta\rangle + \langle\omega, *\partial *\eta\rangle$.

$\square$

One then defines Laplacians by

$$\begin{cases} \left.\begin{aligned} \Delta_{\bar\partial} &= \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \\ \Delta_\partial &= \partial\partial^* + \partial^*\partial \end{aligned}\right\} \text{ on } A^{p,q}(M) \quad (\text{or } A^k(M)) \\ \\ \Delta_E = \bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E \qquad \text{on } A^{0,q}(M,E) \end{cases}$$

and checks that

$$\text{(C.5)} \qquad \begin{cases} \ker(\Delta_{\bar\partial}) = \ker(\bar\partial)\cap\ker(\bar\partial^*) \\ \text{etc.} \end{cases}$$

Lemma 4: $\Delta_{\bar\partial}$, Δ_∂, and Δ_E are elliptic (& self-adjoint).

Proof (for $\bar\partial$, Ex/rest): Same as for d except:

$$\Delta_{\bar\partial}(\varphi^2(\nu))\big|_{p_0} = -\{*\partial*\bar\partial + \bar\partial*\partial*\}(\varphi^2\nu)\big|_{p_0} = -\frac{1}{2}\{\overbrace{*\mu_\chi*\mu_{\bar\chi} + \mu_{\bar\chi}*\mu_\chi*}^{\text{injective by lemmas } 1+2}\}(\nu_p)$$

$$\neq 0.$$

$(d\varphi)(p_0)=\xi\neq 0$
2Re ↑ ⇓
$(\partial\varphi)(p_0)=:\chi\neq 0$
$\bar\partial\varphi(p_0)=\bar\chi\neq 0$

(local (p,q)-form $\neq 0$ at p_0)

$\square$

Definition 6:
$$\mathcal{H}^{0,q}(M,E) := \ker(\Delta_E) \subset A^{0,q}(M,E)$$
$$\mathcal{H}^{p,q}(M) := \ker(\Delta_{\bar\partial}) \; (=\ker(\Delta_{\Omega^p})) \subset A^{p,q}(M)$$

Remark 2: $\Delta_{E=\Omega^p}$ and $\Delta_{\bar\partial}$ differ by a constant on $A^{0,q}(\Omega^p)=A^{p,q}$ (cf. Voisin)

Theorem 3: Assume M compact. Then one has isomorphisms

$$\mathcal{H}^{0,q}(M,E) \xrightarrow{\cong} H^q(M, \mathcal{O}(E))$$

$$\mathcal{H}^{p,q}(M) \xrightarrow{\cong} H^q(M, \Omega^p) \cong H^{p,q}_{\bar\partial}(M)$$

(special case of)

and these spaces are finite-dimensional.

Proof: By Lemma 4 & the Decomposition Theorem,

$$A^{0,q}(M,E) = \mathcal{H}^{0,q}(M,E) \overset{\perp}{\oplus} \Delta_E(A^{0,q}(M,E)).$$

Given $\beta \in$ $\bar\partial_E$-closed, hence giving a class in $H^q(M, \mathcal{O}(E))$,

we have $\beta = \alpha + \Delta_E \gamma = \alpha + \bar\partial_E \bar\partial_E^* \gamma + \bar\partial_E^* \bar\partial_E \gamma$

all $\bar\partial_E$-closed!

$\Rightarrow \bar\partial_E^* \bar\partial_E \gamma$ is also $\bar\partial_E$-closed; clearly it is $\bar\partial_E^*$-closed (as $\bar\partial_E^* \circ \bar\partial_E^* = 0$), so it must be harmonic. But it is also in the space $\perp$ to harmonic forms! Hence, it is zero, and $\beta \equiv \alpha$ mod a $\bar\partial_E$-exact form. This proves the maps of Thm. 3 are surjective.

For injectivity, let $\beta \in \mathcal{H}^{0,q}(M,E)$, assume β $\bar\partial_E$-exact $(=\bar\partial_E \alpha)$. Then for any $\gamma \in \mathcal{H}^{0,q}(M,E)$, $\langle \beta, \gamma \rangle_E = \langle \bar\partial_E \alpha, \gamma \rangle_E = \langle \alpha, \cancel{\bar\partial_E^* \gamma} \rangle = 0 \Rightarrow$ $\beta \in (\mathcal{H}^{0,q}(M,E))^\perp \Rightarrow \beta = 0$. $\square$

We need the following

Lemma 5: $H^n(M, \Omega^n_M) \cong \mathbb{C}$.

Proof: $\text{LHS} \underset{\text{Thm. 3}}{=} \mathcal{H}^{n,n}(M) = \ker(\bar\partial^*) = \{\ker(\partial *) \subset A^{n,n}(M)\} \cong \{\ker(\partial) \subset A^{0,0}(M)\} = \mathbb{C}$

(antiholo. fcns.)

$\mathcal{H}^{n,n}(M) \xrightarrow{*} \mathbb{C}$

(From this, it is also clear that $\int_M$ gives an $\cong$ from LHS to $\mathbb{C}$.)

Now consider the pairings

$$(C.6)\quad H^q(M,\mathcal{O}(E))\otimes H^{n-q}(M,\mathcal{O}(K_M\otimes E^\vee)) \xrightarrow{\wedge} H^n(M,\underset{\mathcal{O}(K_M)}{\Omega^n}) \underset{\cong}{\xrightarrow{\int}} \mathbb{C}\,.$$

a special case of which is

$$(C.7)\quad H^{p,q}_{\bar\partial}(M)\otimes H^{n-p,n-q}_{\bar\partial}(M) \xrightarrow{\int(\cdot)\wedge(\cdot)} \mathbb{C}\,.$$

<u>Corollary 6 (Serre duality)</u>: (C.6) & (C.7) are perfect pairings.

<u>Proof</u>: Let $\alpha\in\mathcal{H}^{q,q}(M,E)$. Then

$$0=\bar{*}_E\,\Delta_E\,\alpha \overset{(\dagger)}{=} \Delta_{K_X\otimes E^\vee}\,\bar{*}_E\,\alpha \implies \bar{*}_E\alpha \text{ is harmonic}$$
$$\implies \bar{*}_E\alpha \text{ is } \bar\partial_E\text{-closed}.$$

So it represents a class in $\mathcal{H}^{0,n-q}(M,\mathcal{O}(K_X\otimes E^\vee))$, and

$$\int_M \alpha\wedge\bar{*}_E\alpha = \|\alpha\|^2\neq 0 \implies \text{this class pairs nontrivially with } [\alpha].$$

$\square$

<u>Remark 3</u>: There is also a proof (by harmonic theory) of the Künneth formula

$$H^k(M\times N)\cong\bigoplus_{a+b=k}H^a(M)\otimes H^b(N)\,.\quad\text{(cf. Griffiths-Harris)}.$$