G. Hodge structures

Let $V_{\mathbb{Z}}$ = finitely-generated abelian group

$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ (also write $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes \mathbb{R}$, $V := V_{\mathbb{Z}} \otimes \mathbb{Q}$)

Definition 1: A Hodge structure (HS) of weight n on $V_{\mathbb{Z}}$ is a decomposition*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

s.t. $\overline{V^{p,q}} = V^{q,p}$. [Note: the rank of a HS is that of $V_{\mathbb{Z}}$.]

Example 1: M = compact Kähler manifold: the Hodge decomposition theorem puts a HS of weight k on $V_{\mathbb{Z}} = H^k(M, \mathbb{Z})$.

Definition 2: A weight n HS on $V_{\mathbb{Z}}$ is a decreasing filtration** (or "flag")

$$V_{\mathbb{C}} = F^0 V_{\mathbb{C}} \supset F^1 V_{\mathbb{C}} \supset \cdots \supset F^n V_{\mathbb{C}} \supset F^{n+1} V_{\mathbb{C}} = \{0\}$$

s.t. $V_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$ $(\forall p)$. [the "p-opposed condition"]

Proposition 1: The 2 definitions are equivalent.

Proof: $(1 \Rightarrow 2)$ $F^p := \bigoplus_{p' \geq p} V^{p', n-p'}$ $(\forall p)$ $\Longrightarrow$

$$F^{n-p+1} = \bigoplus_{q' \geq n-p+1} V^{q', \overset{p'}{\overset{\shortparallel}{n-q'}}} = \bigoplus_{p'<p} V^{n-p', p'} \Longrightarrow \overline{F^{n-p+1}} = \bigoplus_{p'<p} V^{p', n-p'}.$$

** when there is no ambiguity possible, sometimes $F^k V_{\mathbb{C}}$ is abbreviated F^k

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* technically, what we are about to define is an effective HS. In general, one does allow negative p & q.

$(2 \Rightarrow 1)$ $V^{p,n-p} := F^p \cap \overline{F^{n-p}}$

Ex/ finish the proof. $\square$

Definition 3: A morphism* of HS is a map $\Theta : V_{\mathbb{Z}} \to \tilde{V}_{\mathbb{Z}}$ of abelian groups s.t. [the complex linear extension] $\Theta_{\mathbb{C}} : V_{\mathbb{C}} \to \tilde{V}_{\mathbb{C}}$ satisfies $\Theta_{\mathbb{C}}(V^{p,q}) \subset \tilde{V}^{p,q}$ $(\forall p,q)$.

Proposition 2: This is equivalent to requiring $\Theta_{\mathbb{C}}(F^p) \subset \tilde{F}^p$ $(\forall p)$.

Proof: Because Θ is defined $/\mathbb{Z}\,(\subset \mathbb{R})$,

$$\Theta_{\mathbb{C}}(F^p) \subset \tilde{F}^p \ (\forall p) \Rightarrow \Theta_{\mathbb{C}}(\overline{F^{n-p+1}}) \subset \overline{\tilde{F}^{n-p+1}} \ (\forall p). \quad \square$$

Definition 3 gives rise to an obvious notion of sub-HS, isomorphism of HS, etc.

Ex/ $U_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ yields a sub HS $\iff U_{\mathbb{C}} = \bigoplus_{p+q=n} U_{\mathbb{C}} \cap V^{p,q}$ //

Ex/ the category of HS of weight n form an abelian category //

Remark 1: One obtains $\mathbb{Q}$-HS resp. $\mathbb{R}$-HS (w./results identical to above) by replacing $V_{\mathbb{Z}}$ by a $\mathbb{Q}$-vector sp. V or $\mathbb{R}$-v.s. $V_{\mathbb{R}}$. We'll use "V" to denote both the HS and the $\mathbb{Q}$-v.s. //

Proposition 3: Morphisms of HS are strict for the Hodge filtration:

$$\mathrm{im}(\Theta) \cap \tilde{F}^k = \Theta(F^k).$$

Proof: Given $\Theta(\alpha) \in \tilde{F}^k$, write $\alpha = \sum \alpha^{p,q} \rightsquigarrow \Theta(\alpha) = \sum \Theta(\alpha^{p,q})$ type (p,q) or zero

$\Rightarrow \Theta(\alpha^{p,q}) = 0$ for $p<k \Rightarrow \Theta(\alpha) = \Theta(\sum_{p \geq k} \alpha^{p,q})$. $\square$

Example 2:
- $M \xrightarrow{\phi} N$ holomorphic $\Rightarrow H^k(N) \xrightarrow{\phi^*} H^k(M)$ morphism of HS
- $N \overset{\iota}{\hookrightarrow} M$ codim. r $\Rightarrow H^k(N) \underset{\text{"Gy"}}{\xrightarrow{\iota_*}} H^{k+2r}(M)$ morphism of HS of type (r,r)

(we did for $r=1$; cf Voisin for general case)

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* There is also the notion of a morphism of type (r,r): then one demands $\Theta_{\mathbb{C}}(V^{p,q}) \subset \tilde{V}^{p+r,q+r}$.

Proposition 4: ϕ surjective $\Rightarrow$ ϕ^* injective $\left(H^k(N) \overset{\text{sub HS}}{\subset} H^k(M)\right)$

Sketch of Proof: $\alpha \in H^k(N) \xRightarrow[\text{duality}]{\text{Poincaré}} \exists\, \beta \in H^{2n-k}(N)$ ($n = \dim N$) s.t. $\alpha \cup \beta = [dV_N]$; and

$$\phi^*\alpha \cup \phi^*\beta \cup [\omega_M^r] = [\underbrace{\phi^* dV_N \wedge \omega_N^r}_{\text{everywhere} \geq 0 \text{ or } 0}] \neq 0. \qquad (r = \dim M - \dim N)$$

□

Polarized Hodge structures

Definition 4. A polarization of a weight n HS V, is a nondegenerate $(-1)^n$-symmetric* bilinear form

$$Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \longrightarrow \mathbb{Z}$$

(also write Q for its $\mathbb{C}$-linear extension)

satisfying

(HR I) $\quad Q(V^{p,q}, V^{p',q'}) = 0 \quad$ unless $\quad (p', q') = (q, p)$

(HR II) $\quad i^{p-q} Q(\xi, \bar{\xi}) > 0 \quad \forall\, \xi \in V^{p,q} \setminus \{0\}$.

A polarized Hodge structure (PHS) is often written as a triple $(V_{\mathbb{Z}}, F^\bullet, Q)$. □

Example 3: For X projective, $\exists$ integral Kähler class (i.e. $[\omega] \in H^2(X, \mathbb{Z})$) (pullback of ω_{FS} from $\mathbb{P}^N$)

$\Rightarrow$ the $(-1)^{\binom{k+1}{2}} Q_k(\alpha, \beta) = \int_X \alpha \wedge \beta \wedge \omega^{n-k}$ are integral, hence (by ~~Thm~~. E.3) polarize $H^k_{pr}(X)$. In fact, writing $H^k(X) = \oplus L^j H^{k-2j}_{pr}(X)$, the orthogonal direct sum of the $(-1)^{\left(j + \binom{k+1}{2}\right)} Q_{k-2j}$ polarizes $H^k(X)$. □

The category of wt. n PHS is not only abelian but semisimple (upon tensoring with $\mathbb{Q}$):

Proposition 5: Given a $\mathbb{Q}$-PHS V with sub-$\mathbb{Q}$-HS V'. Then V' is polarized and $V \cong \underbrace{V'' \oplus V'}_{\text{sub-}\mathbb{Q}\text{-HS}}$ as a HS.

Remark 2: There are actually a lot of non-polarizable HS (even in weight 1 rank 4).

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* my silly notation for $\begin{cases} \text{symmetric}, & n \text{ even} \\ \text{alternating}, & n \text{ odd} \end{cases}$

Proof: Since V' defined $/\mathbb{Q}$, $\overline{V'_{\mathbb{C}}} = V'_{\mathbb{C}}$; by "Ex/" above, $V'^{p,q} = V'_{\mathbb{C}} \cap V^{p,q}$. Hence, HR II for Q on $V_{\mathbb{C}} \Rightarrow Q|_{V'_{\mathbb{C}}}$ is nondegenerate (hence polarizes V'). Now take $V'' = (V')^{\perp}$ (under Q); nondegeneracy of Q & $Q|_{V'} \Rightarrow V = V' \oplus V''$, and rationality of $Q \Rightarrow V''$ defined $/\mathbb{Q}$. Finally, since Q is nondegenerate on $V^{p,q} \times V^{q,p}$ and $(V')^{p,q} \times (V')^{q,p}$, $V^{p,q} = (V')^{p,q} \oplus (\underbrace{(V')^{\perp}}_{V''} \cap V^{p,q}) \Rightarrow V'' = \oplus V'' \cap V^{p,q}$
$\Rightarrow V''$ sub HS. □

$\boxed{\text{Classifying spaces for PHS}}$

Let $V_{\mathbb{Z}}$ = f.g. abelian group (rank $= r$)

$Q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ $(-1)^n$-symmetric nondegenerate bilinear form

$h^{n,0}, h^{n-1,0}, \ldots, h^{0,n} \in \mathbb{Z}_{\geq 0}$ s.t. $h^{p,q} = h^{q,p}$ & $\sum h^{p,q} = r$.

Clearly, a classifying space should parametrize weight n HS on $V_{\mathbb{Z}}$ polarized by Q.

Definition 5: The period domain D associated to the above data is

$$D = \left\{ \text{decompositions } V_{\mathbb{C}} = \oplus V^{p,q} \text{ s.t. } \begin{array}{l} \bullet\ \overline{V^{p,q}} = V^{q,p} \\ \bullet\ \dim_{\mathbb{C}} V^{p,q} = h^{p,q} \\ \text{i.e. HR I \& II hold} \rightsquigarrow \bullet\ (V_{\mathbb{Z}}, V^{\bullet,\bullet}, Q) \text{ is a PHS} \end{array} \right\}$$

Ex/ In terms of the Hodge filtration $F^{\bullet}$, the Hodge-Riemann condition I in Defn. 4 can be rewritten

(HR I′) $\quad Q(F^p, F^{n-p+1}) = 0 \quad (\forall p)$ //

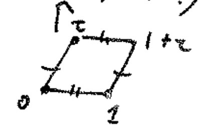
One can rewrite (HR II) as $\boxed{Q(\underset{\text{Weil}}{C}(\cdot), \overline{(\cdot)}) > 0}$.

<u>Definition 6</u>: The "compact dual" $\check{D}$ of D is a projective algebraic variety containing D, defined by

$$\check{D} = \left\{ \text{flags } F^\bullet V_{\mathbb{C}} \text{ s.t. } \begin{array}{l} \bullet \ \dim_{\mathbb{C}} F^p = \sum_{p' \geq p} h^{p', n-p'} \\ \bullet \ (\mathrm{HR\,I}') \text{ holds} \end{array} \right\}$$

(we do <u>not</u> impose the "p-opposed condition" — these will <u>not</u> all correspond to decompositions, and the ones that do may not be polarized by Q.)

<u>Example 4</u>: $\begin{cases} V = \mathbb{Z}\langle \gamma_1, \gamma_2 \rangle \\ Q(\gamma_i, \gamma_j) = (-1)^j \sigma_{ij} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ h^{1,0} = 1 = h^{0,1} \end{cases}$ (Inspired by $H^1(E_\tau, \mathbb{Z})$.)



Write $V_{\mathbb{C}} = \underbrace{\mathbb{C}\langle \omega \rangle}_{V^{1,0}} \oplus \underbrace{\mathbb{C}\langle \bar{\omega} \rangle}_{V^{0,1}}$ (for Defn. 5), $\omega = z_1 \gamma_1 + z_2 \gamma_2$, $z_1, z_2 \in \mathbb{C}$, not both 0.

(for Defn. 6) $\to U$: $\mathbb{C}\langle \omega \rangle = F^1$

Since Q is antisymmetric, $Q(\omega, \omega) = 0 \implies (\mathrm{HR\,I}^{(\prime)})$ automatic.

$$\implies \boxed{\check{D} = \mathbb{P}^1}$$

(w./coordinate $z_2/z_1 =: \tau$.)

HR II says $0 < i\, Q(\omega, \bar{\omega})$

$= i\,(z_1 \bar{z}_2 - z_2 \bar{z}_1)$, which

$$\iff 0 < i \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{|z_1|^2} = i\underbrace{(\bar{\tau} - \tau)}_{-2i\,\mathrm{Im}(\tau)} = 2\,\mathrm{Im}(\tau)$$

$\iff \tau \in \mathfrak{h}$.

So $\boxed{D = \mathfrak{h}}$. $\square$

Example 5: There are 1-to-1 correspondences

weight 1 HS $\longleftrightarrow$ compact complex tori } cf. the Appendix below

weight 1 PHS $\longleftrightarrow$ abelian varieties

for $\mathrm{rk}(V_{\mathbb{Z}})=2g$, if $Q=\begin{pmatrix} & -I_g \\ I_g & \end{pmatrix}$ then these are parametrized by $\mathfrak{h}_g\ (=D)$, cf. pp. 101-3.

Remark 3: There are general "formulas" for period domains which come from setting

$$G_{\mathbb{R}} := \mathrm{Aut}(V_{\mathbb{R}}, Q) \cong \begin{cases} Sp(r,\mathbb{R})/\{\pm \mathrm{id}\} & \text{if } n \text{ odd} \\ SO(h_{odd}, h_{even}) & \text{if } n \text{ even} \end{cases}$$

and noting that this acts transitively on D. Fixing a "base PHS" $\varphi \in D$, the subgroup fixing it is denoted

$$H_\varphi \cong \begin{cases} \prod_{p\leq m} U(h^{p,q}) & \text{if } n=2m+1 \\ \prod_{p<m} U(h^{p,q}) \times SO(h^{m,m}) & \text{if } n=2m. \end{cases}$$

and so

$$D \cong G_{\mathbb{R}}/H_\varphi .$$

More on this later. $\boxed{\text{Easy examples: } \mathfrak{h} = SL_2(\mathbb{R})/U(1),\ \mathfrak{h}_g = Sp_{2g}(\mathbb{R})/U(g)}$ (corr. to Ex. 5 above)

Ex/ Show that $G_{\mathbb{R}}$ acts transitively on D. //

Remark 4: I forgot: besides direct sums / kernels / cokernels, you can also take tensors & duals of (P)HS;

(add weights & Hodge types) (negative of weight)

Remark 5: One can also define a PHS on V (polarized by Q) as a real representation $\varphi: S^1 \hookrightarrow \mathrm{Aut}(V_{\mathbb{R}}, Q)$ $\left[\text{think: } \varphi(z)|_{V^{p,q}} = z^{p-q}\right]$

$\cap$ $\mathbb{C}^*$

The smallest $\mathbb{Q}$-algebraic group $\underbrace{}_{M_\varphi}$ containing $\varphi(S^1)$ is called φ's Mumford–Tate group.