C. Polarized Variations of Hodge Structure (PVHS)

Recall that a PHS is a triple $(V_{\mathbb{Z}}, F^{\bullet}, Q)$ where $F^{\bullet}$ is a filtration on $V_{\mathbb{C}}$ satisfying HR I & II.

Let

$\mathcal{S}$ = $\mathbb{C}$-manifold

$\mathbb{V}_{\mathbb{Z}}$ = finite-rank local system of abelian groups over $\mathcal{S}$ (with obvious associated $\mathbb{Q}/\mathbb{R}/\mathbb{C}$-local systems via $\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, etc.)

Given

$\{\mathcal{U}_{\alpha}\}$ = good open cover of $\mathcal{S}$,

$\mathbb{V}_{\mathbb{C}}|_{\mathcal{U}_{\alpha}}$ is constant and $\{\underbrace{\mathbb{V}_{\mathbb{C}}(\mathcal{U}_{\alpha})}_{\text{(vect. sp.)}} \times \mathcal{U}_{\alpha}\}$ patch together

via the sheaf-restriction maps. Viewed as transition functions, these are constant ($\Rightarrow$ holomorphic) sections

$$\{\Phi_{\alpha\beta} \in \Gamma(\mathcal{U}_{\alpha\beta}, GL(\mathbb{V}_{\mathbb{C}}(\mathcal{U}_{\alpha\beta})))\}_{\alpha,\beta} ;$$

hence we have constructed:

V = holomorphic vector bundle over $\mathcal{S}$.

One also has its sheaf of holomorphic sections

$$\mathcal{V} := \mathcal{O}_{\mathcal{S}}(V) = \mathbb{V}_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{\mathcal{S}},$$

and the flat ($\Leftrightarrow$ integrable) connection

$$\text{(C.1)} \quad \boxed{\nabla : \mathcal{V} \to \Omega^1_{\mathcal{S}} \otimes \mathcal{V}}$$

defined locally (on $\mathcal{U}_\alpha$, given a basis $\{\sigma_1,\ldots,\sigma_r\} \subset \mathbb{V}_{\mathbb{C}}(\mathcal{U}_\alpha)$) by

$$\nabla(\Sigma f_j \sigma_j) = \Sigma df_j \otimes \sigma_j$$

and compatible with the $\Phi_{\alpha\beta}$ by $\mathbb{C}$-linearity of d.

Definition 1: A PVHS of weight n over $\mathcal{S}$ consists of

- $\mathbb{V}_{\mathbb{Z}}$ (= $\mathbb{Z}$-local system of rank $r < \infty$)
- $Q : \mathbb{V}_{\mathbb{Z}} \times \mathbb{V}_{\mathbb{Z}} \to \underline{\mathbb{Z}}$ $\begin{cases}(-1)^n\text{-symmetric bilinear form} \\ \text{nondegenerate}\end{cases}$ (map of local systems)
- $F^\bullet$ = filtration of V by holomorphic subbundles [write $\mathcal{F}^\bullet = \mathcal{O}_{\mathcal{S}}(F^\bullet V)$]

such that (a) the fiberwise restrictions $(\mathbb{V}_{\mathbb{Z},s}, F^\bullet_s, Q_s)$ yield PHS of weight n and (b) $\nabla(\mathcal{F}^p) \subset \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{p-1}$ $(\forall p)$. $\square$

The ranks $h^{p,n-p} = \mathrm{rk}(\mathcal{F}^p/\mathcal{F}^{p+1})$ are constant, and as in §B one has $\overline{\nabla}$ and its $\mathcal{O}_{\mathcal{S}}$-linearity. Moreover, fixing a base point $s_0 \in \mathcal{S}$, the local system gives rise to a monodromy representation

$$(C.2) \qquad \boxed{\rho : \pi_1(\mathcal{S}) \longrightarrow \mathrm{Aut}(\mathbb{V}_{\mathbb{Z},s_0}, Q_{s_0}) =: G_{\mathbb{Z}}}$$

since $Q(\sigma, \eta)$ remains globally* constant for (multivalued) sections of $\mathbb{V}_{\mathbb{Z}}$.

How about some examples. By Griffiths transversality, we have

Proposition 1: The n^{th} primitive cohomology of a smooth projective family $\pi : \mathcal{X} \to \mathcal{S}$ produces a PVHS $/\mathcal{S}$.

Such a PVHS is said to come from geometry, and if $\pi, \mathcal{X}, \mathcal{S}$ are defined over a field $K \subset \mathbb{C}$, to be motivated over K.

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* $\underline{\mathbb{Z}}$ is a constant sheaf; so locally constant $\Longrightarrow$ globally constant

Here are 2 more concrete examples along these lines, with

$\mathcal{S} = \mathbb{D}^*$, $T := \rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$

<u>Example 1</u> $(n=1, r=2)$†: $\mathbb{V}_{\mathbb{Z},s_0} = \mathbb{Z}\langle\beta,\alpha\rangle$, $Q = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $T = \begin{pmatrix}1&0\\1&1\end{pmatrix}$.

$$F^1_s = \mathbb{C}\langle\omega_s\rangle, \quad \underbrace{\omega_s}_{\text{(single-valued!)}} = \beta + (\underbrace{g(s)}_{\substack{\text{holo.}\\ \text{on } \mathbb{D}}} - \underbrace{\ell(s)}_{=\frac{\log(s)}{2\pi i}})\,\alpha$$

think of as giving <u>multivalued</u> sections of $\mathbb{V}_{\mathbb{Z}}$.

Note that one can reparametrize by $q(s) := \exp\left(2\pi i\,\frac{Q(\beta,\omega_s)}{Q(\alpha,\omega_s)}\right)$
$= s\cdot e^{-2\pi i g(s)}$

to get rid of g. The transversality condition (b) is empty.

<u>Example 2</u> $\overset{\text{type }(1,1,1,1)}{(n=3, r=4)}$: $\mathbb{V}_{\mathbb{Z},s_0} = \mathbb{Z}\langle\gamma_3,\gamma_2,\gamma_1,\gamma_0\rangle$, $Q = \begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$, $T = \begin{pmatrix}1&&&\\1&1&&\\3&-5&1&\\5&8&-1&1\end{pmatrix}$

recall $\delta_q = q\,\partial/\partial q$

write $\tau = \ell(q)$

$$F^3_q = \mathbb{C}\langle\omega_q\rangle$$
$$F^2_q = \mathbb{C}\langle\omega_q, \nabla_{\delta_q}\omega_q\rangle$$
$$F^1_q = \mathbb{C}\langle\omega_q, \nabla_{\delta_q}\omega_q, \nabla^2_{\delta_q}\omega_q\rangle$$

where $\omega_q := \gamma_3 - \tau\gamma_2 + \left(-\frac{5}{2}\tau^2 - \frac{11}{2}\tau + \frac{25}{12}\right)\gamma_1 + \left(-\frac{5}{6}\tau^3 - \frac{25}{12}\tau + \frac{25i}{\pi}\zeta(3)\right)\gamma_0 + O(\tau\log^3\tau)$

is single-valued

and one has the (Griffiths-)<u>Yukawa coupling</u>

$$\mathcal{Y}(q) := Q(\underbrace{\omega_q}_{\in F^3}, \underbrace{\nabla^3_{\delta_q}\omega_q}_{\in \mathcal{F}^0}).$$

<u>Remark 1</u>: Ex. 1 comes from $\mathcal{H}^1$ of a nodally degenerating family of elliptic curves; Ex. 2 from $\mathcal{H}^3$ of the "mirror quintic" family of Calabi-Yau 3-folds. Why "mirror"? Around 1990, [Candelas et al] realized that the coefficients n_d of the power-series expansion of $\mathcal{Y}$ "counted" rational curves of degree d on the Fermat quintic family of CY 3-folds (cf. pp. 174-5, $n=4$), giving rise to <u>mirror symmetry</u>.

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† more succinctly, of type $(1,1)\,(=(h^{1,0},h^{0,1}))$. In general, we call a PVHS of "type $(h^{n,0}, h^{n-1,1}, \dots, h^{0,n}) = \underline{h}$".

Ex/ derive Example 1 from the Picard-Fuchs equation in Problem Set 5 Exercise 8. Also show how it arises from the natural family of complex 1-tori over the upper half-plane. //

Continuing to fix a base point $s_0 \in \mathcal{S}$, we may consider instead $F_s^\bullet$ as a multivalued family of flags on V_{s_0}, with multivaluedness described by

$$\Gamma := \rho(\pi_1(\mathcal{S})) \subseteq G_{\mathbb{Z}}$$

of type $\underline{h}$ / pol. Q

acting on the integral basis γ. Hence from our VHS/ $\mathcal{S}$ we get a well-defined map*

$$\Phi : \mathcal{S} \to {}_{\Gamma}\backslash D = {}_{\Gamma}\backslash G_{\mathbb{R}} / H_0$$

period domain for HS of type $\underline{h}$ pol. by Q

with local liftings

$$\begin{array}{ccc} \cup & & \uparrow \\ \mathcal{U}_\alpha & \longrightarrow & D \end{array} \quad \text{since } \pi_1(\mathcal{U}_\alpha) = \{1\}.$$

Here H_0 = stabilizer of a reference flag $F_0 \in D$, and

$$\begin{array}{ccl} D & = & G_{\mathbb{R}} / H_0 \\ \cap & & \\ \check{D} & = & G_{\mathbb{C}} / B_0 \quad , \quad H_0 = B_0 \cap G_{\mathbb{R}} \\ \cap & & \\ \prod_p \mathrm{Grass}(f^p, V) & & , \quad f_p = \dim F_0^p \end{array}$$

Now $T_{F^p} \mathrm{Grass}(f^p, V) \cong \mathrm{Hom}_{\mathbb{C}}(F^p, V/F^p)$ [holomorphic tangent space]

$$\Longrightarrow T_{F^\bullet} \overset{(\vee)}{D} \subseteq \bigoplus_p \mathrm{Hom}_{\mathbb{C}}(F^p, V/F^p) \; ;$$

and since the flags $F_s^\bullet$ vary holomorphically, this implies <u>Φ is a holomorphic mapping of $\mathbb{C}$-manifolds</u>.

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* Γ acts on the left, H_0 on the right. This makes sense if you think in terms of period matrices: columns are ($\mathbb{C}$-) basis vectors of F^p, written in terms of $\mathbb{Z}$-basis. Acting on the left changes $\mathbb{Z}$-basis, acting on the right moves the flag. (This has to be interpreted carefully, since they are acting on $G_{\mathbb{R}}$ which acts on pd. matrices on left.)

Next consider the <u>horizontal distribution</u>

$$\text{(C.3)} \qquad \boxed{W_{F^\cdot} := T_{F^\cdot}\check{D} \cap \left(\bigoplus_p \mathrm{Hom}_{\mathbb{C}}\left(F^p, F^{p-1}/F^p\right)\right) \subset T_{F^\cdot}\check{D}} \; ;$$

clearly Φ is <u>horizontal</u>:

$$\text{(C.4)} \qquad \boxed{d\Phi(T\mathcal{S}) \subset W.}$$

<u>Definition 2</u>: A <u>period map</u> (from a $\mathbb{C}$-manifold $\mathcal{S}$) is a locally liftable, holomorphic, horizontal mapping $\Phi : \mathcal{S} \to {}_{\Gamma}\backslash^{D}$, where D is a period domain and $\Gamma \leq G_{\mathbb{Z}}$.

<u>Proposition 2</u>: The notions of "period map + compatible local system" and PVHS are equivalent.

One has the dual distribution

$$I = W^{\perp} \subset T^*\check{D}$$

and the differential ideal it generates (= an "exterior differential system")

$$\mathcal{I}_D \subset \bigoplus_j \Omega^j_{\check{D}}$$

is called the <u>IPR</u> (= infinitesimal period relation). An integral ($\mathbb{C}$-) manifold of $\mathcal{I}_D$ is a holomorphic mapping $f : M \hookrightarrow \check{D}$ with

$$f^*(\mathcal{I}) = 0 .$$

<u>Proposition 3</u>: The local liftings of a period map give integral manifolds of the IPR.

<u>Example 3</u>: The PVHS of Example 1 maps $\mathbb{D}^* \to {}_{\langle T\rangle}\backslash^{\mathfrak{h}} \cong \mathbb{D}^*$.

For weight 1 HS, the IPR is trivial (clear).

<u>Definition 3</u>: A period domain is <u>classical</u> $\iff$ $\mathcal{I}_D = 0$.

Ex/ Show that the IPR is trivial for $\begin{cases} \text{weight } n=2 \\ \text{and} \\ h^{2,0} (= h^{0,2}) > 1 \end{cases}$.

Moreover this is the only other example (of classical pd. domain.) //

Example 4: The PVHS of Example 2 maps $\mathbb{D}^* \longrightarrow \langle T \rangle \backslash \underbrace{Sp_4(\mathbb{R}) / U(1)^{\times 2}}_{D}$.

Claim: $\dim(D) = 4$. 2 proofs: ① $\dim_{\mathbb{R}} Sp_4(\mathbb{R}) = 10$
$\Rightarrow \dim_{\mathbb{C}} D = \frac{10-2}{2} = 4$.

② choosing F^3: 3 degrees of freedom

$Q(F^2, F^3) = 0$ (HRI) $\Rightarrow \begin{cases} F^2 \text{ is 2-plane thru line } F^3 \\ \text{in } \underline{3\text{-space}}\ (F^3)^{\perp} \end{cases} \Rightarrow$ 1 degree of freedom

$Q(F^1, F^3) = 0 \Rightarrow F^1 = (F^3)^{\perp} \Rightarrow$ 0 degrees of freedom. //

Now $W \subset TD \cap \{ \mathrm{Hom}(F^3, F^2/F^3) \oplus \mathrm{Hom}(F^2, F^1/F^2) \oplus \mathrm{Hom}(F^1, F^0/F^1) \}$

contracts

evidently has rank 2.

Moreover, if $\omega_j \in V_0^{j, 3-j}$ is a basis and ω a local section of F^3_s (for a general PVHS), it is clear that

$\begin{cases} \omega_3'(0) \longleftrightarrow \text{the } \mathrm{Hom}(F^3, F^2/F^3) \text{ part above} \\ \omega_3''(0) \longleftrightarrow \text{" } \mathrm{Hom}(F^2, F^1/F^2) \text{ " "} \end{cases} \Longrightarrow$ maximal integral manifold has dimension 1 !! //

Remark 2: This example reflects the fact that W is non-involutive (i.e. nonintegrable) whenever it is nontrivial*: So the dimension of integral manifolds is always strictly less than the rank of W (when $\mathrm{rk}\, W < \dim D$). In weight 2, the IPR is related to contact equations. //

* at least under the assumption of no gaps in the nonzero Hodge numbers $\underline{h}$.

We also have the curious

Theorem 1: In a nonclassical period domain D, the set of PHS coming from geometry (cf. Prop. 1) has measure zero.

Proof: Given a family $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ defined $/\mathbb{C}$, it sits inside a larger family $\tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \tilde{\mathcal{S}}$ ($\tilde{\pi}^{-1}(\mathcal{S}) = \mathcal{X}$) defined $/\bar{\mathbb{Q}}$, called the $\bar{\mathbb{Q}}$-spread. [This is constructed as follows (roughly): the original $\pi, \mathcal{X}, \mathcal{S}$ are defined over a finitely generated extension $K \,(\subset \mathbb{C})$ of $\bar{\mathbb{Q}}$, and one has $K(\mathcal{S}) \cong \bar{\mathbb{Q}}(\tilde{\mathcal{S}})$ for some smooth projective $\tilde{\mathcal{S}}/\bar{\mathbb{Q}}$. Hence we may replace coefficients of defining eqn. of $\mathcal{X}$ in $K(\mathcal{S})$ by ones in $\bar{\mathbb{Q}}(\tilde{\mathcal{S}})$; this yields (after a "good compactification") $\tilde{\mathcal{X}}$.] Hence, every motivated PVHS can be embedded in one motivated $/\bar{\mathbb{Q}}$; the latter are countable in number. So if every period map image has dimension $< \dim D$, the points in D coming from geometry are a union of countably many proper submanifolds. $\square$

<u>Remark 3</u>: And yet, not a single explicit non-motivic PHS is known. //

An interesting question is: how can one compute the dimension of the maximal integral submanifolds? To at least hint at this, and to elucidate some of the constructions above, we can look at period domains from a Lie algebra standpoint: writing

$$V_0 = V_{s_0} \,(= \mathbb{C}\text{-vector space}) \supset V_{0,\mathbb{R}} \,(= \text{invariants under complex conj.})$$

we have

$$\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{C}}) = \{ X \in \mathfrak{gl}(V_0) \mid Q(Xu,v) + Q(u,Xv) = 0 \quad (\forall u,v \in V_0)\}$$

$$\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}}) = \mathfrak{g} \cap \mathfrak{gl}(V_{0,\mathbb{R}}).$$

Using our choice of reference flag $F_0 \,(\in D)$, define

$$F^a\mathfrak{g} := \{ X \in \mathfrak{g} \mid X(F_0^p) \subset F_0^{p+a} \; (\forall p)\}.$$

This puts a $\mathbb{Q}$-HS* of weight 0 on $\mathfrak{g}$:

$$\mathfrak{g}^{a,-a} := \{ X \in \mathfrak{g} \mid X(V_0^{p,q}) \subset V_0^{p+a,q-a}\} = F^a\mathfrak{g} \cap \overline{F^{-a}\mathfrak{g}},$$

and clearly

$$\text{(C.5)} \qquad [F^p\mathfrak{g}, F^q\mathfrak{g}] \subset F^{p+q}\mathfrak{g}, \quad [\mathfrak{g}^{a,-a}, \mathfrak{g}^{b,-b}] \subset \mathfrak{g}^{a+b,-(a+b)}.$$

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* in fact, polarized, by the <u>Killing</u> form

Now a flag varies infinitesimally under $(\mathbb{1}+\epsilon X) \iff$

$$X(F^p) \not\subset F^p \text{ for some } p\ ;$$

hence, $\mathfrak{b}_0 = F^0\mathfrak{g}$ and $\mathfrak{h}_0 = F^0\mathfrak{g} \cap \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}$.

It follows that

$$T_{F_0}\check{D} = \mathfrak{g}/\mathfrak{b}_0 \qquad \text{[holo. tangent space]}$$

and

$$T\check{D} = \check{D} \underset{B_0}{\times} \mathfrak{g}/\mathfrak{b}_0\,. \qquad \text{[holo. tangent bundle \underline{as a homogeneous bundle}]}$$

Since $[F^0\mathfrak{g}, F^p\mathfrak{g}] \subset F^p\mathfrak{g}$, the adjoint action of B_0 leaves the F^p invariant and they extend to global subbundles $\mathfrak{F}^p$. In particular, we have

$$W = \mathfrak{F}^{-1}T\check{D} \quad \text{with} \quad W_{F_0} = \mathfrak{g}^{-1,1}$$

[and so $[W_{F_0}, W_{F_0}] \subset \mathfrak{g}^{-2,2}$ (by (C.5)) rather than W_{F_0}]

For D, considered as a real manifold,

$$T_{F_0}D = \mathfrak{g}_{\mathbb{R}}/\mathfrak{h}_0 = \mathfrak{g}_{\mathbb{R}}/\mathfrak{g}^{0,0}\cap\mathfrak{g}_{\mathbb{R}} \qquad \text{[real tangent bundle]}$$

$$\implies T_{F_0}D \otimes_{\mathbb{R}} \mathbb{C} \left(= T^{1,0}_{F_0}D \oplus T^{0,1}_{F_0}D\right) = \underbrace{\left(\bigoplus_{i>0} \mathfrak{g}^{-i,i}\right)}_{\mathfrak{g}^-\ \cong\ \mathfrak{g}/\mathfrak{b}_0} \oplus \underbrace{\left(\bigoplus_{i>0} \mathfrak{g}^{i,-i}\right)}_{\mathfrak{g}^+}. \qquad \text{[choice of } \mathbb{C}\text{-structure]}$$

<u>Remark 4</u>: $\bigoplus_{i>0} \mathfrak{g}^{-i,i}$ is involutive, so Newlander-Nirenberg $\implies$ this complex structure is integrable. (But then, from $\check{D}$, we know that!)

Write $\mathfrak{k} = \bigoplus_{i\text{ even}} \mathfrak{g}^{i,-i}$, $\mathfrak{p} := \bigoplus_{i\text{ odd}} \mathfrak{g}^{i,-i}$ $\implies \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

exp $\rightsquigarrow$ maximal compact sgp. $K \leq G$

So <u>D is a Hermitian symmetric domain</u> $\iff H_0$ maximal compact $\iff \mathfrak{k} = \mathfrak{g}^{0,0}$

and

<u>the IPR is trivial</u> $\iff \mathfrak{F}^{-1}TD = TD \iff \begin{cases} \mathfrak{k} = \mathfrak{g}^{0,0} \text{ \underline{AND}} \\ \mathfrak{p} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}. \end{cases}$

[N.B. For period domains (as opposed to the more general Mumford-Tate domains), however, there are essentially no interesting examples with D HSD and IPR nontrivial: just Hodge #'s $\underline{h} = (1,1,0,1,1)$, $(1,1,0,0,0,1,1)$, etc.]

Finally, to answer the question about integral submanifolds, there is the (fairly obvious)

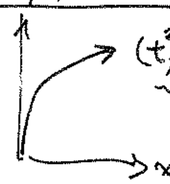
"integral element"

Proposition 4: Given an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}^{-1,1}$, the $\mathbb{C}$-manifold $\exp(\mathfrak{a})$ is an integral manifold of I_D with tangent space $\mathfrak{a}$.

Notes: (i) Regarding Example 4 on p. 204, the point is that if $\mathrm{Hom}(F^3, F^2/F^3) \oplus \mathrm{Hom}(F^2, F^1/F^2)$ is the tangent plane to some surface S integral to W, with local complex coordinates x and y, the relation $x'' = y'x'$ leads to a contradiction.

then ω_3

We should be able, locally, to draw any curve $(x(t), y(t))$ in S, but already (t^2, t) leads to a contradiction.*



There is a paper by Carlson, Griffiths, & Green (recent) which gives an explicit local normal form for I_D in this case.

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* Thus, there are lots of curves thru $F_0^{\bullet}$ integral for W, but there is not an integral surface.