D. Curvature of period domains

Let M = Hermitian manifold w./ complex structure J & metric h,

$p \in M$,

$P = \mathbb{R}\langle \xi, J(\xi)\rangle \subset T_{M,p}$ (= real tangent bundle)

a J-invariant plane,

S = the surface consisting of geodesics emanating from p tangent to P.

The <u>holomorphic sectional curvature</u> $K(\xi, p)$ is the Gaussian curvature* of S at p.

<u>Example 1</u>: Here are 4 <u>1-$\mathbb{C}$-dim'l</u> cases where the Gaussian curvature

$$K := \frac{-\partial_z \partial_{\bar z} \log \rho}{\rho} \equiv -1 \quad (\text{where } h = \rho\, dz \otimes d\bar z):$$

(a) $\mathbb{D}(=M)$, $\frac{1}{(1-|z|^2)^2}$ $(=\rho(z))$

(b) $\mathbb{D}_R$, $\frac{R^2}{(R^2-|z|^2)^2}$

(c) $\mathfrak{h}$, $\frac{1}{y^2}$

(d) $\mathbb{D}^*$, $\frac{1}{|z|^2(\log|z|^2)^2}$

easy Ex/ $\boxed{\text{up to const., with } E = \exp(2\pi i(\cdot)) \quad E^* h_{\mathbb{D}} = h_{\mathfrak{h}}.}$

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<u>Example 2</u>: D period domain, $F_0^\bullet$ reference HS, $T_{D,F_0^\bullet} \cong \mathfrak{g}^-$.

We need, first, a Hermitian metric h_D on D.

<u>Lemma 1</u>: $-B$ polarizes the HS on $\mathfrak{g}$, where

$$B(X,Y) := \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$$

is the <u>Killing form</u>.

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* if M were isometrically embedded in $\mathbb{R}^N$, this would be the product of the reciprocals of the maximal & minimal radii of normal osculating circles.

<u>Proof</u>: Since C (← Weil operator) $\in G_{\mathbb{R}}$ (why?), one can define a Hermitian form by

$$\langle X, Y\rangle := -B((\mathrm{Ad}\, C)X, \bar{Y}).$$

Also set $\mathfrak{n} = i\mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} = \{\xi \in \mathfrak{g}_{\mathbb{C}} \mid (\mathrm{Ad}\, C)\xi = -\bar{\xi}\} \subseteq \mathfrak{g}$

Ex/ For $\xi \in \mathfrak{n}$, $\langle (\mathrm{ad}\,\xi)X, Y\rangle = \langle X, (\mathrm{ad}\,\xi)Y\rangle$. //

So $\mathrm{ad}\,\xi$ self-adjoint $\Longrightarrow$ diagonalizable (on $\mathfrak{g}$) with real eigenvalues

$\Longrightarrow$ for $X \in \mathfrak{g}^{-p,p}$,

$$\langle X, X\rangle = \tfrac{1}{2}\langle X+\bar{X}, X+\bar{X}\rangle \qquad X+\bar{X} =: \begin{cases} \xi, & p \text{ odd} \\ i\xi, & p \text{ even}\end{cases}$$

$$= -\frac{(-1)^{p+1}}{2}(-1)^p \,\mathrm{Tr}\, \mathrm{ad}\,\xi\, \mathrm{ad}\,\xi > 0.$$

$\Rightarrow$ (HR II)

Next, $X \in \mathfrak{g}^{-p,p}$, $Y \in \mathfrak{g}^{-q,q}$ $\Rightarrow$ $\mathrm{ad}\,X\, \mathrm{ad}\,Y$ sends $\mathfrak{g}^{-a,a}$ to $\mathfrak{g}^{-(a+p+q),(a+p+q)}$

$\Rightarrow$ trace $= 0$ unless $p = -q$

$\Rightarrow$ (HR I). □

Extend the inner product $\langle\, ,\rangle$ via translation by $G_{\mathbb{R}}$ to a Hermitian metric on D. ($= h_D$)

<u>Lemma 2</u>: For $X \in \mathfrak{g}^{-p,p}$, $\displaystyle K(X, F_0^{\bullet}) = (-1)^{p-1}\frac{B([X,\bar{X}], \overline{[X,\bar{X}]})}{\langle X, X\rangle^2}$.

"<u>Proof</u>": need theory of Lie algebras + curvature of homogeneous bundles (Maurer-Cartan formulas). or see p. 213

See [Griffiths-Schmid], "Locally homogeneous compact manifolds". □

<u>Theorem 1</u>: The holomorphic sectional curvatures in W (= horizontal distribution) are negative* and bounded away from zero (can normalize so ≤ -1).

"<u>Proof</u>": take $p=1$ in lemma 2. $B(\cdot, \bar{\cdot})$ is negative definite on $\mathfrak{g}^{0,0}$ ($\ni [X,\bar{X}]$), and one can show $[X,\bar{X}] \neq 0$ if $X \neq 0$. To see "bounded above" at $F_0^{\bullet}$, take the maximum over $\begin{cases}\langle X,X\rangle = 1 \\ X \in \mathfrak{g}^{-1,1}\end{cases}$; it is then bounded

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* more generally, they are positive in $\mathfrak{k}$ and negative on $\mathfrak{p}$.

above on all of $\mathbb{D}$ by $G_{\mathbb{R}}$-invariance of $h_{\mathbb{D}}$. $\square$

Write ω_h for the $(1,1)$ form associated to a Hermitian metric.

On a complex 1-manifold,
$$\begin{cases} \omega_h = i\, h\, dz \wedge d\bar{z} \\ \operatorname{Ric} \omega_h = \frac{i}{2} \partial\bar{\partial} \log h = -K \omega_h . \end{cases}$$

<u>Lemma 3 (Ahlfors)</u>: Given $f : \mathbb{D}_R \to M$, $\mathbb{h}$ Hermitian metric on M with $K \leq -1$, we have
$$\boxed{f^* \omega_{\mathbb{h}} \leq \omega_{h_R}} .$$

<u>Proof</u>: writing for any $r < R$
$$\psi := f^* \omega_{\mathbb{h}} = \mu\, \omega_{h_r} ,$$

$\omega_{h_r} \to \infty$ as $|z| \to r$ $\Rightarrow$ μ bounded on $\mathbb{D}_r$

$\Rightarrow$ has interior max (say, at z_0)

$$\Rightarrow \left.\begin{aligned} 0 &\geq i\partial\bar{\partial} \log \mu \\ &= \operatorname{Ric} \psi - \operatorname{Ric} \omega_{h_r} \\ &\overset{\substack{K=-1 \\ \text{resp } \leq -1}}{\geq} \psi - \omega_{h_r} \end{aligned}\right\} (@\, z_0)$$

$\Rightarrow$ $\mu(z_0) \leq 1$ (max value!)

$\Rightarrow$ $\psi \leq \omega_{h_r}$ on $\mathbb{D}_r$. Take $r \to R$. $\square$

<u>Lemma 4 (Schwarz)</u>: Given $f : \mathbb{D} \to \tilde{M}$, with image tangent to directions for which $K \leq -1$, the map is distance decreasing:
$$f^* \omega_{\mathbb{h}} \leq \omega_{\mathbb{D}} .$$

<u>Proof</u>: Apply Lemma 3 to the case $M = f(\mathbb{D}) \subset \tilde{M}$, $\mathbb{h} = \tilde{\mathbb{h}}|_M$. $\square$

The proof easily generalizes to replace $\mathbb{D}$ by $\mathfrak{h}$, $\mathbb{D}^*$.

Consider a VHS $/\mathbb{D}^*$, with period map
and lifting (via $q(\tau) = e^{2\pi i \tau}$)
satisfying $\tilde{\Phi}(\tau+1) = T(\tilde{\Phi}(\tau))$.

$$\begin{tikzcd} \mathbb{D}^* \arrow[r, "\Phi"] & \langle T\rangle\backslash D \\ \mathfrak{h} \arrow[u, "q"] \arrow[r, dashed, "\tilde{\Phi}"] & D \arrow[u, two heads] \end{tikzcd}$$

<u>Lemma 5</u>: $\tilde{\Phi}$ is distance-decreasing with respect to the Riemannian metrics associated to $h_{\mathfrak{h}}$ and h_D ($= G_{\mathbb{R}}$-invariant metric above):

$$d_D(\tilde{\Phi}(p_1), \tilde{\Phi}(p_2)) \leq d_{\mathfrak{h}}(p_1, p_2).$$

<u>Proof</u>: Apply Theorem 1 and Lemma 4. $\square$

<u>Theorem 2</u>: Any VHS over $\mathbb{C}^*$ is <u>isotrivial</u>: i.e. Φ is constant.*

<u>Proof</u>: Start with $\mathbb{C} \xrightarrow{\Phi} D$. By Ahlfors, $\Phi_R := \Phi|_{\mathbb{D}_R}$ satisfies

$$\underset{\substack{\| \\ i\, J(z)\, dz\wedge d\bar{z}}}{\Phi_R^* \omega_{h_D}} \leq C\cdot \omega_{h_R} \quad \Rightarrow \quad (0 \leq)\ J(0) \leq C\cdot \frac{1}{R^2} \quad (\forall R)$$
$$\Rightarrow \quad J(0) = 0.$$

Of course, we can do this about any other point of $\mathbb{C}$. So $J \equiv 0 \Rightarrow \Phi_R$ constant. To see the result for $\mathbb{C}^*$, lift

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\Phi"] & \langle T\rangle\backslash D \\ \mathbb{C} \arrow[u] \arrow[r, dashed] & D \arrow[u, two heads] \end{tikzcd}$$

$\square$

<u>Corollary 1</u>: Let $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$ be a projective family with smooth general fiber ($\Rightarrow$ only finitely many singular fibers) inducing a non-isotrivial VHS (from $\mathcal{H}^n_{\mathcal{X}/\mathbb{P}^1\setminus\{pts\}}$). Then π has <u>at least 3</u> singular fibers.

Ex/ Find a family of elliptic curves with only 3 singular fibers and nonconstant J-invariant. This proves Cor. 1 sharp. Why? //

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* the local system can still have finite monodromies: hence "isotrivial" and not "trivial".

<u>Theorem 3 (Monodromy Theorem)</u>: Given a VHS $/ \mathbb{D}^*$, T as above.

Then T is quasi-unipotent: $\exists\, m, N \in \mathbb{N}$ s.t. $(T^m - \mathbb{1})^N = 0$.

<u>Proof</u>: By lemma 5, $d_D(\tilde{\Phi}(in), \underbrace{T(\tilde{\Phi}(in))}_{(=\tilde{\Phi}(in+1))}) \leq d_{\mathfrak{h}}(in, in+1) \underset{\uparrow \text{(why?)}}{=} \frac{1}{n}$.

Writing $\tilde{\Phi}(in) = g_n H$ (cosets: $g_n \in G_{\mathbb{R}}$), $G_{\mathbb{R}}$-invariance of h_D $\Longrightarrow$

$d_D(eH, g_n^{-1} T g_n H) = d_D(g_n H, T g_n H) \leq \frac{1}{n} \Rightarrow \lim_{n\to\infty} g_n^{-1} T g_n \in H$.

$\Longrightarrow$ ccl of T has a limit point in the compact subgroup $H_{(0)} \leq G$

$\overset{*}{\Longrightarrow}$ $|\text{eigenvalues}(T)| = 1$.

Moreover, $T \in G_{\mathbb{Z}} \Rightarrow$ eigenvalues(T) solve $\det(\lambda \mathbb{1} - T) = 0$, which (monic integral eqn.)

together w./above $\Rightarrow$ " $\begin{cases} \text{(a) belong to } \bar{\mathbb{Z}} \\ \text{(b) have } |\cdot| = 1 \\ \text{(c) } \underbrace{\underline{\text{all}} \text{ Galois conjugates have } |\cdot| = 1}_{} \end{cases}$.

$\underset{\text{Thm}}{\overset{\text{Kronecker}}{\Longrightarrow}}$ they are roots of 1! (b/c these conjugates are also eigenvalues of T).
(# thry)

$\Rightarrow$ some T^m has eigenvalues $\equiv 1$.

Putting it in Jordan form, we see $T^m - \mathbb{1}$ is nilpotent. $\square$

There are geometric proofs of this theorem (for $\mathcal{X} \xrightarrow[\pi]{} \Delta \subset \mathbb{P}^1$), but as you can see, it is valid more generally. This is why the geometry, algebraic structure, and cohomology of period domains is worth studying!

$*$ $h \in H$ with $|\text{eigenvalues}| \neq 1$ would contradict compactness of H; and 2 elts. in same ccl have same eigenvalues.

Appendix to curve (with a proof of lemma 2)