C. Extensions of MHS

It is convenient at this point to officially drop our convention that in Hodge structures V, the weight n and (p,q)'s are non-negative. The preferred measure of complexity of V is then its

$\underline{\text{level}}(V) :=$ difference between lowest & highest values of p.

When Hodge theorists talk about properties which emerge in "higher weight" they really mean "higher level". This notion extends to MHS, where one also has the

$\underline{\text{length}}(V) :=$ difference between lowest & highest weights.

Now let $\mathcal{C} :=$ an abelian category with enough injectives, $X \in \mathcal{C}$ and

$$0 \to A \to B \to C \to 0$$

a short-exact sequence in $\mathcal{C}$. By Example I.F.11 & Prop. I.F.1, we have a long-exact sequence

$$\text{(C.1)} \quad 0 \to \mathrm{Hom}_{\mathcal{C}}(X,A) \to \mathrm{Hom}_{\mathcal{C}}(X,B) \to \mathrm{Hom}_{\mathcal{C}}(X,C) \to \mathrm{Ext}^1_{\mathcal{C}}(X,A) \to \mathrm{Ext}^1_{\mathcal{C}}(X,B) \to \mathrm{Ext}^1_{\mathcal{C}}(X,C) \to \mathrm{Ext}^2_{\mathcal{C}}(X,A) \to \cdots$$

Taking an injective resolution $A \to I^\bullet$, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(X,A) := \frac{\ker\{\mathrm{Hom}_{\mathcal{C}}(X,I^1) \to \mathrm{Hom}_{\mathcal{C}}(X,I^2)\}}{\mathrm{im}\{\mathrm{Hom}_{\mathcal{C}}(X,I^0) \to \mathrm{Hom}_{\mathcal{C}}(X,I^1)\}}$$

$$\cong \frac{\mathrm{Hom}_{\mathcal{C}}(X,K)}{\text{homs factoring thru } I^0} \quad (\text{writing } K := \ker(I^1 \to I^2))$$

$$\text{(C.2)} \qquad \cong \frac{\text{s.e.s.'s } A \to E \to X}{\text{split s.e.s.'s}},$$

since in
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] & K \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\phi"] & 0 \end{tikzcd}$$
ϕ ⟵ given elt. of $\mathrm{Hom}_{\mathcal{C}}(X, K)$

$E = \ker\{I^0 \oplus X \to K\}$

see appendix to this section

one has a splitting $X \to E \iff \phi$ factors through I^0.

Now <u>MHS</u> does not have enough injectives, but there is a general theory due to Verdier & Yoneda* that extends (C.1) & (C.2) to our setting. I'll give a more down-to-earth presentation that doesn't prove everything. Let A, B be MHS:

<u>Definition 1</u>: $Hg(A) := \mathrm{Hom}_{MHS}(\mathbb{Z}(0), A) \xrightarrow[\text{(image of 1)}]{\cong} A_{\mathbb{Z}} \cap W_0 A \cap F^0 A_{\mathbb{C}}$

More generally, $Hg^p(A) := \mathrm{Hom}_{MHS}(\mathbb{Z}(-p), A) \cong A_{\mathbb{Z}} \cap W_{2p} A \cap F^p A_{\mathbb{C}}$

These are called the $\left\{\begin{array}{l}\text{Hodge classes} \\ \text{resp.} \\ \text{Hodge } (p,p) \text{ classes}\end{array}\right.$ in A, since integral & in $W_{2p} \cap F^p \Rightarrow$ also in $\overline{F^p} \Rightarrow$ in $I^{p,p}$.

<u>Definition 2</u>: (a) An <u>extension of MHS</u> is just an exact sequence of MHS $0 \to A \xrightarrow{\imath} H \xrightarrow{\pi} B \to 0$. A <u>section</u> is a morphism $s: B \to H$ s.t. $\pi \circ s = id_B$; an extension with section is <u>split</u>.

A <u>morphism</u> of extensions is a diagram
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & E' \arrow[r] & B' \arrow[r] & 0 \end{tikzcd}.$$

A <u>congruence</u> of extensions is an isomorphism s.t. α, β = identity. (Split extensions are congruent to $0 \to A \to A \oplus B \to B \to 0$.)

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* cf. the appendix A in [Peters–Steenbrink]

(b) $\underline{\mathrm{Ext}^1_{MHS}(B,A)} := \dfrac{\text{extensions}}{\text{congruence}}$, with the abelian group structure given by Baer summation (w./split extension as "0")

$$\left.\begin{array}{c} 0 \to A \to E \to B \to 0 \\ + \\ 0 \to A \to E' \to B \to 0 \end{array}\right\} = \quad
\begin{tikzcd}
0 \arrow[r] & A\oplus A \arrow[r] & E\oplus E' \arrow[r] & B\oplus B \arrow[r] & 0 \\
0 \arrow[r] & A\oplus A \arrow[r] \arrow[u, equal] \arrow[d, "\text{sum}"] & E'' \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u, "\Delta\ (\text{diagonal})"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & E''' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

(This requires some work, but is the same as the proof for R-modules in Maclane's book *Homology*.)

(c) $J(A) := \dfrac{W_0 A_{\mathbb{C}}}{F^0 W_0 A_{\mathbb{C}} + (W_0 A \cap A_{\mathbb{Z}})}$ [generalized Jacobian of A]

$J^p(A) := \dfrac{W_{2p} A_{\mathbb{C}}}{F^p W_{2p} A_{\mathbb{C}} + (W_{2p} A \cap A_{\mathbb{Z}})}$ [p^{th} " " " "]

(These are not in general algebraic, even for pure HS's of weight $2p-1$ where they are compact complex tori. The problem is the indefiniteness of the polarizing form in higher level.)

Ex/ $J^p(A)$ is a Lie group $\iff W_{2p}A = W_{2p-1}A$.

[Hint: draw the picture, and determine the conditions under which $W_{2p}A_{\mathbb{Z}}$ is a lattice in $W_{2p}A_{\mathbb{C}} / (W_{2p} \cap F^p) A_{\mathbb{C}}$.]

Theorem 1 (J. Carlson): There is a canonical & functorial isomorphism of groups (Ex)

$$\underbrace{\mathrm{Ext}^1_{MHS}(B,A) \cong J(\mathrm{Hom}(B,A))}_{\text{viewed as MHS}}$$

viewed as MHS: $\begin{cases} W_m \mathrm{Hom}(B,A) = \{\phi \in \mathrm{Hom} \mid \phi(W_k) \subset W_{k+m}\} \\ F^p \text{ defined similarly} \end{cases}$

Proof: Let $0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0 \in \mathrm{Ext}^1$, and choose a section $\sigma_{\mathbb{Z}} : B_{\mathbb{Z}} \to E_{\mathbb{Z}}$ strictly preserving $W_\bullet$. Any 2 differ by $\mathrm{Hom}(B,A)_{\mathbb{Z}} \cap \underbrace{W_0}_{\text{preserve weights}}$, and we define

$$\left.\begin{array}{ccc} A_{\mathbb{Z}} \oplus B_{\mathbb{Z}} & \underset{\cong}{\xrightarrow{f(\sigma_{\mathbb{Z}})}} & E_{\mathbb{Z}} \\ (a,b) & \longmapsto & \alpha(a) + \sigma_{\mathbb{Z}}(b) \end{array}\right\} \text{(preserves weights)}$$

Next, choose a section $\sigma_F : B_{\mathbb{C}} \to E_{\mathbb{C}}$ strictly preserving $F^\bullet$ & $W_\bullet$ (use Deligne's $I^{p,q}$'s). Any 2 differ by $\mathrm{Hom}(B,A)_{\mathbb{C}} \cap W_0 \cap F^0$, and we define

$$\begin{array}{ccc} A_{\mathbb{C}} \oplus B_{\mathbb{C}} & \underset{\cong}{\xrightarrow{f(\sigma_F)}} & E_{\mathbb{C}} \\ (a,b) & \longmapsto & \alpha(a) + \sigma_F(b) . \end{array}$$

Consider
$$f(\sigma_F)^{-1} \circ f(\sigma_{\mathbb{Z}})_{\mathbb{C}} \in \mathrm{Aut}_{(\mathbb{C})}(A_{\mathbb{C}} \oplus B_{\mathbb{C}}) \cap W_0$$
$$\begin{pmatrix} 1_A & \phi \\ 0 & 1_B \end{pmatrix}, \quad \text{where } \phi = \alpha^{-1} \circ (\sigma_{\mathbb{Z}} - \sigma_F) \in W_0 \mathrm{Hom}_{\mathbb{C}}(B,A).$$

Clearly the <u>extension class</u>
$$[\phi] \in \frac{W_0 \,\mathrm{Hom}(B,A)_{\mathbb{C}}}{F^0(\mathrm{hom}) + (\mathrm{hom})_{\mathbb{Z}}} = J(\mathrm{Hom}(B,A)),$$
is well-defined and if the extension is split gives zero.

To show that all classes in $J(\dots)$ occur: let $\phi \in \mathrm{Hom}(B,A)_{\mathbb{C}} \cap W_0$. Then $g_\phi := \begin{pmatrix} 1_A & \phi \\ 0 & 1_B \end{pmatrix} \in W_0 \,\mathrm{Aut}(A \oplus B)_{\mathbb{C}}$, and we define a MHS E by

$$\begin{aligned} F^\bullet_\phi (A_{\mathbb{C}} \oplus B_{\mathbb{C}}) &:= g_\phi \left(F^\bullet(A_{\mathbb{C}}) \oplus F^\bullet(B_{\mathbb{C}})\right) \\ &= F^\bullet(A_{\mathbb{C}}) + (\mathbb{1}_B + \phi) F^\bullet(B_{\mathbb{C}}) \end{aligned}$$
$\square$

<u>Example 1</u>: $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), \mathbb{Z}(n)) \cong J(\mathrm{Hom}(\mathbb{Z}(0), \mathbb{Z}(n))) \cong J(\mathbb{Z}(n)) = \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$

$n \geq 1$ (otherwise get 0)

or "$\mathbb{C}/\mathbb{Z}(n)$" $= \mathbb{C}/(2\pi i)^n \mathbb{Z}$.

<u>Example 2</u>: $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), A) \cong J(A)$

$\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-p), A) \cong J^p(A)$

<u>Example 3</u>: $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), H^1(\underset{\uparrow \text{curve}}{C})) \cong \mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), H^1(C)(1))$

$$\cong \frac{H^1(C,\mathbb{C})}{F^1 + H_{\mathbb{Z}}} \cong J(C) \text{ (Jacobian of } C).$$

<u>Theorem 2</u>: All Ext^i_{MHS} for $i \geq 2$ vanish.

<u>"Proof"</u>: This is a formal consequence of the surjectivity of $J(B) \twoheadrightarrow J(C)$ when $B \twoheadrightarrow C$ (clear from Thm. 1), using the aforementioned general theory. $\square$

A nice result in Carlson's article on Ext_{MHS} is a Torelli theorem for open curves. That is, the extensions in Deligne's MHS capture "where the missing points lie" (up to $\cong$) on $\overline{C}$ (= compactification of open curve). The content of this, beyond Torelli for curves, is really just Abel's theorem, which we shall prove in §V.1.

<u>Appendix</u>: at top of p. 229, if $\exists$ lift $\tilde{\phi}: X \to I^0$ of ϕ, you don't need to use injectivity of I^0 to get a splitting $X \to E$. It's more trivial than that: just recall $E := \ker\{I^0 \oplus X \to K\}$ and map X to E by $x \longmapsto (x, -\tilde{\phi}(x))$. That's it.