V. Algebraic cycles

A. Abel's theorem

Let M = compact Riemann surface (smooth projective curve), and recall

$$\begin{array}{ccc} \mathrm{Div}(M) = \text{divisors} & \overset{\text{finite}}{\sum} n_p[p] & \\ \downarrow \deg & \downarrow & \\ \mathbb{Z} & \sum n_p & \end{array}$$

} write $\mathrm{Div}^0(M)$ for $\ker(\deg)$.

We have the ~~map~~ homomorphism "take divisor" from meromorphic functions to divisors of degree 0

$$\mathcal{M}^*(M) \xrightarrow{(\cdot)} \mathrm{Div}^0(M)$$

$$f \longmapsto (f) := \sum \nu_p(f)[p]$$

and denote its image "$\mathrm{PDiv}(M)$" = <u>principal divisors</u>. The question "when is a degree-0 divisor the divisor of a function" can be rephrased in terms of the structure of the Picard group

$$\mathrm{Pic}^0(M) := \frac{\mathrm{Div}^0(M)}{\mathrm{PDiv}(M)}.$$

Let $\{\omega_1, \ldots, \omega_g\}$ denote a basis for $\Omega^1(M)$, and consider the map

$$\begin{tikzcd} \mathrm{Div}^0(M) \arrow[r] \arrow[rr, bend left, "\widetilde{AJ}"] & \frac{\Omega^1(M)^\vee}{\int_{H_1(M,\mathbb{Z})}} \arrow[r, "\cong"', "ev_{\{\omega_i\}}"] & \frac{\mathbb{C}^g}{\Lambda} = J(M) \text{ (Jacobian)} \end{tikzcd}$$

Λ ↑ (2g lattice)

sending

$$D \longmapsto \int_\Gamma \longmapsto \left(\int_\Gamma \omega_1, \ldots, \int_\Gamma \omega_g\right)$$

where $\Gamma \in C_1(M)$ is any 1-chain with $\partial\Gamma = D$. If $f \in \mathcal{M}^*(M)$, $D = (f) = f^{-1}(0) - f^{-1}(\infty)$, then

$$t \longmapsto \int_{f^{-1}(\overrightarrow{0.t})} (\cdot)$$

induces a holomorphic* map

$$\mathbb{P}^1 \longrightarrow J(M).$$

This must be constant, since otherwise (by pullback) it would define a nontrivial element of $\Omega^1(\mathbb{P}^1)\,(=\{0\})$. By evaluating at $t=0$, one sees the map is zero. Hence $\widetilde{AJ}$ drops to the <u>Abel–Jacobi map</u>

$$AJ : \mathrm{Pic}^0(M) \to J(M).$$

<u>Theorem 1 (Abel's Theorem)</u>: AJ is injective. (That is, a divisor is principal iff it is (a) of degree 0 and (b) in $\ker(AJ)$.)

<u>Proof</u>: There are three (probably more). The classical one is in Chap. 28–29 of my AG book on-line; it involves cutting open curves and integrating on the boundary. Here are the other two.

(A) By Riemann–Roch, for any $p, q \in M$

$$i(-[p]-[q]) = g-(-2)-1+\cancel{\ell(-[p]-[q])}^{\,0} = g+1 > g$$

$\Rightarrow \exists\ \eta \in \mathcal{L}(-[p]-[q]) \setminus \Omega^1(M)$; by Residue theorem $0 = \mathrm{Res}_p\,\eta + \mathrm{Res}_q\,\eta$.

Hence, for any $\underset{\sum n_p[p]}{\underset{\|}{D}} \in \mathrm{Div}^0(M)$, $\exists$ mero. 1-form $\hat{\omega}$ with simple poles and $\underline{\mathrm{Res}_p(\hat{\omega}) = n_p\ (\forall p)}$.

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* by the holomorphic implicit function theorem + the Riemann extension theorem

The 1-current $\kappa := \hat{\omega} - 2\pi i\,\delta_\Gamma$ is closed. Moreover, if $\dot{AJ}(D) = 0$ then Γ may be chosen so that $\int_\Gamma \omega_i = 0$ $(\forall i)$
$\Rightarrow \int \kappa \wedge \omega_i = 0$ $(\forall i)$. Smoothing κ in its cohomology class, to $\omega = \kappa - d\eta \in \Omega^1(M)$, we have
↑distribution

$$f := e^{\int \hat{\omega} - \omega}$$

(meromorphic but possibly multivalued)

$$= e^{2\pi i \int \delta_\Gamma}\, e^{\eta}$$

(single-valued but possibly discontinuous)

jumps up by integer upon crossing Γ

$e^{2\pi i \mathbb{Z}} = 1$

So $f \in \mathcal{M}^*(M)$; in particular at $p \in M$ (loc. coord. z),

$$f = \underset{\neq 0}{\text{holo.}} \times \underbrace{e^{\int \frac{n_p}{z} dz}}_{z^{n_p}}$$

has the right $\nu_p(f)$ $\Longrightarrow$ $(f) = D$. //

(B) Use the two exact sequences $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$ (sheaf of divisors)

$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$

to produce a diagram w/ exact rows & column

takes $D \mapsto L_D$ (line bundle); by the last result in Appendix A, this is surjective

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathbb{C}^* \arrow[d] & & \\
 & & \mathcal{M}^*(M) \arrow[d, "(\cdot)"] \arrow[dl, dashed] & & \\
0 \arrow[r] & \mathrm{Div}^0(M) \arrow[r] \arrow[d, dotted, "AJ"] & \mathrm{Div}(M) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J(M) = \dfrac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \arrow[r] & H^1(M,\mathcal{O}^*) \arrow[r] \arrow[d] & H^2(M,\mathbb{Z}) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

Given $D \in \mathrm{Div}^0(M)$ with $AJ(D) = 0$, it clearly must come from $\mathcal{M}^*(M)$.

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$\square$

<u>Theorem 2 (Jacobi inversion)</u>: AJ is surjective. More precisely, fixing $q_1, \dots, q_g \in M$, the morphism

$$\mathrm{Sym}^g M \to J(M)$$

induced by $$p_1 + \cdots + p_g \longmapsto \int_{\underbrace{\partial^{-1}}_{\text{take a 1-chain bounding on}}(\sum_i p_i - q_i)} (\cdot)$$

is surjective & generically injective (i.e. birational).

<u>Sketch of Pf.</u>:

<u>Step 1</u>: Let D = divisor of degree d on M

$$\alpha_D : \mathbb{P}(\mathcal{L}(D)) \to \mathrm{Sym}^d M$$
$$[f] \longmapsto (f) + D \quad (\geq 0, \text{ of deg. } d, \text{ hence consists of sum of } d \text{ points})$$

write $|D|$ = set of all <u>effective</u> divisors differing from D by a principal divisor (i.e. "rationally equivalent to D")

Then we have

$$|D| = \mathrm{image}(\alpha_D) \cong \mathbb{P}^{\ell(D)-1} \subset \mathrm{Sym}^d(M),$$

since $\{(f) + D = (g) + D \Rightarrow (f/g) = 0 \Rightarrow f/g \text{ const.} \Rightarrow [f] = [g]\}$
$\Rightarrow \alpha_D$ injective (& holomorphic*).

<u>Step 2</u>: Let $u_d : \mathrm{Sym}^d M \to J(M)$ [also holo.]
$$p_1 + \cdots + p_d \longmapsto AJ\left(\sum [p_j] - d[q]\right) \quad (q \in M \text{ fixed})$$

Then

$$u_d^{-1}(u_d(D)) = |D|,$$

since $u_d(E) = u_d(D) \Rightarrow AJ(E-D) = 0 \xRightarrow{\text{Abel}} E - D = (f) \Rightarrow$
$(f) + D = E \geq 0 \Rightarrow f \in \mathcal{L}(D) \Rightarrow E = \alpha_D(f) \ (\in \mathrm{im}\, \alpha_D)$.

Ex/ prove the $\supset$ inclusion.

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* this requires putting an explicit $\mathbb{C}$-analytic structure on $\mathrm{Sym}^d M$, then using holo. IFT / Rouché / Riemann.

Step 3: For $D = [p_1] + \dots + [p_g]$ general $(d = g)$,

$$(du_g)_D : T_{\mathrm{Sym}^g M, D} \longrightarrow T_{J(M), u_g(D)}$$

has (maximal) rank g — hence surjective + injective.

This follows from linear independence of $\omega_1, \dots, \omega_g$. (Ex/)

Step 4: By $\overset{\text{surj. of } (du_g)_{D_0}}{\text{Step 3}}$, if B is a ball about a general $D_0 \in \mathrm{Sym}^g M$, then $u_g(B)$ is open. Since $u_g(\mathrm{Sym}^g M)$ is closed, u_g is onto.

Step 5: Also by $\underset{\text{inj. of } (du_g)_{D_0}}{\text{Step 3}}$, D_0 isolated in $u_g^{-1}(u_g(D))$. But

$$u_g^{-1}(u_g(D)) \underset{\text{Step 2}}{=\!=} |D_0| \underset{\text{Step 1}}{\cong} \mathbb{P}^m \implies m = 0.$$

This shows generic injectivity of u_g. $\square$

The interesting point in this result is that both $\mathrm{Sym}^g M$ and $J(M)$ are proj. algebraic (the latter b/c polarized + Kodaira's thm.), and the "graph" of u_g is a closed analytic subvariety — hence, by GAGA, algebraic.