Problem Set 2 (solutions)

1) • $\bar\partial \omega, \bar\partial \eta = 0 \Rightarrow \bar\partial(\omega \wedge \eta) = \bar\partial\omega \wedge \eta \pm \omega \wedge \bar\partial \eta = 0$

• $\left.\begin{array}{l} \omega' = \omega + \bar\partial\alpha \\ \eta' = \eta + \bar\partial\beta \end{array}\right\} \Rightarrow \omega' \wedge \eta' = \omega\wedge\eta + \omega \wedge \bar\partial\beta + \bar\partial\alpha \wedge \eta'$

$$= \omega \wedge \eta + \bar\partial\{\alpha\wedge\eta' \pm \omega \wedge \beta\}.$$

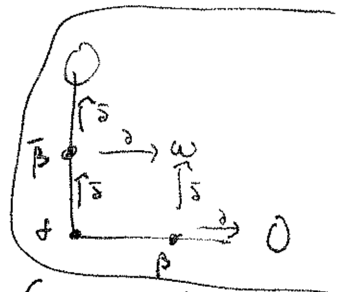
2) Since $\boxed{d\omega = 0}$, d-Poincaré $\Rightarrow \exists_{p \in} U' \subset M$ where $\omega = d\alpha$ (for some 1-form α).

$\boxed{\omega = \bar\omega} \Longrightarrow$ we may assume $\alpha = \bar\alpha$, i.e. $\alpha = \beta + \bar\beta$ where β is of type $(1,0)$.

So we can write $\omega = d\alpha = \partial\beta + \bar\partial\beta + \partial\bar\beta + \bar\partial\bar\beta$,

but since $\boxed{\omega \text{ is of type } (1,1)}$,

$\omega = \bar\partial\beta + \partial\bar\beta$ and $\partial\beta = 0 = \underbrace{\bar\partial\bar\beta}$.



By $\bar\partial$-Poincaré, $\exists_{p \in} U \subset U'$ where $\bar\beta = \bar\partial f$ for some ($\mathbb{C}$-valued) C^∞ fcn. f. But then,

$$\omega = \bar\partial \underbrace{\partial \bar f}_{\beta} + \partial \underbrace{\bar\partial f}_{\bar\beta} = -\partial\bar\partial\bar f + \partial\bar\partial f = \partial\bar\partial(f - \bar f)$$

$$= 2i \cdot \partial\bar\partial(\operatorname{Im} f).$$

3)(a) Let $\{\eta_\alpha\}$ be a partition of unity subordinate to some open cover $\mathcal{U}_\alpha$ and $\{(x_1^\alpha, \ldots, x_m^\alpha)\}$ systems of coordinates s.t. the $(\phi_{\alpha\beta})_*$'s (the Jacobians of transition fcns) have determinants > 0.

Then set $\omega := \sum_\alpha \eta_\alpha \, dx_1^\alpha \wedge \cdots \wedge dx_m^\alpha$, and notice that (for any β)

$$\omega|_{\mathcal{U}_\beta} = \sum_{\substack{\alpha(\neq\beta)\\ \mathcal{U}_\alpha\cap\mathcal{U}_\beta\neq\emptyset}} \eta_\alpha\, \phi_{\alpha\beta}^{*}\,(dx_1^\alpha\wedge\cdots\wedge dx_m^\alpha) \quad + \eta_\beta\, dx_1^\beta\wedge\cdots\wedge dx_m^\beta$$

$$= \underbrace{\left(\sum_{\substack{\alpha(\neq\beta)\\ \mathcal{U}_\alpha\cap\mathcal{U}_\beta\neq\emptyset}} \eta_\alpha \det \overbrace{\left((\phi_{\alpha\beta})_*\right)}^{>0 \text{ where defined}} \quad + \eta_\beta \right)} \; dx_1^\beta\wedge\cdots\wedge dx_m^\beta$$

and clearly this >0. It follows that $\int_M \omega > 0$.

Now $d\omega = 0$ since ω is of top degree; and we cannot have $\omega = d\eta$ since then $0 < \int_M \omega = \int_M d\eta \overset{\text{Stokes}}{=} \int_{\underbrace{\partial M}_{\emptyset}} \eta = 0$.

write $m=2n$

(b) We can choose C^∞ coordinates $\{x_i^\alpha\}$ about any point $p\in U_\alpha$ s.t. $J_\alpha(\partial/\partial x^\alpha_{i\le n}) = \partial/\partial x^\alpha_{i+n}$ at p, but not on the whole neighborhood. However, replacing the bases with respect to which we compute $\det((\phi_{\alpha\beta})_*)$ by a coframe $\{\theta_i^\alpha\}$ (equal to $\{\partial/\partial x_i^\alpha\}$ at p, but not necessarily integrable) does not change the signs of the determinants. Then for every α we have simply $[J_\alpha]_{\theta^\alpha} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, and the compatibility conditions

$$J_\alpha \circ (\phi_{\alpha\beta})_* = (\phi_{\alpha\beta})_* \circ J_\beta$$

these ensure the $\{J_\alpha\}$ piece together to give a global section J of $\mathrm{End}(T_M)$

read "$M_{\alpha\beta} := {}_{\theta^\alpha}[(\phi_{\alpha\beta})_*]_{\theta^\beta}$ commutes with $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$"

$\Rightarrow M_{\alpha\beta}$ is of the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \Rightarrow \det M_{\alpha\beta} = |\det(A+Bi)|^2 > 0$.

4) (a) antisymmetry is obvious and for $f\in C^\infty(M)$,

$$N(fX,Y) = f N(X,Y) \cancel{-(JY)(f).JX} + \cancel{Y(f).X} + \cancel{Y(f).\underbrace{JJX}_{-X}} + \cancel{(JY)(f).JX}$$

(using e.g. $[fX,Y] = f[X,Y] - Y(f).X$)

$$= f N(X,Y)$$

which demonstrates (on one side, hence the other) $C^\infty(M)$-linearity.

(b) Recall that on $T^{1,0}$ resp. $T^{0,1}$, J is mult. by i resp. $(-i)$.
So if $X \in T^{1,0}$, $Y \in T^{0,1}$, $N(X,Y) = \cancel{[iX,-iY] - [X,Y]} - \cancel{J[iX,Y] - J[X,-iY]}$
$= 0$.

Now if $X, Y \in T^{1,0}$ then

$$0 = N(X,Y) = [iX,iY] - [X,Y] - J[iX,Y] - J[X,iY]$$
$$= -2[X,Y] - 2iJ[X,Y]$$

$$\iff i[X,Y] = J[X,Y] \iff [X,Y] \in T^{1,0}.$$

The same thing holds for $T^{0,1}$. We conclude that

$$N = 0 \iff [T^{1,0}, T^{1,0}] \subset T^{1,0} \iff J \text{ is integrable.}$$
(and its "conjugate"). (Newlander--Nirenberg)

(quasi-sketch)

5 By the $C^\infty(M)$-bilinearity of N ($M = S^2$ or S^6) it turns out that we can compute it (i.e. N) for J on the space of imaginary quaternions/octonions and then evaluate on a pair of vectors in TS^2 resp. TS^6 at a point. This is much easier than trying to work on the sphere. Goal is simply to show that, for S^6, there is a point $p \in S^6$ & a pair of vectors $\in T_pS^6$ on which N gives a nonzero answer. Alas, I haven't had time to sit down with a table of the octonions...