Problem Set 4 (Solutions)

already proved

(1)
$$\begin{tikzcd}
0 \arrow[r] & \alpha \arrow[r, "j_1"] \arrow[d, "d"'] & \beta \arrow[r, "j_2"] \arrow[d, "e"'] & \gamma \arrow[r, "j_3"] \arrow[d, "f"'] & \delta \arrow[r, "j_4"] \arrow[d, "g"'] & \epsilon \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i_1"] & B \arrow[r, "i_2"] & C \arrow[r, "i_3"] & D \arrow[r, "i_4"] & E \arrow[r] & 0
\end{tikzcd}$$

Given $c \in C$, $j_4(g^{-1}(i_3(c))) = i_4(i_3(c)) = 0 \Rightarrow$

$\exists\ \tilde{c} \in \gamma$ s.t. $j_3(\tilde{c}) = g^{-1}(i_3(c)) \Rightarrow$

$i_3(f(\tilde{c})) = g(j_3(\tilde{c})) = i_3(c) \Rightarrow i_3(f(\tilde{c}) - c) = 0 \Rightarrow$

$\exists\ b \in B$ s.t. $i_2(b) = f(\tilde{c}) - c$ and $\tilde{b}\ (\in \beta)$ s.t. $e(\tilde{b}) = b \Rightarrow$

$$f(\tilde{c} - j_2(\tilde{b})) = f(\tilde{c}) - f(j_2(\tilde{b})) = f(\tilde{c}) - i_2(\underbrace{e(\tilde{b})}_{b})$$
$$= f(\tilde{c}) - (f(\tilde{c}) - c)$$
$$= c.$$

(2) (a) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact

i.e. $\alpha \circ (\cdot)$ i.e. $\beta \circ (\cdot)$

$$0 \to \mathrm{Hom}(X, A) \xrightarrow{\alpha_*} \mathrm{Hom}(X, B) \xrightarrow{\beta_*} \mathrm{Hom}(X, C)$$

exactness here easy

To prove exactness here, main point is: given $f \in \mathrm{Hom}(X, B)$ with $\beta \circ f = 0$, define $g \in \mathrm{Hom}(X, A)$ by $g(x) :=$ $\underbrace{\text{unique}}_{\text{by injectivity of } \alpha}$ element of A mapping by α to $f(x)$ ($\exists$ since $\beta(f(x)) = 0$).

(b) $0 \to \mathbb{Z}_m \xrightarrow[(\cdot)/m]{} \mathbb{Q}/\mathbb{Z} \xrightarrow[\cdot m]{} \mathbb{Q}/\mathbb{Z} \to 0$

(c) $$\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n) = \left(R^1 \mathrm{Hom}(\mathbb{Z}_m, -)\right)(\mathbb{Z}_n)$$

$$= H^1\left\{ \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cdot n} \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z}) \right\}$$

$$= \frac{\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z})}{n \cdot \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Q}/\mathbb{Z})} \begin{array}{l} \leftarrow \text{identifies with } \left\{ \frac{a}{m} \mid a \in [0, m-1] \right\} \\ \leftarrow \text{identifies with } \left\{ \frac{na}{m} \mid a \in [0, m-1] \right\} \end{array}$$

via looking at where $\bar{1} \in \mathbb{Z}_m$ is sent

$$= \frac{\mathbb{Z}_m}{n\mathbb{Z}_m} \cong \mathbb{Z}_{(m,n)}.$$

(3)

- $\mathcal{D}$ flasque b/c sections $/U$ are divisors with support on U. These are also divisors w/support on any open set $W \supset U$, so $\mathcal{D}(W) \twoheadrightarrow \mathcal{D}(U)$. Note that this is only true for $\dim M = 1$
- we have $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$ (shvs $/M$)

  so $0 \to H^0(\mathcal{O}^*) \to H^0(\mathcal{M}^*) \to H^0(\mathcal{D}) \to H^1(\mathcal{O}^*) \to H^1(\mathcal{M}^*) \to \underbrace{H^1(\mathcal{D})}_{\substack{0 \text{ since} \\ \mathcal{D} \text{ flasque}}}$

  $$= 0 \to \mathbb{C} \to \mathcal{M}(M)^* \to \mathrm{Div}(M) \xrightarrow{\delta} \underset{\cong}{\underline{\text{holo. line bundles}}} \xrightarrow{\beta} ?? \to 0$$

  and $\underbrace{\delta \text{ surjective}}_{\text{what we want}} \Leftrightarrow \beta = 0 \Leftrightarrow H^1(\mathcal{M}^*) = \{0\}$

- need a little argument to show that δ sends a divisor D to $\mathcal{O}(D)$. Done in Čech cohomology, this is almost a tautology (left to you).

(4) Identify $\Omega(D) = H^0(\Omega^1(-D))$.

(a) Let $\omega \in K^1(M)$. If $\omega \in \Omega(D)\setminus\{0\}$ then

$(\omega) \geq D$, which yields

$$2g-2 \underset{P-H}{=} \deg((\omega)) \geq \deg D.$$

If $\deg D > 2g-2$, this is impossible & so there are no (nonzero) meromorphic forms in $\Omega(D)$.

(b) Likewise, if $f \in \mathcal{L}(D)\setminus\{0\}$ then $(f)+D \geq 0$

$$\Longrightarrow \quad 0 = \deg((f)) \geq -\deg D$$

So if $\deg D < 0$ (i.e. $-\deg D > 0$), we again get a contradiction.

(5) (a) $p \in M$

$$\ell((g+1)[p]) = i((g+1)[p]) + \overbrace{(g+1)}^{\deg D} - g+1 \geq 2$$

$\Rightarrow \mathcal{L}((g+1)[p])$ contains a <u>nonconstant</u> fcn. f.

The mapping degree is the "cardinality w/multiplicity" of $f^{-1}(\alpha)$ for any $\alpha \in \mathbb{P}^1$. For $\alpha = 0$ or ∞ that means counting <u>order</u> of 0's or poles.

mapping degree, <u>not</u> degree of divisor $\deg((f))$

Now f has only pole at p, so

$$\deg(f) = \underbrace{|\mathrm{ord}_p(f)|}_{\text{or } \deg(\underbrace{f^{-1}([\infty])}_{\text{pullback divisor}}))} \leq g+1.$$

(b) By Riemann–Roch, for any $p \in M$,

$$i((g-2)[p]) = g-(g-2)-1 + \underbrace{\ell((g-2)[p])}_{\geq 1 \text{ (contains constant fcns.)}}$$

$$\geq 2.$$

Hence, there are 2 linearly independent holomorphic forms ω_1, ω_2 with $(\omega_j) - (g-2)[p] =: D_j \geq 0$ $(j=1,2)$, and $\deg D_j \underset{p\text{-H}}{=} (2g-2) - (g-2) = g$. Set $f := \frac{\omega_1}{\omega_2} \in \mathcal{M}(M)^*$, so that $(f) = D_1 - D_2 = D_1' - D_2'$ where D_1' & D_2' are effective divisors of degrees $\leq g$ with no points in common.

We have that the mapping degree

$$\deg(f) = \deg(f^{-1}(\{0\})) = \deg D_1' \leq g.$$