Néron models and limits of Abel-Jacobi mappings

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Abstract

We show that the limit of a 1-parameter admissible normal function with no singularities lies in a non-classical sub-object of the limiting intermediate Jacobian. Using this, we construct a Hausdorff slit-analytic space, with complex Lie group fibres, which "graphs" such normal functions. For singular normal functions, an extension of the sub-object by a finite group leads to the Néron models of the title.

When the normal function comes from geometry, i.e. a family of algebraic cycles on a semistably degenerating family of varieties, its limit may be interpreted via the AJ map on motivic cohomology of the singular fiber, hence via regulators on K-groups of its substrata. Two examples are worked out in detail, for families of 1-cycles on CY and abelian 3-folds, where this produces interesting arithmetic constraints on such limits. We also show how to compute the finite "singularity group" in the geometric setting.

I. Introduction

There are two main objectives in this paper. For the first, we denote by $S = \{s \in \mathbb{C} : |s| < 1\}$ the disc with origin $s_0 = \{0\}$ and by $S^* = S\backslash\{s_0\}$ the punctured disc. Let $\mathcal{H}$ be a polarized variation of Hodge structure (VHS) over the punctured disc S^* with torsion free integral structure $\mathcal{H}_{\mathbb{Z}}$, of odd weight $2n-1$ and unipotent monodromy T. We want to construct over the whole disc S a slit analytic fibre space of connected, abelian complex Lie groups

$$\mathbf{J}_e \to S$$

that (i) fills in over the origin the family of intermediate Jacobians associated to the VHS, (ii) is a Hausdorff space and (iii) which graphs *normal functions*. Additionally, we want to construct a Néron model $\tilde{\mathbf{J}}_e$ which fits in an exact sequence

$$0 \to \mathbf{J}_e \to \tilde{\mathbf{J}}_e \to G \to 0$$

where (i) G is a finite abelian group constructed from T and sitting over the origin and (ii) $\tilde{\mathbf{J}}_e$ graphs *admissible normal functions* (ANF's). The various terminologies will be explained in the text. Here we remark that usual or "classical" normal functions, as defined for example in [EZ] and [Z2], are admissible normal functions, but not conversely. Admissable normal functions were introduced in [Sa].

Secondly, if the VHS arises from a family $\{X_s\}_{s\in S}$ of varieties of dimension $2n-1$ with X_s smooth for $s \neq s_0$, and if $Z_s \in Z^n(X_s)$ is a family of algebraic cycles with $Z_s \equiv_{\text{hom}} 0$ for $s \neq s_0$, then the Abel-Jacobi images

$$\mathrm{AJ}_{X_s}(Z_s) \in J(X_s)\ , \qquad s \neq s_0$$

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will be proved to give an ANF and therefore a section of $\tilde{\mathbf{J}}_e \to S$. In case the singular fibre X_{s_0} is a reduced normal crossing variety and the cohomology class of Z_{s_0} is zero, one may define the generalized Jacobian $J(X_{s_0})$ and the Abel-Jacobi image $\mathrm{AJ}_{X_{s_0}}(Z_{s_0}) \in J(X_{s_0})$. Moreover, (i) the $\mathrm{AJ}_{X_s}(Z_s)$ for $s \in S$ define a normal function and therefore a section of $\mathbf{J}_e \to S$, (ii) there is a natural map

$$J(X_{s_0}) \xrightarrow{\mathrm{AJ}(\rho)} J_{e,s_0}$$

from the generalized intermediate Jacobian $J(X_{s_0})$ to the fibre of $\mathbf{J}_e$ over the origin, and (iii) in a manner to be made precise

$$\lim_{s \to s_0} \mathrm{AJ}_{X_s}(Z_s) = \mathrm{AJ}(\rho)\left(\mathrm{AJ}_{X_{s_0}}(Z_{s_0})\right) .$$

Here, $\mathrm{AJ}(\rho)$ is induced from a map ρ in the Clemens-Schmid exact sequence in a manner to be explained below.

In more detail, in the classical case we let

$$\mathcal{X} \xrightarrow{\pi} S \tag{I.1}$$

be a family of algebraic curves $X_s = \pi^{-1}(s)$ over the disc. We assume that the curves X_s are smooth for $s \neq s_0$ and let $J_s = J(X_s)$ be the Jacobian variety of X_s. Then as above letting $S^* = S \backslash \{s_0\}$ denote the punctured disc, the union

$$\bigcup_{s \in S^*} J_s =: \mathbf{J} \to S^* \tag{I.2}$$

forms naturally an analytic fibre space of principally polarized abelian varieties (PPAV's) over S^*. The classical Néron model [BLR] gives an extension of (I.2) to an analytic fibre space of abelian complex Lie groups

$$\tilde{\mathbf{J}}_e \xrightarrow{p} S \tag{I.3}$$

with the following property: Let ν be an admissible normal function, which is given by a holomorphic section of (I.2) that satisfies certain conditions, to be specified below, as $s \to s_0$. If $\mathcal{Z} \in Z^1(\mathcal{X})$ is a codimension-1 algebraic cycle such that a general intersection $Z_s = \mathcal{Z} \cdot X_s$ has degree zero, then

$$\nu_{\mathcal{Z}}(s) = \mathrm{AJ}_{X_s}(Z_s) \in J(X_s) \tag{I.4}$$

gives an ANF. Then a property of Néron models is

An admissible normal function extends to a holomorphic section of (I.3). (I.5)

We will informally phrase this as saying that *the Néron model graphs admissible normal functions.*

Remark. In the earlier works [GG1] and [GG2] of the first two authors the term *extended normal functions* was used. However, since as pointed out to us by Greg Pearlstein, exended normal functions correspond to admissible variations of mixed Hodge structure in the sense of M. Saito, we have changed to that terminology. The admissibility of normal functions arising from geometry follows from the work of M. Saito [Sa]. For discussion of admissible variations of mixed Hodge structure see [Pe2] and the original reference [SZ].

Remark on notations. Throughout this paper the subscript "e" means that an object has been extended from S^* to S. The family of identity components of the fibres of the extension of (I.3) to the general case will be denoted by

$$\mathbf{J}_e \to S .$$

There is then an exact sequence

$$0 \to \mathbf{J}_e \to \tilde{\mathbf{J}}_e \to G \to 0$$

where

$$G \cong \tilde{J}_{e,s_0}/J_{e,s_0}$$

is the group of components, which will be shown to be finite, of the fibre of the Néron model over the origin.

As indicated above, one objective of this paper is to extend this story to the case when (I.1) is replaced by a family of projective varieties of complex dimension $2n-1$ and, for $s \neq s_0$, $J_s = J(X_s)$ is the n^{th} intermediate Jacobian of X_s. We remark that if for $s \neq s_0$ we denote by $Z^n(X_s)_{\text{hom}}$ the codimension-n algebraic cycles on X_s whose fundamental class is zero, the Abel-Jacobi map will be denoted by

$$\mathrm{AJ}_{X_s} : Z^n(X_s)_{\text{hom}} \to J(X_s) \ .$$

As in the $n = 1$ case, it is a result that an algebraic cycle $\mathcal{Z} \in Z^n(\mathcal{X})$ such that $Z_s \in Z^n(X_s)_{\text{hom}}$ for $s \neq s_0$ gives an admissible normal function by the same formula (I.4).

In constructing the Néron model we shall make the (perhaps inessential) assumption that the monodromy is unipotent.

Returning to the general discussion, in the classical case the Néron model may be constructed for a general family of PPAV's, not just those that arise as Jacobians of curves. More generally, given over the punctured disc S^* any principally polarized variation of Hodge structure $(\mathcal{H}, \mathcal{F}^p, \mathcal{H}_{\mathbb{Z}}, \nabla)$ of weight $2n-1$ and with unipotent monodromy, there is a corresponding family of compact complex tori

$$J_s = F^n_s \backslash H_s/(\mathcal{H}_{\mathbb{Z}})_s \ ,$$

and we shall construct a corresponding Néron model (I.3). There are a number of noteworthy features of the construction:

(i) The group G may be identified in terms of the monodromy transformation T as

$$G \cong \frac{\mathrm{Im}(T_{\mathbb{Q}} - I)_{\mathbb{Z}}}{\mathrm{Im}(T_{\mathbb{Z}} - I)} \ . \tag{I.6}$$

We note that

$$G \cong H^1(S^*, \mathcal{H}_{\mathbb{Z}})_{\text{tor}} \ ;$$

a similar formula is stated for the example in [Sa, (2.5.4)], where G is denoted by $C_0(\mathbf{H})$. Using the polarization we have the equivalent expression

$$G \cong \frac{(\mathrm{Ker}(T_{\mathbb{Q}} - I)^{\perp})_{\mathbb{Z}}}{\mathrm{Im}(T_{\mathbb{Z}} - I)}$$

where the outer subscript $\mathbb{Z}$ means both in (I.6) and above that we take the integral classes in $V_{\mathbb{Q}} = Z_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ for any lattice $V_{\mathbb{Z}} \cong \mathbb{Z}^r$. This formulation is useful in computing examples. It is also used in the description of G in Theorem (III.C.3).

To explain the next point, we recall that there is a canonical extension (cf. [De]) $(\mathcal{H}_e, \mathcal{F}^p_e, \mathcal{H}_{\mathbb{Z},e}, \nabla)$ of the VHS, where denoting by $j : S^* \hookrightarrow S$ the inclusion we have $\mathcal{H}_{\mathbb{Z},e} = j_*(\mathcal{H}_{\mathbb{Z}})$, and the Gauss-Manin connection satisfies

$$\nabla : \mathcal{F}^p_e \to \mathcal{F}^{p-1}_e \otimes \Omega^1_S(\log\{s_0\}) \ .$$

We denote by $\mathbf{H}_e$ and $\mathbf{F}^p_e$ the extended Hodge bundles, and we define the sheaves

$$\begin{cases} \mathcal{J} = \mathcal{F}^n \backslash \mathcal{H}/\mathcal{H}_{\mathbb{Z}} & \text{(over } S^*) \\ \hat{\mathcal{J}}_e = \mathcal{F}^n_e \backslash \mathcal{H}_e/\mathcal{H}_{\mathbb{Z},e} & \text{(over } S) \end{cases}$$

The reason for the $\hat{}$ will appear below. Finally, we set

$$\hat{J}_{e,s_0} = F^n_{e,s_0} \backslash H_{e,s_0} / (\mathcal{H}_{\mathbb{Z}})_{s_0} \ ;$$

we may informally think of $\hat{J}_{e,s_0}$ as the "fibre" of $\hat{\mathcal{J}}_e$ at s_0. A natural candidate for $\mathbf{J}_e \to S$ would be to set

$$\hat{\mathbf{J}}_e = \mathbf{J} \cup \hat{J}_{e,s_0} \ ; \tag{I.7}$$

i.e., we simply insert $\hat{J}_{e,s_0}$ over the origin. For example

- this is what is suggested by the extension theorems of Zucker [Z1], [Z2] and El Zein-Zucker [EZ];
- in the classical case $n = 1$, or more generally for any n and with the assumption (which is automatic when $n = 1$)
$$(T - I)^2 = 0 \tag{I.8}$$
plus a technical assumption (cf. Propostion II.A.8 below) on the limiting mixed Hodge structure (LMHS), the construction (I.7) is the correct one (cf. Clemens [Cl2] and M. Saito [Sa]).

However, our second point is

(ii) In general, (I.7) is the *wrong* object.

Rather, denoting as usual by $N = \log T$ the logarithm of T, it turns out that we should set

$$J_{e,s_0} = F^n \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}} / (\mathcal{H}_{\mathbb{Z},e})_{s_0} \tag{I.9}$$

and

$$\mathbf{J}_e = \mathbf{J} \cup J_{e,s_0} \to S \ . \tag{I.10}$$

A more suggestive notation might be $J(\operatorname{Ker} N)$, meaning that we construct the intermediate Jacobian using only the $\operatorname{Ker} N$ part of H_{e,s_0}.

We note that $\mathbf{J}_e$ is a subset of the "wrong object" $\hat{\mathbf{J}}_e$, and that unless (I.8) is satisfied

$$\dim J_{e,s_0} < \dim J_s \ , \qquad s \neq s_0$$

so that (I.10) is *not* an analytic fibre space of complex Lie groups in the usual sense, but rather is what we shall call a *slit analytic fibre space of complex Lie groups.* The concept of a slit analytic space first appeared in the fundamental work [KU]. There the reason was also to obtain a Hausdorff quotient. The space $\mathbf{J}_e$ will be seen to have the properties:

- It is a Hausdorff topological space. By contrast, we shall see that, with the natural topology, $\hat{\mathbf{J}}_e$ is not separated unless (I.8) is satisfied (cf. [Sa]);
- The fibres of $\mathbf{J}_e \to S$ are connected abelian complex Lie groups, composed of compact, complex tori and algebraic tori — no vector groups;
- One may "do geometry" on $\mathbf{J}_e \to S$.

This point will be elaborated on in a future work; cf. [KU] where "doing geometry" on a slit analytic space is discussed in their context.

Remark. The referee observes that, after adjusting the VHS to have weight -1,

$$J_{e,s_0} \cong \operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), K) \tag{I.11}$$

where K is the induced limit mixed Hodge structure on the kernel of N. From a formal point of view this description is preferable. For the computation of examples in our setting, taking the classical description

$$\nu_Z(s) \in J_{e,s} = \check{F}^n_{e,s} / \mathcal{H}_{\mathbb{Z},e,s} \qquad (s \neq s_0)$$

and analyzing the limit as $s \to s_0$ turns out for us to be more convenient.

(iii) To get a feeling for this, we let ν be an admissible normal function. Then we shall show that for some integer m the value

$$m\nu(s_0) \in J_{e,s_0}$$

is defined. *This puts a constraint, not visible in the classical case (I.8), on the values of an admissible normal function at the singular point.* In particular, we may define the subsheaf

$$\mathcal{J}_e \subset \hat{\mathcal{J}}_e$$

to be given by $\mathcal{O}_S(\mathbf{J}_e)$, and the results of Zucker and El Zein-Zucker may be refined to give an extension of a normal function satisfying their assumptions to the smaller sheaf $\mathcal{J}_e$.

Remark. The result that for a normal function ν the value

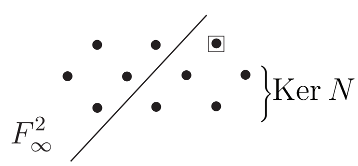
$$\nu(s_0) \in J_{e,s_0} \subset \hat{J}_{e,s_0}$$

has been found independently by Greg Pearlstein. Indeed, our proof is a direct application of his extension of Schmid's nilpotent orbit theorem to admissible variations of mixed Hodge structure [Pe1]. It is to be expected that this work, together with the extension of Schmid's SL_2-orbit theorem to AVMHS's [Pe2], will have important applications to the study of normal functions. In fact, in addition to the above result one such is the result of Brosnan-Pearlstein [BP] that will be commented on below.

The first case that illustrates this non-classical behaviour is when $n = 2$ and, using the standard notation from mixed Hodge theory,

$$\begin{cases} N^2 = 0, \quad N \neq 0 \\ \qquad I^{1,3} \neq 0\,. \end{cases}$$

One may schematically represent any LMHS as a Hodge diamond with N playing the role of the Lefschetz operator. In this case the picture is



where N is an isomorphism from the top row to the bottom row. Then one has

$$0 \to J_{e,s_0} \to \hat{J}_{e,s_0} \to I^{1,3} \to 0$$

where $I^{1,3}$ is represented by the dot in the box. The constraint is that $\nu(s_0)$ projects to zero in this term. In the geometric case this result is non-trivial and may be thought of as some sort of relative version of the local invariant cycle theorem.

Since the category of mixed Hodge structure is abelian, $\operatorname{Ker} N_{\mathbb{C}} \subset H_{e,s_0}$ inherits a weight filtration $W_\bullet$ defined over $\mathbb{Q}$ and a Hodge filtration $F^p \operatorname{Ker} N_{\mathbb{C}}$, and these then induce a filtration

$$W_{-m}(J_{e,s_0}), \qquad\qquad 0 \leqq m \leqq 2n-1$$

on the analytic group J_{e,s_0}. We will then see that

$$\mathrm{Gr}_{-m}(J_{e,s_0}) \cong F^n \backslash H^{2n-1-m}/H^{2n-1-m}_{\mathbb{Z}}$$

where $(H^{2n-1-m}, F^p, H^{2n-1-m}_{\mathbb{Z}})$ is a polarized Hodge structure of weight $2n-1-m$. Filtrations of this form were introduced by Carlson, cf. [Ca2, formulas (1.5)–(1.6)]. Thus, as indicated above, when $n = 2$

$$\begin{cases} \mathrm{Gr}_0(J_{e,s_0}) \cong F^2\backslash H^3/H^3_{\mathbb{Z}} \\ \mathrm{Gr}_{-1}(J_{e,s_0}) \cong F^2\backslash H^2/H^2_{\mathbb{Z}} \\ \mathrm{Gr}_{-2}(J_{e,s_0}) \cong H^1/H^1_{\mathbb{Z}} \\ \mathrm{Gr}_{-3}(J_{e,s_0}) \cong H^0/H^0_{\mathbb{Z}} \end{cases}$$

from which it follows that

$$\begin{cases} \mathrm{Gr}_0(J_{e,s_0}) \cong \mathbb{C}^h/\Lambda_{2h} \\ \mathrm{Gr}_{-1}(J_{e,s_0}) \cong \mathbb{C}^k/\Lambda_l, \quad k \leqslant l \leqslant 2k \\ \mathrm{Gr}_{-2}(J_{e,s_0}) \cong \mathbb{C}^a/\Lambda_a \\ \mathrm{Gr}_{-3}(J_{e,s_0}) \cong \mathbb{C}^b/\Lambda_b \end{cases}$$

where $\Lambda_i \cong \mathbb{Z}^i$ is a discrete subgroup; therefore, as previously remarked, the graded quotients are composed of extensions of compact analytic tori by algebraic tori.

Two very natural questions to ask are:

(i) When the VHS arises from a geometric family, how are H^0, H^1, H^2, H^3 related to the geometry of the singular fiber X_{s_0}?

(ii) When we have a geometric family of cycles, what is the geometric content of the part of $\lim_{s\to s_0} \mathrm{AJ}_{X_s}(Z_s)$ lying in the various pieces of J_{e,s_0}?

More specifically, we assume given a family (I.1) where now $\mathcal{X}$ is smooth and of dimension $2n$ and $X_{s_0} = \underset{i}{\cup} X_i$ is a reduced strict normal crossings divisor (SNCD; cd. III.A). We denote by $Z^n_\pi(\mathcal{X})$ the group of codimension-n algebraic cycles $\mathcal{Z}$ that are in general position relative to the fibration (I.1). In particular, we assume that

$$\mathcal{Z}\cdot X_{s_0} =: Z_{s_0} \;\; \textit{meets the strata } X_I \textit{ properly,}$$

i.e., the intersections are proper intersections as in the standard theory of algebraic cycles. This assumption will be discussed in detail in section III.A below. Here, we are using the customary notation

$$X_I = \bigcap_{i\in I} X_i$$

for the strata of X_{s_0}. For $\mathcal{Z} \in Z^n_{\#}(\mathcal{X})$ we set $\mathcal{X}^* = \pi^{-1}(S^*) = \mathcal{X}\backslash\{X_{s_0}\}$ and $\mathcal{Z}^* = \mathcal{Z}|_{\mathcal{X}^*}$ and consider the conditions

$$[\mathcal{Z}] = 0 \text{ in } H^{2n}(\mathcal{X},\mathbb{Z}); \tag{I.12.i}$$

$$[\mathcal{Z}^*] = 0 \text{ in } H^{2n}(\mathcal{X}^*,\mathbb{Z}) \tag{I.12.ii}$$

$$[Z_s] = 0 \text{ in } H^{2n}(X_s,\mathbb{Z}) \text{ for } s \neq s_0\,. \tag{I.12.iii}$$

We note that

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}\,,$$

and all implications are easily seen by example to be strict. In cases (i) and (ii) it will follow from our results that the extended normal function

$$\nu_{\mathcal{Z}}(s_0) \in J_{e,s_0}$$

and therefore takes its value in the identity component of $\tilde{J}_{e,s_0}$. These are refinements of the results of Zucker [Z1], [Z2] and El Zein-Zucker [EZ].

In case (i), we will define $\mathrm{AJ}_{X_{s_0}}(Z_{s_0}) \in J(X_{s_0})$, where by definition the (generalized) intermediate Jacobian of X_{s_0} is

$$J(X_{s_0}) = F^n H^{2n-1}(X_{s_0}) \backslash H^{2n-1}(X_{s_0}) / H^{2n-1}(X_{s_0}, \mathbb{Z}) \, .$$

We recall the Clemens-Schmid (C-S) (cf. [Mo1] and [KK]) exact sequence of MHS's with $\mathbb{Q}$-coefficients

$$\to H^{2n-1}(X_{s_0}) \xrightarrow{\rho} H^{2n-1}_{\lim}(X_s) \xrightarrow{N} H^{2n-1}_{\lim}(X_s) \xrightarrow{\sigma} H_{2n-1}(X_{s_0}) \to$$

where $H^q_{\lim}(X_s)$ is the limiting mixed Hodge structure on the cohomology of a general fibre. There is an induced map

$$\mathrm{AJ}(\rho) : J(X_{s_0}) \to J_{e,s_0} \, , \tag{I.13}$$

and one of our main results is:

In case (1.12.i) we have

$$\nu_{\mathcal{Z}}(s_0) = \mathrm{AJ}(\rho)\left(\mathrm{AJ}_{X_{s_0}}(Z_{s_0})\right) \, .$$

We shall write this as

$$\boxed{\lim_{s \to s_0} \mathrm{AJ}_{X_s}(Z_s) = \mathrm{AJ}_{X_{s_0}}(Z_{s_0}) \, .} \tag{I.14}$$

In a precise sense, in case (i) the Abel-Jacobi mappings on the smooth fibres specialize to the Abel-Jacobi mapping on the NCD over the origin.

The mixed Hodge structure on $H^{2n-1}(X_{s_0})$ induces a weight filtration on $J(X_{s_0})$ and (I.13) is a map of filtered analytic groups. Using (I.14) we will be able to interpret the induced graded pieces of $\lim_{s \to s_0} \mathrm{AJ}_{X_s}$ as follows:

There will be subquotients $A^n(l)$ of

$$\bigoplus_{|I|=l+1} \mathrm{CH}^n(X_I, l)$$

together with (Beilinson) regulator maps

$$A^n(l) \to \mathrm{Gr}_{-l}(J(X_{s_0})) \, .$$

In a manner to be made precise we will then have the

Conclusion. $\lim_{s \to s_0} \mathrm{AJ}_{X_s}$ *involves the regulator maps, defined on the higher Chow groups of the strata of* X_{s_0}.

In fact, although so far as we know it has not generally been formulated in quite this way, this result is already the case classically when X_{s_0} is a nodal curve: the $\mathbb{C}^*$'s in the generalized Jacobian of the nodal curve are the images of maps

$$\begin{array}{c} \mathrm{CH}^1(\mathrm{node}, 1) \to \mathbb{C}^* \\ \wr\| \\ \mathrm{CH}^1(\mathbb{C}, 1) \, . \end{array}$$

In section IV we turn to the analysis of examples. The first is the Fermat-pentahedron (F/P) pencil

$$s(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) - x_0 x_1 x_2 x_3 x_4 = 0 \tag{I.15}$$

which has been much studied. After applying semi-stable reduction to (I.15) we will find that

$$\mathrm{AJ}(\rho) : J(X_{s_0}) \xrightarrow{\sim} J_{e,s_0}$$

is an isomorphism. Then the graded pieces turn out to be

$$\begin{aligned}\mathrm{Gr}_0 &= \overset{10}{\oplus} J(C)(-1)\\ \mathrm{Gr}_{-1} &= 0\\ \mathrm{Gr}_{-2} &\cong \overset{40}{\oplus} \mathbb{C}^*\\ \mathrm{Gr}_{-3} &\cong \mathbb{C}^*\end{aligned}$$

where C is a Fermat quintic curve $\{x^5 + y^5 = 1\}$. The group of components of the Néron model turns out to be

$$G \cong (\mathbb{Z}/5\mathbb{Z})^3 . \tag{I.16}$$

Of particular interest are the limits

$$\lim_{s\to s_0} \mathrm{AJ}_{X_s}(L'_s - L''_s)$$

where L'_s and L''_s are lines in X_s whose limiting positions L'_{s_0}, L''_{s_0} are in different components of the pentahedron P. Denoting by [ ] the quotient map

$$\tilde{J}_{e,s_0} \xrightarrow{[\quad]} G$$

we shall prove in section IV.B that, under the above conditions,

$$\left[\lim_{s\to s_0} \mathrm{AJ}_{X_s}(L'_s - L''_s)\right] \neq 0 ,$$

and that taken over all pairs L'_s, L''_s these limits generate G.

Remark. The Abel-Jacobi map on the difference of lines on a quintic was one of the first examples studied of non-classical Abel-Jacobi maps (cf. [Gr1] and [Cl2]). It was shown by a global monodromy argument that for a generic quintic the Abel-Jacobi image of a difference of lines was non-torsion and therefore non-trivial. But (I.15) is not a generic quintic for a generic choice of s; moreover, the last relation above does not imply that the ANF $\mathrm{AJ}_{X_s}(L'_s - L''_s)$ is nontorsion. The construction in section IV.C, in contrast, *does* yield nontorsion normal functions.

We shall also construct in section IV.C a cycle $W_{s_0} \in Z^2(X_{s_0})$ in general position and with the properties

(i) W_{s_0} deforms to a cycle $W_s \in Z^2(X_s)_{\mathrm{hom}}$ for s near to s_0;

(ii) $\lim_{s\to s_0} \mathrm{AJ}_{X_s}(W_s) \in W_{-3}(J_{e,s_0})$;

(iii) this limit is the image under the map (I.13) of a map

$$\mathrm{CH}^2(\mathbb{C}, 3) \to \mathbb{C}/\mathbb{Z}(2) .$$

This illustrates in an interesting example how the higher Chow groups, beyond the classical cases, enter naturally into limits of Abel-Jacobi mappings.

Finally, the Abel-Jacobi images of the degeneration of the Collino variant of the Ceresa cycle — on the Jacobian of a genus three curve specializing to one with three independent nodes — gives another interesting example where the regulator image of a higher Chow group naturally appears (in the form of Catalan's constant). This story is worked out in section IV.D.

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Note added in proof. The paper [DM] contains results related to those presented in this paper, especially those dealing with the integral structure on monodromy of the classical mirror quintic family.

II. Construction of the Néron model and graphing of admissible normal functions

II.A Graphing normal functions

We denote by S a smooth projective variety and by $D \subset S$ a divisor with local normal crossings (LNC's) with complement $S^* = S \backslash D$. Over S^* we assume given a polarized variation of Hodge structure (VHS) $(\mathcal{H}, \mathcal{F}^p, \nabla, \mathcal{H}_{\mathbb{Z}})$ of weight $2n-1$. Here, $\mathcal{H}_{\mathbb{Z}}$ is a torsion free local system $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{C}} \mathcal{O}_{S^*}$. The transversality condition

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{S^*} \tag{II.A.1}$$

gives the usual differential constraint on how the Hodge filtration varies.

In a neighborhood U of $s_0 \in D \subset S$ where $D \cap U = \underset{i}{\cup} D_i$ has normal crossings, we assume that the local monodromies $T_i : \mathcal{H}_{\mathbb{Z},s} \to \mathcal{H}_{\mathbb{Z},s}$, $s \in U^* = U \backslash U \cap D$ are unipotent with logarithms N_i. Then it is well-known [De] that there are canonical extensions $\mathcal{H}_e, \mathcal{F}^p_e$ of the sheaves associated to the VHS such that

$$\begin{cases} \nabla : \mathcal{H}_e \to \mathcal{H}_e \otimes \Omega^1_S(\log D) \\ \mathrm{Res}_{s_i}(\nabla) = N_i \ . \end{cases}$$

We define $\mathcal{H}_{\mathbb{Z},e}$ by $\mathcal{H}_{\mathbb{Z},e} = j_*(\mathcal{H}_{\mathbb{Z}})$ where $j : S^* \hookrightarrow S$ is the inclusion. We have the natural identification

$$\mathcal{H}_{\mathbb{Z},e;s_0} = \bigcap_i \mathrm{Ker}(T_i - I)_{\mathbb{Z}} \ .$$

In this work we shall primarily be concerned with the case

$$\dim S = 1 \ . \tag{II.A.2}$$

Then there is only one local monodromy transformation T with $N = \log T$. The neighborhood U of $s_0 \in D$ may be taken to be a disc $S \cong \Delta$, and we shall denote by s both a point in U and a coordinate on S with $s_0 = \{0\}$. All the constructions described below will be canonical up to scaling of s.

Unless mentioned to the contrary we shall make the assumption (II.A.2).

For $s \in S^*$ we denote by $J_s = F^n_s \backslash H_s / \mathcal{H}_{\mathbb{Z},s}$ the compact, complex torus that, in the geometric case, will be the intermediate Jacobian of the smooth fibre X_s. Then $\mathbf{J} = \bigcup_{s \in S^*} J_s$ has naturally the structure of a complex manifold such that

$$\mathbf{J} \to S^* \tag{II.A.3}$$

is an analytic fibre space of complex Lie groups. We denote by

$$\mathcal{J} = \mathcal{F}^n \backslash \mathcal{H} / \mathcal{H}_{\mathbb{Z}} \cong \mathcal{O}_{S^*}(\mathbf{J})$$

the sheaf of holomorphic sections of (II.A.3). We may locally lift a section ν of $\mathcal{J}$ to a section $\tilde{\nu}$ of $\mathcal{H}$, and then by (II.A.1) the condition

$$\nabla \tilde{\nu} \in \mathcal{F}^{n-1} \otimes \Omega^1_{S^*} \tag{II.A.4}$$

is well-defined. We denote by $\mathcal{J}_\nabla \subset \mathcal{J}$ the subsheaf of $\mathcal{J}$ defined by the transversality condition (II.A.4).

Using the canonical extensions defined above we set $\hat{\mathcal{J}}_e = \mathcal{F}^n_e \backslash \mathcal{H}_e / \mathcal{H}_{\mathbb{Z},e}$, and then define

$$\hat{\mathcal{J}}_{e,\nabla} = \hat{\mathcal{J}}_e \cap j_*(\mathcal{J}_\nabla) \ .$$

Definition. A normal function ν is given by a section over S of $\hat{\mathfrak{J}}_{e,\nabla}$.

More precisely, over an open set $U \subset S$ normal functions are given by $\Gamma(U, \hat{\mathfrak{J}}_{e,\nabla})$. For U a neighborhood of $s_0 \in D$, assuming that it can be defined one would expect that the "value" of ν at s_0 will be given by

$$\nu(s_0) \in \hat{J}_{e,s_0} = F^n_{e,s_0} \backslash H_{e,s_0} / \mathcal{H}_{\mathbb{Z},e;s_0}\ . \tag{II.A.5}$$

We shall see below that

$$\mathcal{H}_{\mathbb{Z},e;s_0} \subset F^n_{e,s_0} \backslash H_{e,s_0}$$

is a discrete subgroup, so that the RHS of (II.A.5) is in fact a connected, abelian complex Lie group. However, if we add $\hat{J}_{e,s_0}$ to $\mathbf{J}$ to obtain

$$\hat{\mathbf{J}}_e =: \mathbf{J} \cup \left(\bigcup_{s_0 \in D} \hat{J}_{e,s_0} \right)$$

we do not obtain a complex manifold, since as will be seen below the equivalence relation on $\mathbf{F}^n_e \backslash \mathbf{H}_e$ defined by the subgroups $\mathcal{H}_{\mathbb{Z},s} \subset F^n_{e,s} \backslash H_{e,s}$, $s \in S$, is, except in the very special circumstances described below, *not* closed so that the natural induced topology on $\hat{\mathbf{J}}_e$ is non-Hausdorff (see M. Saito [Sa] or the example in §II.B below).

As a consequence of the nilpotent orbit theorem (recalled below) on the limiting Hodge structures as $s \to s_0$, we see that the sub-MHS $\operatorname{Ker} N_{\mathbb{C}} \subset H^n_{e,s_0}$ is defined independently of the choice of coordinate and that $\mathcal{H}_{\mathbb{Z},e;s_0} \subset \operatorname{Ker} N_{\mathbb{C}}$. Therefore, we may define

$$J_{e,s_0} = F^n \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}} / \mathcal{H}_{\mathbb{Z},e;s_0}\ . \tag{II.A.6}$$

We note that

$$J_{e,s_0} \subset \hat{J}_{e,s_0}\ , \tag{II.A.7}$$

and below we shall show

Proposition II.A.8. *Equality holds in (II.A.7) if, and only if, the limiting mixed Hodge structure has weight filtration*

$$W_{2n-2} \subset W_{2n-1} \subset W_{2n}$$

with

$$\mathrm{Gr}_{2n}(W_\bullet) \cong \mathrm{Gr}_{2n-2}(W_\bullet) \text{ of Tate type }.$$

The isomorphism results from the structure of the limiting MHS. To say that a Hodge structure H of even weight $2p$ is of "Tate type" means, by definition, that only $H^{p,p} \neq 0$. Equivalently, $H_{\mathbb{Q}} \cong \oplus \mathbb{Q}(-p)$. The conditions in the Proposition are realized in the geometric setting in case the degenerate X_{s_0} has only (possibly several) nodes.

We now define the set

$$\mathbf{J}_e = \bigcup_{s \in S} J_{e,s}\ ;$$

the main result of this section is the

Theorem II.A.9. *(i) For a normal function ν, the value at $s_0 \in S$ may be defined and $\nu(s_0) \in J_{e,s_0}$.*

(ii) There is a natural topology on $\mathbf{J}_e \subset \hat{\mathbf{J}}_e$ so that it is Hausdorff, and

$$\mathbf{J}_e \to S \tag{II.A.10}$$

forms a slit analytic fibre space of complex Lie groups.

Remark. This topology is modeled on the "strong topology" in [KU]. The infinite sequences constructed in §II.B will not have any convergent subsequences in this topology, although it does have subsequences which converge in the natural non-Hausdorff topology on $\hat{\mathbf{J}}_e$.

We shall explain the terminology "slit analytic fibre space" below. We note that in general $\dim J_{e,s_0} < \dim J_{e,s}$, $s \neq s_0$, so that $\mathbf{J}_e \to S$ is not a complex manifold and is not a fibre space in the usual sense. Informally, we may state the theorem as saying

The slit analytic fibre space (II.A.10) graphs normal functions.

More precisely, one may define the tangent bundle $T\mathbf{J}$ and the sub-bundle $\mathbf{H} \subset T\mathbf{J}_e$ corresponding to (II.A.4). Then normal functions are given by holomorphic sections of (II.A.10), which will be seen to be a well-defined concept, whose graphs have tangent spaces lying in $\mathbf{H}$.

We will give two proofs of the theorem. Both are in essence very similar. The first is more self-contained, deducing the result from standard results in the classical literature, together with an extension of Schmid's nilpotent orbit theorem to the situation at hand. The second, which was kindly provided by a referee, uses more recent results on admissible variations of MHS's.

The first proof will consist of two steps.

(i) The observation due to Greg Pearlstein et al. [PBNF] that a normal function gives an admissible variation of mixed Hodge structure (AVMHS), and Greg Pearlstein's extension[Pe1], [Pe2] of Schmid's nilpotent orbit and Sl_2 orbit theorems to AVMHS's, which will lead to part (i) of the theorem; and

(ii) The structure of the limiting MHS associated to a VHS, which is a consequence of Schmid's Sl_2-orbit theorem.

As noted in the introduction, part (i) of Theorem II.A.9 has been obtained independently by Pearlstein (private communication).

Remark. Before commencing the proof we note that any section of $\hat{\mathcal{J}}_e$ has a well-defined value in $\hat{J}_{e,s_0}$ at s_0 (cf. the argument following III.B.4). Theorem II.A.9 constrains this value for a *quasi-horizontal* section. In particular, recalling $\mathcal{J}_e := \mathcal{O}_S(\mathbf{J}_e)$ from §I, we have that the obvious inclusion $\mathcal{J}_{e,\nabla} \subseteq \hat{\mathcal{J}}_{e,\nabla}$ *is an equality.* We shall therefore write $\mathcal{J}_{e,\nabla}$ below for the sheaf of normal functions over Δ.

We note that an admissible normal function — which is more general than a classical normal function — also gives an admissible VMHS. This is due to M. Saito [Sa], where the definition of ANF's first appears.

Step one: We begin by recalling Schmid's nilpotent orbit theorem [Sc] associated to a polarized VHS over the punctured disc S^* with unipotent monodromy. For this we let $\mathcal{U} = \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$ be the usual upper half plane, $\mathbf{D}$ the classifying space for polarized Hodge structures of weight $2n-1$ and $\mathbf{D} \subset \check{\mathbf{D}}$ the inclusion into the dual classifying space (cf. [CMP] for a general reference). We have

$$\begin{array}{ccc} \mathbf{D} & = & G/H \\ \cap & & \cap \\ \check{\mathbf{D}} & = & G_{\mathbb{C}}/P \end{array}$$

where G is a real semi-simple Lie group with complexification $G_{\mathbb{C}}$. Upon the choice of a reference Hodge structure, the polarized VHS is given by a locally liftable holomorphic map

$$\varphi : S^* \to \{T^m\}\backslash\mathbf{D} \tag{II.A.11}$$

where $\{T^m\}$ is the subgroup of $G_{\mathbb{Z}}$ given by the powers of T, and where the differential of φ satisfies the condition corresponding to (II.A.1). Setting $s = e^{2\pi\sqrt{-1}t}$ (II.A.11) leads to a diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathbf{D} \arrow[d] \\ S^* \arrow[r, "\varphi"] & \{T^n\}\backslash\mathbf{D} \end{tikzcd}$$

where $\tilde{\varphi}(t+1) = T\tilde{\varphi}(t)$. One then may define a map $\tilde{\psi} : \mathcal{U} \to \check{\mathbf{D}}$ into the dual classifying space by $\tilde{\psi}(t) = \exp(-tN)\tilde{\varphi}(t)$. Because of $\tilde{\psi}(t+1) = \tilde{\psi}(t)$ there is an induced map $\psi : S^* \to \check{\mathbf{D}}$ and Schmid [Sc] showed that:

(a) the map ψ extends across the origin, and if we set

$$\psi(s_0) = \{F^p_{s_0}\} \in \check{\mathbf{D}}$$

then

(b) the original period mapping φ is approximated (cf. [Sc] for the precise meaning) by the nilpotent orbit

$$\mathcal{O}(s) = \big(\log s(2\pi\sqrt{-1})N\big) \cdot \{F^p_{s_0}\}\ .$$

More explicitly, for some positive ϵ and $0 < |s| < \epsilon$ we have $\tilde{\mathcal{O}}(s) \in \mathbf{D} \subset \check{\mathbf{D}}$ for any point $\tilde{\mathcal{O}}(s)$ lying over $\mathcal{O}(s)$ under the projection $\check{\mathbf{D}} \to \{T^m\}\backslash\check{\mathbf{D}}$, and rescaling to have $\epsilon = 1$ the map $\mathcal{O} : S \to \{T^m\}\backslash\mathbf{D}$ gives a VHS that approximates (II.A.11). Denoting by H_0 the vector space on which the reference Hodge structure is defined, we may write

$$\mathcal{O}(s) = \{F^p_s\} \qquad (\text{modulo}\{T^m\} \text{ acting on } H_0)$$

where $F^p_s \subset H_0$ is a multi-valued, holomorphically varying filtration. By abuse of notation, we will denote the nilpotent orbit by (F^p_s, H_s) where $F^p_s \subset H_s$. Taking the limit as $s \to s_0$ of (II.A.1) in the form $\frac{dF^p_s}{ds} \subseteq F^{p-1}_s$ gives the important fact that

$$NF^p_{s_0} \subseteq F^{p-1}_{s_0}\ . \tag{II.A.12}$$

Next, the nilpotent endomorphism $N \in \mathrm{Hom}(H_0, H_0)$ defines the monodromy weight filtration $W_\bullet(H_0)$ such that

$$\left\{ \begin{array}{c} (H_0, W_\bullet, F^p_{s_0}) \text{defines a} \\ \text{mixed Hodge structure} \end{array} \right\}$$

with the property that

$$N \text{ has type } (-1,-1)\ . \tag{II.A.13}$$

This result is a consequence of Schmid's SL_2-orbit theorem [Sc], which goes far deeper than the nilpotent orbit theorem and leads to precise conclusions about the limiting mixed Hodge structure (LMHS). The integral and rational structures involved will be discussed at the end of this section.

For the next step, we use the natural identification due to Carlson [Ca1].

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-n), H_s) \cong F^n_s\backslash H_s/\mathcal{H}_{\mathbb{Z},s} \tag{II.A.14}$$

for $s \neq s_0$ to think of $\nu(s)$ as giving a variation of mixed Hodge structure (VMHS)

$$0 \to H_s \to \tilde{H}_s \to \mathbb{Z}(-n) \to 0 \tag{II.A.15}$$

with the evident weight filtration $W_{2n-1}(\tilde{H}_s) = H_s$ and $W_{2n}(\tilde{H}_s) = \tilde{H}_s$, and with Hodge filtration $\tilde{F}^p_s \subset \tilde{H}_s$. The transversality condition (II.A.1) on the original VHS together with the differential condition (II.A.4) on ν combine to give the conditions that (II.A.15) be an admissible variation of mixed Hodge structure (AVMHS) (cf. [Pe2], [BP], and [Sa], and the references cited there).

The monodromy associated to this AVMHS is again unipotent with logarithm

$$\tilde{N} = \begin{pmatrix} N & \lambda \\ 0 & 0 \end{pmatrix}$$

where $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}(-n), H_{\mathbb{Z},s})$. Writing $\tilde{N}$ as above means we have chosen a splitting over $\mathbb{Z}$ of (II.A.15), and a different change in splitting induces the substitution

$$\lambda \to \lambda + Nu\ , \qquad u \in H_{\mathbb{Z},s}\ . \tag{II.A.16}$$

An important property (cf. [Sa] and [SZ]) of AVMHS is that, by the existence of the relative weight filtration, the exact sequence

$$0 \to H_s \to \tilde{H}_{s_0} \to \mathbb{Q}(-n) \to 0$$

splits as a $\mathbb{Q}$-vector space in a manner that makes $\lambda = 0$. Since a change of basis sends $\lambda \mapsto \lambda - Nu$, $u \in H_{\mathbb{Q},s_0}$ we have that $\lambda = Nu$ for some $u \in H_{\mathbb{Q}}$. Note that u is defined modulo $H_{\mathbb{Z},s_0}$. Now

$$\tilde{N} = \begin{pmatrix} N & Nu \\ 0 & 0 \end{pmatrix}\ .$$

We also have that $\tilde{T} = e^{\tilde{N}}$ is integral. Now

$$\tilde{T} = e^{\tilde{N}} = \begin{pmatrix} e^N & (e^N - I)u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T & (T-I)u \\ 0 & 1 \end{pmatrix}\ .$$

Thus

$$u \in \frac{\{w \in H_{\mathbb{Q},s_0} : (T-I)w \in H_{\mathbb{Z},s_0}\}}{H_{\mathbb{Z},s_0}}\ .$$

According to Pearlstein [Pe2] there is a nilpotent orbit of MHS's that approximates the above AVMHS. The constant weight filtration will be written $H_0 \subset \tilde{H}_0$ for the reference vector spaces. We will write the nilpotent orbit as

$$0 \to H_s \to \tilde{H}_s \to \mathbb{Z}(-n) \to 0 \tag{II.A.17}$$

where $\tilde{H}_s$ means the above weight filtration on $\tilde{H}_0$ together with the Hodge filtration

$$\tilde{F}_s = \exp\left(\frac{\log s}{2\pi\sqrt{-1}}\tilde{N}\right)\tilde{F}_{s_0}\ . \tag{II.A.18}$$

There is an induced nilpotent orbit on the reference vector space H_0 which agrees with the previous one provided by Schmid's theorem.

Again from Pearlstein's work, the nilpotent orbit (II.A.17) and (II.A.18) defines an AVMHS which in the limit as $s \to s_0$ gives a MHS satisfying the analogous conditions to (II.A.12) and (II.A.13)

$$\tilde{N}\tilde{F}^p_{s_0} \subseteq \tilde{F}^{p-1}_{s_0} \tag{II.A.19}$$

$$\tilde{N} \text{ has type } (-1,-1) \text{ on } \tilde{H}_0\ . \tag{II.A.20}$$

If

$$\begin{cases} v_{\mathbb{Z}} \in \tilde{H}_{\mathbb{Z},s_0} \\ v_h \in \tilde{F}^n_{s_0} \end{cases}$$

both project to the generator of $\mathbb{Z}(-n)$, then $v_0 = v_h - v_{\mathbb{Z}} \in H_{\mathbb{C},s_0}$. Now $Nv_0 = \tilde{N}v_h - \tilde{N}v_{\mathbb{Z}}$ so that $Nv_0 \in F^{n-1}_{s_0} + H_{\mathbb{Q},s_0}$.

We now want to use this to show that

$$v_0 \in \operatorname{Ker} N + F^n_{s_0} + H_{\mathbb{Q},s_0}\ .$$

For this the following general result will be used:

Proposition. *Let H be a limit MHS of weight m. Let*

$$H^i = \operatorname{Ker} N \subseteq W_i/W_{i-1}, \qquad i = 0, \dots, m$$

which is a pure Hodge structure of weight i. Then as a $\mathbb{Q}-\mathrm{HS}$,

$$W_i/W_{i-1} \simeq \bigoplus_{k=0}^{[i/2]} H^{i-2k}(-k)$$

and

$$H^{i-2k}(-k) \simeq \operatorname{Ker} N^{k+1} \cap \operatorname{Im} N^k \subseteq W_i/W_{i-1} \ .$$

Further,

$$W_i/W_{i-1} \xrightarrow{N} W_{i-2}/W_{i-3}$$

is diagonal with respect to this decomposition, and takes

$$H^{i-2k}(-k) \xrightarrow{\mathrm{id}} H^{i-2k}(-(k-1))$$

with a shift of -1 in the Hodge filtration.

Proof. Since we work over $\mathbb{Q}$, we apply $\otimes_{\mathbb{Z}}\mathbb{Q}$ to all objects. We know that for $j = 1, \dots, n$,

$$W_{n+j}/W_{n+j-1} \xrightarrow{N^j} W_{n-j}/W_{n-j-1}$$

is an isomorphism over $\mathbb{Q}$. Also, in the category of $\mathbb{Q}-\mathrm{HS}$,

$$M = \bigoplus_{j=-m}^{m} W_{m+j}/W_{m+j-1}$$

is a $\mathbb{Q}[x]$-module where x acts by N, and thus by the classification of finitely generated modules over a PID, $M \simeq \bigoplus_i M_i$ where

$$M_i \simeq V_i \otimes \mathbb{Q}[x]/x^{i+1}$$

for some vector space V_i. It follows that

$$V_i = \operatorname{Ker} N^{i+1} \subseteq W_{m+i}/W_{m+i-1}$$

and

$$W_{m+j}/W_{m+j-1} \simeq \bigoplus_{\substack{i \geqslant j \\ i-j \text{ even}}} N^{\frac{i-j}{2}} V_i$$

It follows that if H^i is defined as in the proposition, we have $V_i \simeq H^i(-(n-i))$ and the proposition follows. □

Using the above proposition we may now conclude that

$$v_0 \in \operatorname{Ker} N + F^n_{s_0} + H_{\mathbb{Q},s_0}$$

as desired.

Now v_0 represents the extension class of

$$0 \to H_{0,s_0} \to \tilde{H}_{0,s_0} \to \mathbb{Z}(-n) \to 0 \ ,$$

and this extension class lies in $H_{\mathbb{C},s_0}/F^n_{s_0} + H_{\mathbb{Z},s_0}$. The result above gives that, if $m \in \mathbb{Z}$ clears denominators,

$$mv_0 \in \operatorname{Ker} N_{\mathbb{C}}/F^n \operatorname{Ker} N_{\mathbb{C}} + \operatorname{Ker} N_{\mathbb{Z}} = J_{e,s_0} \ .$$

Referring to (II.A.16) we may replace the normal function ν by $m\nu$ to get a new $\mathbb{Z}$-splitting of (II.A.15) with $\lambda = 0$. Since J_{e,s_0} is divisible and $\hat{J}_{e,s_0}/J_{e,s_0}$ is torsion free

$$m\nu(s_0) \in J_{e,s_0} \Rightarrow \nu(s_0) \in J_{e,s_0} \ .$$

The nilpotent orbit of AVMHS corresponds, under (II.A.14), to a normal function given by

$$\left[\exp\left(\frac{\log s}{2\pi\sqrt{-1}}N\right)\cdot v_0\right]$$

where the bracket means the quotient in the corresponding intermediate Jacobian.

Remark. At the suggestion of a referee, we will elaborate on and illustrate this point. The nilpotent orbit is asymptotic to $e^{(\log s/2\pi\sqrt{-1})\cdot N)}g_0$ where g_0 is a vector space with a filtration and lattice. For example, an elliptic curve with node is asymptotic to

$$e^{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\left(\frac{-\log s}{2\pi\sqrt{-1}}\right)}g_0 = \begin{pmatrix} 1 & \frac{\log s}{2\pi\sqrt{-1}} \\ 0 & 1 \end{pmatrix} g_0$$

and g_0 has $F^1 = \binom{0}{1}$, lattice $\binom{n_1}{n_2}$. In our picture

$$\begin{array}{ccc} & N & \\ \mathbb{Z}(-1) & \longrightarrow & F_1 \\ & \mathbb{Z}(0) & \end{array}$$

is the limit MHS. Now for a normal function, the nilpotent orbit looks like

$$e^{\tilde{N}\left(\frac{-\log s}{2\pi\sqrt{-1}}\right)}\overline{g}_0$$

where $\tilde{N}$ is the monodromy of the extension

$$0 \to H^1(E) \to * \to \mathbb{Z}(-1) \to 0$$

i.e.

$$\tilde{N} = \begin{pmatrix} N & u \\ 0 & 0 \end{pmatrix}$$

where

$$\mathbb{Z}(-1) \xrightarrow{u} H^1(E)\ .$$

We find that

$$u = Nw, \qquad w \in H^1(E)$$

so

$$\tilde{N} = \begin{pmatrix} N & Nw \\ 0 & 0 \end{pmatrix}\ .$$

The limit MHS $\overline{g}_0$ is an extension

$$0 \to H^1(E)_{\lim} \to * \to \mathbb{Z}(-1) \to 0$$

and this is the limit of the Abel-Jacobi map. Now

$$\overline{g}_0 = \begin{pmatrix} \begin{smallmatrix}*\\ *\end{smallmatrix} & H^1(E)_{\lim} \\ 1 & 0 \end{pmatrix}$$

and $\binom{*}{*}$ is the vector corresponding to the extension class. It is defined modulo $F^1H^1(E)_{\lim}$.

Returning to the general discussion, note that $[v_0] \in F^n_{s_0}\backslash H_0/H_{0,\mathbb{Z}}$ is well-defined. The condition

(II.A.4) for $s \neq s_0$, a coordinate on the disc, is

$$\frac{d}{ds}\left(\exp\left(\frac{\log s}{2\pi\sqrt{-1}}N\right)v_0\right) \in F_s^{n-1} ,$$

which, in the limit as $s \to s_0$ implies that

$$Nv_0 \in F_{s_0}^{n-1} . \tag{II.A.21}$$

Proof. Given that

$$\frac{d}{ds}\exp\left(\frac{\log s}{2\pi\sqrt{-1}}N\right)v_0 = \left(\frac{ds}{s}\right)Nv_0 \in F_s^{n-1} \text{ for } s \neq s_0$$

and that F_s^{n-1} smoothly approaches $F_{s_0}^{n-1}$, then if $\alpha_0 \in (F_{s_0}^{n-1})^{\perp}$ there is

$$\alpha(s) = \alpha_0 + s\alpha_1 + \cdots \in (F_s)^{\perp} .$$

Now

$$\left\langle \alpha(s), \frac{1}{s}Nv_0 + b_0 + sb_1 + \dots \right\rangle \equiv 0 ,$$

where the b_i are vectors and $s \neq 0$. This implies that

$$\left\langle \alpha_0 + \alpha_1 s + \dots, \frac{1}{s}(Nv_0 + b_0 + sb_1 + \dots)\right\rangle \equiv 0, \quad s \neq 0 .$$

Next, multiply through by s to have

$$\langle \alpha_0 + \alpha_1 s + \dots, Nv_0 + sb_0 + \dots\rangle \equiv 0, \qquad s \neq 0 ,$$

which is

$$\langle \alpha, Nv_0\rangle + s\left(\langle \alpha_1, Nv_0\rangle + \langle \alpha_0, b_0\rangle\right) + \cdots \equiv 0, \qquad s \neq 0$$

from which it follows that $\langle \alpha, Nv_0\rangle = 0$ for all $\alpha \in (F_{s_0}^{n-1})^{\perp}$, and thus $Nv_0 \in F_{s_0}^{n-1}$. We want to show that $N\nu_0 \in F_{s_0}^{n-1}$ implies $\nu_0 \in \ker(N) + F_{s_0}^{n}$.

Proof. Strictness of a morphism φ of MHS's says that $\varphi(H) \cap F^i = \varphi(F^i H)$, and similarly for the weight filtration. Moreover, N is an endomorphism of $(H_{s_0}, F_{s_0}^{\bullet}, W_{\bullet})$ of type $(-1,-1)$, meaning that strictness reads

$$(*) \qquad N(H_{s_0}) \cap F_{s_0}^{n-1} = N(F_{s_0}^{n}) .$$

Given $N\nu_0 \in \mathrm{LHS}(*)$, there exists $\mu \in F_{s_0}^{n}$ with $N(\mu) = N\nu_0$, and so $\nu_0 = (\nu_0 - \mu) + \mu \in \ker(N) + F_{s_0}^{n}$. $\square$

This completes the proof of (i) in the theorem; the proof of (ii) will be given in the next section.

For the second proof, as suggested by the referee, we use the canonical Deligne extensions [De] to have the short exact sequence over S

$$0 \to \mathcal{H}_e \to \tilde{\mathcal{H}}_e \to \mathcal{O}_s \to 0 .$$

This has two splittings defined by sections σ_F and $\sigma_{\mathbb{Z}}$ of $\tilde{\mathcal{H}}_e \to \mathcal{O}_S$. The first is compatible with the Hodge filtration F; it exists by the freeness of $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \tilde{\mathcal{H}}_e$, which is one of the conditions for an admissible variation of MHS (cf. [SZ]). We note that the above sequence defines the weight filtration W on $\tilde{\mathcal{H}}_e$. The second splitting is compatible with the integral structure; it exists because the normal function is defined over S. We apply both to the canonical generator 1 of $\mathcal{O}_S$, and take the difference $\nu = \sigma_F(1) - \sigma_{\mathbb{Z}}(1) \in \mathcal{H}_e$. This gives the corresponding normal function, by definition. By transversality together with the freeness of $\mathcal{F}_e^{n-1}\backslash\mathcal{H}_e$ we have

$$(s\nabla_{d/ds})v \in \mathcal{F}_e^{n-1} .$$

Since $\mathrm{Res}_{s_0}\nabla = N$, we may restrict to s_0 to get $N(v(s_0)) \in F^{n-1}$. Using the strict compatibility of $N : (H_{e,s_0}, F) \to (H_{e,s_0}, F[-1])$ we infer that $v(s_0) \in \operatorname{Ker} N + F^n$. The rest of the proof now proceeds as before.

Remark on integral and rational structures. In defining the approximating nilpotent orbit and subsequent limiting MHS, one first chooses a reference Hodge structure, which we may think of as choosing a base point $s^* \in S^*$, built on a complex vector space H_0 obtained as $H_0 = H_{0,\mathbb{Z}} \otimes \mathbb{C}$ where $H_{0,\mathbb{Z}} \cong \mathbb{Z}^{2h}$. The integral structure at $s \in S^*$ in the nilpotent orbit is, by definition,

$$\exp\left(\frac{\log s}{2\pi\sqrt{-1}} N\right) \cdot H_{0,\mathbb{Z}}\ .$$

Rescaling by $s \to e^{2\pi\sqrt{-1}\alpha} \cdot s$ induces a change in the integral structure by

$$(T^\alpha - I)H_{0,\mathbb{Z}} := \left(\sum_{k\geqslant 1} \frac{\alpha^k}{k!} N^k\right) H_{0,\mathbb{Z}}\ .$$

Conclusion. Only the part

$$\operatorname{Ker} N_{\mathbb{Z}} = \operatorname{Ker} N \cap H_{0,\mathbb{Z}}$$

of the integral structure of the approximating nilpotent orbit is well-defined independently of the scaling of the parameter.

Rescaling as above also changes the limiting Hodge filtration by $F^p_{s_0} \to e^{-\alpha N} F^p_{s_0}$. Since

$$\begin{cases} N(W_k) \subseteq W_{k-2} \\ N F^p_{s_0} \subseteq F^{p-1}_{s_0} \end{cases}$$

we have the

Conclusion. The Hodge structures on the associated graded of the monodromy weight filtrations are well-defined independently of the parameterization.

Finally, due to the denominators in the standard series expansion of $\log T$ around $T = I$ — in general N does not preserve $H_{0,\mathbb{Z}}$ — we only have $N \cdot H_{0,\mathbb{Q}} \subseteq H_{0,\mathbb{Q}}$. For this reason the monodromy weight filtration is usually considered over $\mathbb{Q}$. Later on, in the discussion of the group of components of the Néron model, the important issue of defining it integrally will arise.

II.B Néron models and graphing admissible normal functions

We begin by defining an admissible normal function (called an extended normal function in [GG1], [GG2]). Over $S^* \cong \Delta^*$ we assume given a polarized VHS of weight $2n-1$, and for convenience we assume that the polarization Q is principal. Then we have

$$F^n_s \backslash H_s / \mathcal{H}_{\mathbb{Z},s} \cong \check{F}^n_s / \mathcal{H}_{\mathbb{Z},s} \tag{II.B.1}$$

where the map $\mathcal{H}_{\mathbb{Z},s} \hookrightarrow \check{F}^n_s$ is given for $\lambda \in \mathcal{H}_{\mathbb{Z},s}$ by $\lambda(\omega) = Q(\omega, \lambda)$, $\omega \in F^n_s$.

Using (II.B.1) we may give a normal function over S^* by a section ν of $\check{\mathcal{F}}^n / \mathcal{H}_{\mathbb{Z}}$. Locally in S^* we may lift ν to a section $\tilde{\nu}$ of $\check{\mathcal{F}}^n$. We denote by $(T-I)\tilde{\nu}$ the analytic continuation around the origin s_0 of such a local lifting. We then have

- $(T-I)\tilde{\nu}(s) \in \mathcal{H}_{\mathbb{Z},s}$,
- $(T-I)\tilde{\nu}(s)$ is well-defined modulo $(T-I)\mathcal{H}_{\mathbb{Z},s}$.

We shall write $(T-I)\mathcal{H}_{\mathbb{Z},s}$ as $\operatorname{Im}(T_{\mathbb{Z}} - I)$.

In general in what follows we shall use the following notation: Given a lattice $\Lambda \cong \mathbb{Z}^r$ and a subgroup $\Lambda' \subset \Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, we set $\Lambda'_{\mathbb{Z}} = \Lambda' \cap \Lambda_{\mathbb{Z}}$. $(T-I)\mathcal{H}_{\mathbb{Q},s}$ shall be written $\operatorname{Im}(T-I)_{\mathbb{Z}}$.

Definition. An admissible normal function (ANF) is a global section $\nu \in \Gamma(S^*, \mathcal{J}_\nabla)$ such that for any local lifting $\tilde{\nu}$ to a section of $\check{\mathcal{J}}^n$, the following two conditions are satisfied:

(i) $\tilde{\nu}$ has moderate — i.e. logarithmic — growth as a section of $\check{\mathcal{F}}_e$; and

(ii) we have

$$(T - I)\tilde{\nu} \in \mathrm{Im}(T - I)_{\mathbb{Z}} \ .$$

Since sections of $\mathcal{H}_{\mathbb{Z}}$ have moderate growth and $\mathrm{Im}(T_{\mathbb{Z}} - I) \subseteq \mathrm{Im}(T - I)_{\mathbb{Z}}$ the conditions (i) and (ii) are well-defined.

We note that this definition is equivalent to the one given in [Sa]. The two conditions (i) and (ii) above correspond to the two conditions there for an admissible variation of mixed Hodge structure, given there respectively as the freeness of $\mathrm{Gr}_F^p \mathrm{Gr}_k^W$ for the Deligne extension and the existence of the relative monodromy filtration.

We denote by $\tilde{\mathcal{J}}_{e,\nabla}$ the sheaf of ANF's. There is then an exact sheaf sequence

$$0 \to \mathcal{J}_{e,\nabla} \to \tilde{\mathcal{J}}_{e,\nabla} \to G_{s_0} \to 0 \qquad \text{(II.B.2)}$$

where G_{s_0} is a skyscraper sheaf supported at s_0 with stalk G.

Theorem II.B.3. *We have that*

$$G \cong \frac{\mathrm{Im}(T_{\mathbb{Q}} - I)_{\mathbb{Z}}}{\mathrm{Im}(T_{\mathbb{Z}} - I)} \ ,$$

which is a finite abelian group.

Proof. As previously noted, it follows directly from the definitions that $G = H^1(\Delta^*, \mathcal{H}_{\mathbb{Z}})_{\mathrm{tor}}$ is a finite abelian group. □

We have also noted above the identification

$$G \cong \frac{(\mathrm{Ker}(T_{\mathbb{Q}} - I)^{\perp})_{\mathbb{Z}}}{\mathrm{Im}(T_{\mathbb{Z}} - I)} \ , \qquad \text{(II.B.4)}$$

which uses the principal polarization. Because it will illustrate some of the concepts to be used below, and also will give an opportunity to recall some facts about the limiting mixed Hodge structure (LMHS), we shall point out in the case $n = 2$ that G as given by (II.B.4) is a finite group. For this, it will suffice to prove that over the rationals

$$\left(\mathrm{Ker}(T - I)^{\perp}\right)_{\mathbb{Q}} = \mathrm{Im}(T - I)_{\mathbb{Q}} \ .$$

Since

$$N = (T - I)A = A(T - I)$$

where A is invertible, it will suffice to show that either over $\mathbb{Q}$ or $\mathbb{C}$ $\mathrm{Ker}\, N^{\perp} = \mathrm{Im}\, N$. For this we shall use the fact that the limiting MHS is *polarized* (cf. [Sc]; this is stronger than saying that the LMHS is graded polarized).

Specifically, recall that for the monodromy weight filtration one has

- $N(W_l) \subseteq W_{l-2}$
- $N^j : \mathrm{Gr}_{2n-1+j} \xrightarrow{\sim} \mathrm{Gr}_{2n-1-j}$ is an isomorphism,

and if we then define the *primitive spaces*

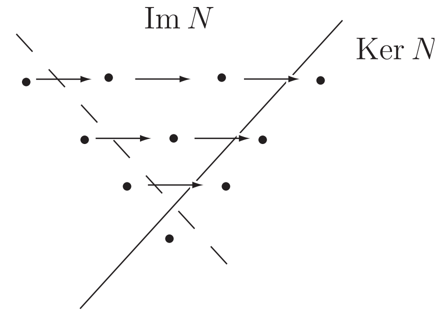
$$P_{2n-1+j} = \mathrm{Ker}\, N^{j+1} \subset \mathrm{Gr}_{2n-1+j} \qquad (j \geqq 0)$$

the bilinear forms on P_{2n-1+j} defined by

$$Q_j(u, u) = Q(u, N^j u) \qquad \text{(II.B.5)}$$

are non-degenerate.

For our present purposes of illustration we will only do the case $n = 2$. Then the picture of the LMHS gives



i.e., everything to the lower right of the solid line is $\operatorname{Ker} N$ and everything to the upper right of the dotted line is $\operatorname{Im} N$. From the non-degeneracy of (II.B.5) it follows that everything to the lower left of the dotted line pairs non-degenerately under Q with $\operatorname{Ker} N$. This gives $\operatorname{Ker} N^{\perp} \subseteq \operatorname{Im} N$, and the reverse inclusion is evident. This shows that, in the case $n = 2$, G as given by (II.B.4) is a finite group. $\square$

Next, we recall that we have defined

$$\begin{array}{rc} J_{e,s_0} = & F^n \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}} / \mathcal{H}_{\mathbb{Z},e;s_0} \\ & \cap \\ \hat{J}_{e,s_0} = & F^n_{e,s_0} \backslash H_{e,s_0} / \mathcal{H}_{\mathbb{Z},e;s_0} \; . \end{array}$$

We shall now prove Proposition II.A.8, and in addition show that

Proposition II.B.6. *$\mathcal{H}_{\mathbb{Z},e;s_0}$ projects to a discrete subgroup in $F^n \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}}$.*

Proof. The monodromy weight filtration on H_{e,s_0} induces a filtration

$$\{0\} \subset W_0(V) \subset W_1(V) \subset \cdots \subset W_n(V) = V$$

on $V =: F^n \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}}$. Let Λ be the image of $\mathcal{H}_{\mathbb{Z},e;s_0} \to V$. We shall use the following elementary

Lemma. *If $\Lambda \cap W_l(V)$ projects to a discrete subgroup in $W_l(V)/W_{l-1}(V)$ for all l, then Λ is a discrete subgroup of V.*

The converse does not hold, as illustrated by

$$V = \mathbb{R}^2, \quad \Lambda = \mathbb{Z}^2, \quad W_0 = \text{ irrational line through the origin } .$$

For the proof of the lemma, we have that $\operatorname{Ker} N_{\mathbb{C}} \subseteq W_{2n-1}(V)$, and thus is a sub-MHS of $W_{2n-1}(V)$. Its quotients

$$\frac{\operatorname{Ker} N_{\mathbb{C}} \cap W_l(V)}{\operatorname{Ker} N_{\mathbb{C}} \cap W_{l-1}(V)}$$

are sub-HS of weight l in $W_l(V)/W_{l-1}(V)$. The integral lattice $\mathcal{H}_{\mathbb{Z},e,s_0}$ projects to the integral lattices in these Hodge structures. However, in any Hodge structure H of weight l, the integral lattice projects to a discrete subgroup of $F^n H/H$ when $l \leqslant 2n-1$, as happens here. The lemma above completes the proof of Proposition II.B.6. $\square$

We now define

$$\begin{aligned}\mathbf{J}_e &= \left(\bigcup_{s\in S^*} J_s\right) \cup J_{e,s_0}\\ &= \mathbf{J} \cup J_{e,s_0}\end{aligned}$$

using the notations from section II.A. Setting

$$\begin{cases} V_s = F_s^n \backslash H_s & s \neq s_0 \\ V_{s_0} = F^n_{e,s_0} \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}} \end{cases}$$

and

$$\mathbf{V}_e = \bigcup_{s\in S} V_s$$

we may view $\mathbf{J}_e$ as a quotient $\mathbf{J}_e = \mathbf{V}_e/\sim$ where the equivalence relation $\sim$ is given fiberwise by equating two elements $\alpha, \alpha' \in V_s$ if and only if

$$\alpha - \alpha' \in \begin{cases} \mathcal{H}_{\mathbb{Z},s} & \text{for } s \neq s_0 \\ \mathcal{H}_{\mathbb{Z},e;s_0} & \text{for } s = s_0\ . \end{cases}$$

We now define what is meant by a slit analytic fibre space of complex Lie groups. It is just a formalization of what has just been discussed. For the local model we take an open set $U \subseteq \mathbb{C}$ with distinguished point $u_0 \in U$. We then set

$$W = (U\backslash\{u_0\} \times \mathbb{C}^m) \cup (\{u_0\} \times \mathbb{C}^k)$$

where $\mathbb{C}^k \subset \mathbb{C}^m$ consists of the first k coordinates. We then take the quotient $W/\sim$ where $\sim$ is an equivalence relation given fiberwise by a discrete group, and where the overall quotient is Hausdorff.

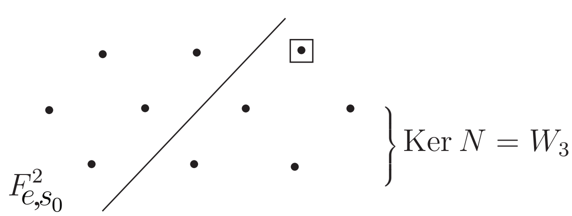
Definition. A slit analytic fibre space of complex Lie groups is a Hausdorff topological space F together with a map

$$\pi : F \to S \tag{II.B.7}$$

to a 1-dimensional complex manifold S having a marked point s_0 such that (i) the fibres $\pi^{-1}(s)$ are commutative and connected complex Lie groups, (ii) $F^* =: F\backslash\pi^{-1}(s_0)$ is a complex analytic fibre space of complex Lie groups in the usual sense, and (iii) locally around points of $\pi^{-1}(s_0)$ we have the local model described above.

At this point we have completed the discussion of (ii) in Theorem II.A.9.

Discussion. We want to explain intuitively the idea behind the above construction. For the purposes of illustration we consider a VHS over $S^* = \Delta^*$ of weight three and whose limiting MHS has the picture



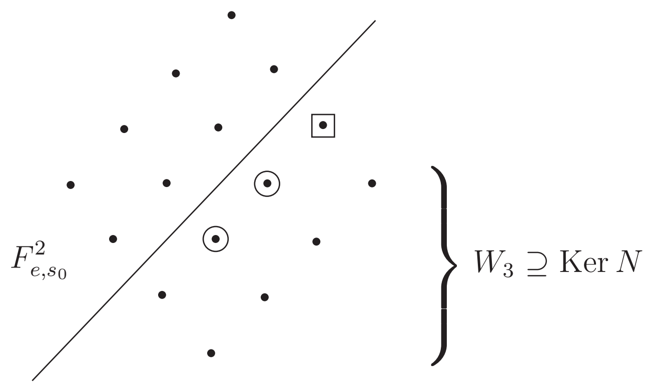
where the boxed term $I^{1,3}$ is non-zero. Setting

$$\begin{aligned} V_s &= F^2_{e,s}\backslash H_{e,s}\\ &\cup\\ \Lambda_s &= \text{image of } \mathcal{H}_{\mathbb{Z},e;s} \to V_s\ , \end{aligned}$$

and recalling that $\mathcal{H}_{\mathbb{Z},e;s_0}$ are the invariant cycles in $\mathcal{H}_{\mathbb{Z},e;s}$ for $s \neq s_0$, we see from the picture that

$$\Lambda_{s_0} \subsetneqq \lim_{s \to s_0} \Lambda_s \;, \tag{II.B.8}$$

where the limit is taken along a ray or even just a *sequence* $\{s_m\}$ going to the origin. More precisely, by the limit we mean taking the closure inside the Deligne extensions, and then (II.B.8) follows from the example in the following figure (cf. [Sa, Remark (3.5)(iv)]).

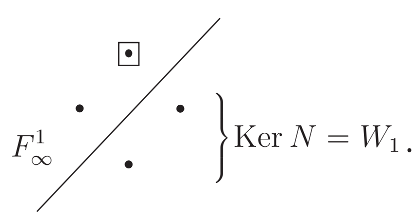


The question is: what should we "plug in" to $\mathbf{J}$ over the origin for purposes of graphing normal functions? Since the RHS of (II.B.8) is not necessarily even a sub-abelian-group of V_{s_0}, taking $V_{s_0}/\lim_{s\to s_0}\Lambda_s$ is completely unsuitable. If we augment $\lim_{s\to s_0}\Lambda_s$ to the subgroup it generates, this can actually include all of $\mathrm{Ker}(N)$ — again unsuitable since that is esentially where $\nu(s_0)$ should lie by Theorem II.A.9. Finally, if we factor V_{s_0} only by Λ_{s_0}, then since by (II.B.8) the equivalence relation on $\bigcup_{s\in S} V_s$ given by the Λ_s is not closed, the resulting total space will be non-Hausdorff. This exhausts the possibilities which are quotients of V_{s_0}. So we conclude that we should replace V_{s_0} by

$$F^2_{e,s_0} \cap \mathrm{Ker}\, N_{\mathbb{C}} \backslash \mathrm{Ker}\, N_{\mathbb{C}} \subsetneqq V_{s_0}$$

to get a closed equivalence relation with a Hausdorff quotient. The price we pay is that we only obtain a slit analytic fibre space.

In the classical $n = 1$ case the above picture is replaced by



In the limit $\lim_{s\to s_0}\Lambda_s$ the non-invariant part of Λ_s (corresponding to the boxed term) "goes to infinity" as in the familiar picture of an elliptic curve acquiring a node where $s = e^{2\pi\sqrt{-1}t}$ and $\mathrm{Im}\, t \to \infty$.

To convince the reader of the claimed intractability of $\lim_{s\to s_0}\Lambda_s \subseteq V_{s_0}$ in the $n = 2$ case, we offer the following geometrically motivated

Example. We shall reproduce in detail and give a geometric example of the observation in Remark 3.5(iv) of [Sa].

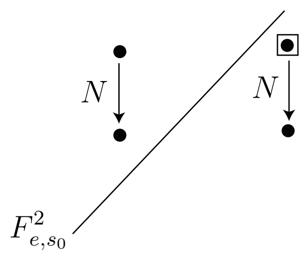
Let E_s be a family of elliptic curves acquiring a node, with $\{\delta_s, \gamma_s\} \subseteq H^1(E_s, \mathbb{Z})$ satisfying $N\gamma = -\delta$, $N\delta = 0$. We take $X_s := E_s \times W$, where W is a fixed exceptional $K3$ surface, $H^1_s := H^1(E_s)$ and

$$H^3_s := H^1_s \otimes H^2_{\mathrm{tr}}(W) \subseteq H^3(X_s)\ .$$

H^3_s has rank 4. The limit MHS on $H^1_{s_0}$ is an extension of $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$ which can be split by choosing $s := \exp(2\pi\sqrt{-1}\int_{\gamma_s}\omega_s)$, where $\omega_s \in F^1_{e,s}H^1_{e,s}$ is normalized to have $\int_{\delta_s}\omega_s = 1$ for all $s \in S$. A clearer notation would be to denote the point by P and then the coordinate s is defined by $s(P) =: \exp(2\pi\sqrt{-1}\int_{\gamma_P}\omega_p)$. Then $\int_{\gamma_s}\omega_s = \frac{\log s}{2\pi\sqrt{-1}}$. This choice splits the limit MHS

$$H^3_{s_0} = \mathrm{Gr}^W_2 \oplus \mathrm{Gr}^W_4 = \mathbb{Z}\langle\delta_0\rangle \otimes H^2_{\mathrm{tr}}(W) \oplus \mathbb{Z}\langle\tilde{\gamma}_0\rangle \otimes H^2_{\mathrm{tr}}(W)$$

where $\tilde{\gamma}_s := \gamma_s + \frac{\log s}{2\pi\sqrt{-1}}\delta_s (\equiv \omega_s)$. This gives the picture



of the limit MHS of H^3_s.

To compute the periods for $s \neq s_0$, let $\Omega \in F^2H^2_{\mathrm{tr}}(W, \mathbb{C})$, $\{\tau_1, \tau_2\} \subseteq H^2_{\mathrm{tr}}(W, \mathbb{Z})$ be a basis, and assume $\int_{\tau_1}\Omega = -1$, $\int_{\tau_2}\Omega = \sqrt{-1}$. Specifically, one may take the very simplest example of a supersingular K3 surface, which is well known to have period ratio $i = \sqrt{-1}$. Taking periods of a generating set for $\mathcal{H}^3_{\mathbb{Z},s}$ with respect to the basis $\delta_s \times \Omega$, $\tilde{\gamma}_s \times \Omega$ of $F^2_{e,s}$, we have

$$\begin{aligned}\Lambda_s :=& \mathbb{Z}\left\langle \pi_s(\delta\times\tau_1),\ \pi_s(\delta\times\tau_2),\ \pi_s(\gamma\times\tau_1),\ \pi_s(\gamma\times\tau_2)\right\rangle \\ =& \mathbb{Z}\left\langle (0,-1), (0,-\sqrt{-1}), \left(1, \frac{\log s}{2\pi\sqrt{-1}}\right), \left(\sqrt{-1}, \sqrt{-1}\frac{\log s}{2\pi\sqrt{-1}}\right)\right\rangle\end{aligned}$$

inside $V_s = \check{F}^2_s$ for $s \neq s_0$. Note that

$$(\mathrm{Ker}\, N)_s = \{(0, \zeta) | \zeta \in \mathbb{C}\} \subset V_s\ .$$

Now consider, for any $\alpha \in \mathbb{C}$, the sequence

$$s_m := e^{2\pi\sqrt{-1}\alpha}/e^{2\pi m} \to 0\ .$$

We have $\frac{\log s_m}{2\pi\sqrt{-1}} = \alpha - \sqrt{-1}m$ and for all integers m,

$$\begin{aligned}&\pi_{s_m}\{\gamma\times\tau_1 - m(\delta\times\tau_2)\} \\ =& \left(1, \frac{\log s_m}{2\pi\sqrt{-1}}\right) - m(0, -\sqrt{-1}) \\ =& (1, \alpha)\ ,\end{aligned}$$

which implies $(1, \alpha) \in \lim_{s\to s_0}\Lambda_s$. More generally, one has

$$\lim_{s\to s_0}\Lambda_s = \{(\kappa, \alpha) | \alpha \in \mathbb{C}, \quad \kappa \in \mathbb{Z}[\sqrt{-1}]\backslash\{0\}\}$$

which is not a subgroup of V_{s_0} and whose abelian-group closure contains $(\mathrm{Ker}\, N)_{s_0}$.

In contrast to this sitaution with H^3_s for H^1_s one has $\Lambda_s = \mathbb{Z}\left\langle 1, \frac{\log s}{2\pi\sqrt{-1}}\right\rangle$ and there is no way to take a comparable sequence since $l(s) \to \sqrt{-1}\infty$.

Returning to the general discussion, the remaining step in the construction of the Néron model is the following:

Theorem II.B.9. *There exists a canonical analytic fibre space*

$$\tilde{\mathbf{J}}_e \to S$$

that graphs admissible normal functions. We have for the fibre over s_0 the exact sequence

$$0 \to J_{e,s_0} \to \tilde{J}_{e,s_0} \to G \to 0$$

where G is as in Theorem II.B.3.

Thus we may speak of the value $\nu(s_0) \in \tilde{J}_{e,s_0}$ of an admissible normal function.

Proof of Theorem II.B.9. We want to define the analytic group $\tilde{J}_{e,s_0}$ that (i) sits naturally in an exact sequence

$$0 \to J_{e,s_0} \to \tilde{J}_{e,s_0} \to G \to 0\ , \tag{II.B.10}$$

and (ii) has the property that

$$\tilde{\mathbf{J}}_e =: \mathbf{J}_e \cup \tilde{J}_{e,s_0} \to S$$

forms naturally a slit analytic fibre space of complex Lie groups that graphs admissible normal functions. We note that the slit analytic fibre space $\tilde{\mathbf{J}}_e$ is a subset of the Clemens extension $J_S^C(\mathbf{H})$ constructed in [Sa, Prop. (2.7)]; they coincide in the abelian variety case.

We first point out that graphs of admissible normal functions in $\tilde{\mathbf{J}}_e$ will be continuous since the "torsion singularities" in G of ANF's are always "limits of something." More precisely, since by definition $\Gamma(\Delta, \tilde{\mathcal{J}}_{e,\nabla}) \subseteq \Gamma(\Delta^*, \mathcal{J}_\nabla)$, the sheaf $\tilde{\mathcal{J}}_{e,\nabla}$ of ANF's has no skyscraper subsheaf and there can be no map of sheaves $G_{s_0} \to \tilde{\mathcal{J}}_{e,\nabla}$ splitting (II.B.2).

On the other hand, let A be a divisible abelian group; then in the category $\mathcal{A}$ of abelian groups $\mathrm{Ext}^1_{\mathcal{A}}(G,A) = 0$. Consequently the extension of stalks

$$0 \to \mathcal{J}_{e,\nabla;s_0} \xrightarrow{\tilde{\imath}} \tilde{\mathcal{J}}_{e,\nabla;s_0} \xrightarrow{\tilde{\jmath}} G \to 0 \tag{II.B.11}$$

splits, but there is no "canonical" choice from among the $|\operatorname{Hom}_{\mathcal{A}}(G, \mathcal{J}_{\nabla,e;s_0})|$ many splittings; the same should go for (II.B.10).

To define $\tilde{J}_{e,s_0}$ without making such a choice, simply push the extension (II.B.11) forward along the morphism $\mathcal{J}_{e,\nabla;s_0} \to J_{e,s_0}$ guaranteed by Theorem II.A.9, to obtain

$$0 \to J_{e,s_0} \xrightarrow{\imath} \tilde{J}_{e,s_0} \xrightarrow{\jmath} G \to 0\ .$$

Equivalently, define

$$\begin{array}{ccl} (\alpha, v) & \in & \tilde{J}_{e,s_0} := \frac{J_{e,s_0} \times \tilde{\mathcal{J}}_{e,\nabla;s_0}}{\{(\sigma(s_0), \tilde{\imath}(\sigma)) \mid \sigma \in \mathcal{J}_{e,\nabla;s_0}\}} \\ \downarrow & & \downarrow \\ \tilde{\jmath}(v) & \in & G \end{array}$$

For the benefit of the reader we recall the definition of $\tilde{\jmath}$: given an admissible variation of MHS $\tilde{H}_s$ corresponding to $\nu \in \tilde{J}_{e,\nabla;s_0}$, write the action of $T - I$

$$\begin{tikzcd} H_s \arrow[r, "i"] & \tilde{H}_s \arrow[r, "j"] \arrow[dl, phantom] & \mathbb{Z}(-n) \\ H_s \arrow[u, "T-I"] \arrow[r] & \tilde{H}_s \arrow[u, "\tilde{T}-I"'] \arrow[ul, dashed] \arrow[r] & \mathbb{Z}(-n)\ . \arrow[u, "0"'] \end{tikzcd}$$

We must define an element of $(\mathrm{Ker}(T-I))^{\perp}_{\mathbb{Z}} \subseteq \mathcal{H}_{\mathbb{Z},s})$. A local lifting $\tilde{\nu}$ of ν is obtained by using maps

$$\begin{cases} \varphi_{\mathbb{Z},s} : \mathbb{Z}(n) \longrightarrow \tilde{\mathcal{H}}_{\mathbb{Z},s} \\ \varphi_{H,s} : \mathbb{Z}(n) \longrightarrow F^n \tilde{H}_s \end{cases}$$

and setting (for $s \neq s_0$) $i(\tilde{v}(s)) := \varphi_{\mathbb{Z},s}(1) - \varphi_{H,s}(1)$. Denoting by $(T-I)\tilde{\nu}(s)$ the change in $\tilde{\nu}$ under analytic continuation around s_0, the definition of ANF says

$$(T-I)\tilde{\nu}(s) \in (\mathrm{Ker}(T-I))^{\perp}_{\mathbb{Z}} \ .$$

Since $\varphi_{H,s}(1)$ is single-valued in s, we may rewrite this as $\tilde{\jmath}(\nu) := i^{-1}(T-I)\varphi_{\mathbb{Z},s}(1)$. In the geometric case where ν arises from a family of cycles Z_s, this simply computes the change in the chain bounding on Z_s as s turns about s_0.

Having constructed $\tilde{J}_{e,s_0}$, which is non-canonically isomorphic to $J_{e,s_0} \times G$, we can set

$$\tilde{\mathbf{J}}_e := \frac{\mathbf{J}_e \coprod \tilde{J}_{e,s_0}}{\sim}$$

where $\sim$ identifies the identity component $i(J_{e,s_0}) \subset \tilde{J}_{e,s_0}$ to the fiber of $\mathbf{J}_e$ over s_0. The local slit analytic structure around a point of this may be transported to the other components simply by translating by ANF's mapping to each $g \in G$.

Example. Specialize to $n = 1$ and the simple case where H_s comes from a family of elliptic curves with singular fiber of type I_m in Kodaira's list [Ko]. We will use a choice of splitting to give an intuitive, analytic construction of the Néron model. Calculations of a similar sort are in [Ko], [Na] and [Nam]. The explicit identification of X with the Néron model is in [Sa, Remark (4.5)(i)]. This argument can be extended to the general case and provided our original, highly non-intrinsic, proof of Theorem II.B.9.

In this case the sequence (II.B.10) is

$$0 \to \mathbb{C}^* \to \tilde{J}_{e,s_0} \to \mathbb{Z}/m\mathbb{Z} \to 0 \ ,$$

which splits as a sequence of abelian groups, the splitting being determined only up to an m^{th} root of unity. In this sense, the sequence (II.B.10) does not split canonically. To explain this, choose $\omega(s) \in H^0(\omega_{X_s})$ so as to have a normalized period matrix

$$\left(1, m\frac{\log s}{2\pi\sqrt{-1}} + h(s)\right)$$

where $h(s)$ is holomorphic. An admissible normal function is given modulo periods by its value on $\omega(s)$

$$\langle \nu(s), \omega(s) \rangle = k\frac{\log s}{2\pi\sqrt{-1}} + g(s) \ , \qquad\qquad k \in \mathbb{Z} \ , \tag{II.B.12}$$

where $g(s)$ is holomorphic. By subtracting a period we may normalize to have $0 \leqslant k < m$. Rescaling s gives a change

$$\begin{cases} h(0) \to h(0) + m\lambda \\ g(0) \to g(0) + k\lambda \end{cases}$$

where $\lambda \in \mathbb{C}$.

The map $\tilde{\mathcal{J}}_{\nabla,e,s_0} \xrightarrow{\tilde{\jmath}} G$ is given by $\nu \to [k] \in \mathbb{Z}/m\mathbb{Z}$. The splitting is obtained first by multiplying (II.B.12) by m to kill the image in $\mathbb{Z}/m\mathbb{Z}$. Normalizing the result by subtracting the period $k(m \cdot l(s) + h(s))$, we get $m \cdot g(s) - k \cdot h(s)$. Evaluating this (in $\mathbb{C}/\mathbb{Z}$) at $s = s_0$, we may then "use divisibility of $\mathbb{C}/\mathbb{Z}$" and divide by m to get

$$g(s_0) - \frac{k}{m} \cdot h(s_0) + \frac{l}{m} \qquad\qquad (l \in \mathbb{Z},\ 0 \leqslant l < m) \ . \tag{II.B.13}$$

No choice of l here is "more natural"; on the other hand, (II.B.13) is evidently not affected by choice of the local parameter. So one obtains a well-defined, but non-canonical, "splitting" sending

$$\text{(II.B.12)} \mapsto (\text{(II.B.13)}, [k]) \ ,$$

and this computes limits of ANF's

$$\tilde{\mathcal{J}}_{e,\nabla;s_0} \to J_{e,s_0} \times G \cong \mathbb{C}/\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \ .$$

Remark on the work of Brosnan-Pearlstein. We have shown in the first part of this section that

$$\mathbf{J}_e \to S$$

is a fibration of a Hausdorff topological space whose fibres are connected, complex Lie groups and where the restriction of the total space to S^* is a complex analytic fibre space in the usual sense. Moreover, a normal function ν gives a continuous section of $\mathbf{J}_e \to S$ which is holomorphic over S^*. It follows that for any closed subset $Z \subset \mathbf{J}_e$ $\nu^{-1}(Z) \subset S$ is closed. Moreover, if Z^* denotes the part of Z lying over S^*, then $\nu^{-1}(Z^*)$ is an analytic subvariety whose closure is $\nu^{-1}(Z)$ and one may suspect that $\nu^{-1}(Z)$ is itself an analytic subvariety.

This brings us to important recent work of Brosnan-Pearlstein [BP] who have proved that $\nu^{-1}(Z)$ is an analytic variety in the case where $\dim S = 1$ and Z is the zero-section. What has to be proved is that $\nu^{-1}(Z^*)$ does not contain an infinite sequence s_n of distinct points with $\lim_{n\to\infty} s_n = s_0$. We think it is quite possible that the extension of the Brosnan-Pearlstein methods to the case when $\dim S$ is arbitrary may not only lead to a proof that $\nu^{-1}(Z)$ is analytic in general, but moreover will provide the tools to show that $\tilde{\mathbf{J}}_e \to S$ really does "behave" like a usual complex-analytic fibre space of complex Lie groups insofar as admissible normal functions are concerned. In particular, objects such as $(\nu \times \nu)^{-1}(\mathcal{P})$ where $\mathcal{P} \to \tilde{\mathbf{J}}_e \times_S \tilde{\mathbf{J}}_e$ is the extension of $\tilde{\mathbf{J}}_e \times_S \tilde{\mathbf{J}}_e$ of the Poincaré line bundle over $\mathbf{J} \times_{S^*} \mathbf{J}$ may have naturally the structure of an analytic line bundle over S that is canonically associated to an admissible normal function.

II.C Computation of the group G of components of the Néron model in Hodge-theoretic examples

We consider a principally polarized variation of Hodge structure of weight $2n-1$ over $S = \{|s| < 1\}$ with unipotent monodromy. We shall give an analysis of G in the case $n = 1$ and in some cases when $n = 2$, $h^{3,0} = 1$ — corresponding to degeneration of Calabi-Yau's — and where in the first instance we assume that the limiting MHS is of the Tate type. Subsequently, we shall drop this assumption in the sub-case $h^{3,0} = h^{2,1} = 1$.

Case $n = 1$. Then $N = T - I$ is integral, the limiting MHS may be defined over $\mathbb{Z}$ and pictured as

$$\begin{array}{ccc} \mathbb{Z}(-1)^r & & \mathbb{Z}(0)^r \\ \bullet & \xrightarrow{N} & \bullet \\ & & \\ & \bullet & \\ & H^1 & \end{array}$$

Since $(T - I)^2 = 0$ we may choose a symplectic basis $\gamma_1, \dots, \gamma_r, \gamma_{r+1}, \dots, \gamma_g$; $\delta_1, \dots, \delta_r, \delta_{r+1}, \dots, \delta_g$

such that, writing elements of $H_{\mathbb{Z}}$ as row vectors,

$$N = \begin{pmatrix} 0 & \tilde{B} \\ 0 & 0 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} B & 0 \\ \underbrace{0}_{r} & \underbrace{0}_{g-r} \end{pmatrix} \begin{array}{l} \}r \\ \}g-r \end{array}$$

where $B = {}^tB > 0$ is integral and positive definite. Thus, for example $\gamma_1 = (1, 0, \ldots, 0)$ and $N\gamma_1 = \sum_{j=1}^r b_{1j}\delta_j$. Then

$$\begin{cases} W_{1,\mathbb{Z}} =: \operatorname{Ker} N \cap H_{\mathbb{Z}} = \operatorname{span}_{\mathbb{Z}}\{\gamma_{r+1}, \ldots, \gamma_g, \delta_1, \ldots, \delta_g\} \\ W_{0,\mathbb{Z}} =: \operatorname{Im} N \cap H_{\mathbb{Z}} = \operatorname{span}_{\mathbb{Z}}\{\delta_1, \ldots, \delta_r\} \ . \end{cases}$$

It follows that

$$\begin{aligned} G &= \frac{\left((\operatorname{Ker} N)^{\perp}\right)_{\mathbb{Z}}}{\operatorname{Im} N_{\mathbb{Z}}} \\ &\cong \frac{\operatorname{Span}_{\mathbb{Z}}\{\delta_1, \ldots, \delta_r\}}{\operatorname{Span}_{\mathbb{Z}}\{N\gamma_1, \ldots, N\gamma_r\}} \\ &\cong \mathbb{Z}^r / \operatorname{Im} B \ . \end{aligned} \tag{II.C.1}$$

In particular

$$|G| = \det B \tag{II.C.2}$$

where $|G|$ is the order of G.

This result is, of course, classical (cf. [Na]).

Case $n = 2$. We recall our notations

$$\begin{cases} W_{m,\mathbb{Z}} = W_m \cap H_{\mathbb{Z}} \\ \operatorname{Gr}_{m,\mathbb{Z}} = W_{m,\mathbb{Z}}/W_{m-1,\mathbb{Z}} \ . \end{cases}$$

We set

$$\begin{cases} h = h^{3,0} + h^{2,1} \\ k = h^{2,1} - 1 \ . \end{cases}$$

Our assumption is that the limiting MHS looks like

$$\begin{tikzcd} \overset{\mathbb{Z}(-3)}{\bullet} \arrow[r] & \overset{\mathbb{Z}(-2)}{\bullet} \arrow[r] & \overset{\mathbb{Z}(-1)}{\bullet} \arrow[r] & \overset{\mathbb{Z}(0)}{\bullet} & \\ & \overset{\mathbb{Z}(-2)^k}{\bullet} \arrow[r] & \overset{\mathbb{Z}(-1)^k}{\bullet} & & \operatorname{Ker}(T-I) \end{tikzcd}$$

where $\operatorname{Ker}(T-I)$ is everything to the lower right of the dotted line. For the identity component of the Néron model we have

$$J_{e,s_0} \cong (\mathbb{C}^*(-1))^k \oplus \mathbb{C}^* \ .$$

We shall use the following (cf. [Mo2]):

Proposition II.C.3. *There exists a symplectic basis*

$$\gamma_1, \gamma_2, \ldots, \gamma_h; \delta_2, \ldots, \delta_h, \delta_1$$

adapted to the integral weight filtration.

Thus the intersection form is

$$Q = \begin{array}{c} \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} \underbrace{\quad}_{h^{3,0}} & \underbrace{\quad}_{h^{2,1}} & \underbrace{\quad}_{h^{1,2}} & \underbrace{\quad}_{h^{0,3}} \end{array} \end{array} \begin{array}{l} \}h^{3,0}=1 \\ \}h^{2,1} \\ \}h^{1,2} \\ \}h^{0,3}=1 \\ \\ \end{array}$$

and

$$\begin{cases} W_{5,\mathbb{Z}} = W_{4,\mathbb{Z}} = \operatorname{span}_{\mathbb{Z}}\{\gamma_2, \dots, \gamma_h; \delta_2, \dots, \delta_h, \delta_1\} \\ W_{3,\mathbb{Z}} = W_{2,\mathbb{Z}} = \operatorname{span}_{\mathbb{Z}}\{\delta_2, \dots, \delta_h, \delta_1\} \\ W_{1,\mathbb{Z}} = W_{0,\mathbb{Z}} = \operatorname{span}_{\mathbb{Z}}\{\delta_1\}\ . \end{cases}$$

Writing elements of H as row vectors we shall also assume that the limiting Hodge filtration is

$$\Big(\underbrace{\underbrace{\underbrace{\overbrace{\qquad}^{h^{3,0}}}_{F^3_{e,s_0}}, \overbrace{\bullet\;\bullet\;\bullet}^{h^{2,1}}}_{F^2_{e,s_0}}, \overbrace{\bullet\;\bullet\;\bullet}^{h^{1,2}}}_{F^1_{e,s_0}}, \overbrace{\qquad}^{h^{0,3}} \Big)$$

This is equivalent to the

Assumption. The limiting mixed Hodge structure is split over $\mathbb{Z}$.

Later on we shall examine the non-split case when $h^{2,1} = 1$.

Proposition. *The log of monodromy is*

$$N = \begin{pmatrix} 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & -{}^tA \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where A and B are integral and

$$\begin{cases} A \neq 0 \\ B = {}^tB > 0\ . \end{cases} \tag{II.C.4}$$

Moreover, the matrices

$$\begin{cases} AB/2 & (1 \times h^{2,1} \ \text{matrix}) \\ AB^tA/6 & (1 \times 1 \ \text{matrix}) \end{cases} \tag{II.C.5}$$

are also integral.

Proof. Since $N(W_m) \subseteq W_{m-2}$ we have

$$N = \begin{pmatrix} 0 & A & D & E \\ 0 & 0 & B & F \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the entries are rational. From the last assumption and $NF^p_{e,s_0} \subseteq F^{p-1}_{e,s_0}$ we infer that $D = E = F = 0$ (this is the key step). From $NQ + Q^t N = 0$ we have

$$\begin{cases} C = -{}^t A \\ B = {}^t B \,. \end{cases}$$

From

$$N^2 = \begin{pmatrix} 0 & 0 & AB & 0 \\ 0 & 0 & 0 & -B^t A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = \begin{pmatrix} 0 & 0 & 0 & -AB^t A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the fact that limiting MHS is polarized we may conclude (II.C.4).

Since $T = \exp N = I + N + \frac{N^2}{2} + \frac{N^3}{6}$ is integral we see that A, B are integral and also conclude (II.C.5). □

We shall write vectors in $H_{\mathbb{C}}$ as (v_1, v_2, v_3, v_4) where v_1, v_4 are scalars and v_2, v_3 are row vectors of length $h^{2,1}$. Then $\operatorname{Ker}(T - I) = \{v_1 = 0, v_2 = 0, v_3{}^t A = 0\}$ and

$$(\operatorname{Ker}(T - I)^{\perp})_{\mathbb{Z}} = \{(0, \lambda A, w_3, w_4) : \text{all entries in } \mathbb{Z}\}\,.$$

Writing $A = (\alpha_1, \dots, \alpha_{h^{2,1}})$ we set $\alpha = \text{g.c.d. } \{\alpha_1, \dots, \alpha_{h^{2,1}}\}$ where we take $a > 0$. Then

$$\alpha = 1 \Leftrightarrow A \text{ is primitive.}$$

Thus

$$\left(\operatorname{Ker}(T - I)^{\perp}\right)_{\mathbb{Z}} = \left\{\left(0, \frac{mA}{\alpha}, w_3, w_4\right) \cdot m \in \mathbb{Z}, w_i \text{ integral}\right\}\,.$$

$$\operatorname{Im}(T - I)_{\mathbb{Z}} = \left\{\left(0, v_1 A, v_1 \frac{AB}{2} + v_2 B, -v_1 \frac{AB^t A}{6} - v_2 \frac{B^t A}{2} - v_3{}^t A\right)\right\}$$

where v_1, v_2, v_3 are integral.

We now define a composition series

$$G_3 \subset G_2 \subset G_1 \subset G$$

for G by taking the kernels of the mappings obtained sucessively by

$$\begin{cases} \left(0, \frac{mA}{\alpha}, w_3, w_4\right) \to [m] \in \mathbb{Z}/\alpha\mathbb{Z} \\ \quad (0, 0, w_3, w_4) \to w_3 / \operatorname{Im} B \\ \quad\ (0, 0, 0, w_4) \to [w_4] \in \mathbb{Z}/\alpha\mathbb{Z}\,. \end{cases}$$

Conclusion. G has a composition series with successive quotients $\mathbb{Z}/\alpha\mathbb{Z}$, $\mathbb{Z}^{h^{2,1}} / \operatorname{Im} B$, $\mathbb{Z}/\alpha\mathbb{Z}$. In particular

$$|G| = \alpha^2 \det B\,.$$

We then have the

Corollary. *Under the above assumption the group G is non-trivial.*

Example. $H_s = \mathrm{Sym}^3 H^1(E_s)$ where E_s is a family of elliptic curves with a singular fiber of type I_1 in Kodaira's list (cf. [Ko]) at s_0. One has cycles δ, γ spanning $H^1(E_s, \mathbb{Z})$ with $T(\delta) = \delta$, $T(\gamma) = \gamma + \delta$; it follows from $(T - I)^3(\gamma \times \gamma \times \gamma) = 6(\delta \times \delta \times \delta)$ that $|G| = 6$.

We shall now compute the group G without making the assumption that the limiting mixed Hodge structure is split, where now we shall assume that $h^{3,0} = h^{2,1} = 1$. Proposition II.C.3 in this case follows from Lemma 2 of [Mo2]. We have then

$$N = \begin{pmatrix} 0 & a & e & f \\ 0 & 0 & b & g \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the entries are rational numbers. From $NQ + Q\,{}^tN = 0$ we infer that

$$N = \begin{pmatrix} 0 & a & e & f \\ 0 & 0 & b & e \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

We shall show that

Proposition. *(i) a and b are integral, and (ii) G has a composition series successive quotients*

$$\mathbb{Z}/a\mathbb{Z},\ \ \mathbb{Z}/b\mathbb{Z},\ \ \mathbb{Z}/a\mathbb{Z}\ .$$

Proof. We compute that

$$T - I = \begin{pmatrix} 0 & a & \frac{ab}{2} + e & \frac{-ae}{2} - \frac{a^2b}{6} + f \\ 0 & 0 & b & -\frac{ab}{2} + e \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and it is an integral matrix. It follows that $\mathrm{Ker}(T - I) = \{(0,0,0,v)\}$

$$\begin{aligned} (\mathrm{Ker}(T-I)^{\perp})_{\mathbb{Z}} &= \{(0, v_1, v_2, v_3) : v_i \in \mathbb{Z}\}, \text{and} \\ \mathrm{Im}(T-I)_{\mathbb{Z}} &= \left\{ \left(0, aw_0, \left(e + \frac{ab}{2}\right) v_0 + bw_1, \left(f - \frac{ae}{2} - \frac{a^2b}{6}\right) w_0 \right.\right. \\ &\qquad \left.\left. + \left(e - \frac{ab}{2}\right) w_1 - aw_2\right) : w_i \in \mathbb{Z} \right\} . \end{aligned}$$

Comparison of the above expressions for $(\mathrm{Ker}(T-I)^{\perp})_{\mathbb{Z}}$ and $\mathrm{Im}(T-I)_{\mathbb{Z}}$ exhibits the composition series as indicated in the statement of the proposition. □

Remark. A somewhat more subtle matter is to relate e and f to the extension data for

$$\mathbb{Z}(-3) \to \mathbb{Z}(-2) \to \mathbb{Z}(-1) \to \mathbb{Z}(0)$$

and to tie this discussion into the boundary component structure of the Kato-Usui spaces. This will be done in a separate work.

III. Abel-Jacobi mappings for normal crossing varieties and degenerations of Abel-Jacobi mappings

III.A Abel-Jacobi mappings for normal crossing varieties

The material in this section is based partly on [KL, §8] and [KLM, §5]. We remark as in the introduction that we expect the moving lemmas in [KL, §8.2] (only proved there rationally) to be valid integrally. The presentation throughout section III.A-B (and IV.C) is done under this assumption.

In this paper a *complex analytic manifold* X will mean an analytic connected open subset of a smooth algebraic variety over $\mathbb{C}$. We shall alternate in the present section between X of a "general" dimension M and a "fixed" dimension $2n$, with results for X in the general case meant to apply to submanifolds of X in the latter. A *strict normal crossing divisor* (SNCD) on X is a union of irreducible codimension-1 submanifolds $Y = \cup Y_i \subset X$ satisfying:

(a) the Y_i are compact (hence are smooth complete algebraic varieties)

(b) [transversality] any point $y \in Y$ has an analytic open neighborhood $U \subset X$ with local holomorphic coordinates $(z_1, \ldots, z_M)$ such that $Y \cap U = \{z_1 \cdot \cdots \cdot z_k = 0\}$ for some $k(y)$; and

(c) the $Y_I := \cap_{i\in I} Y_i$ are smooth (not necessarily irreducible) for all multi-indices I.

The *depth* of a SNCD Y is the smallest $w \in \mathbb{Z}_{\geqslant 0}$ such that $Y_I = \emptyset\ \forall\, |I| > w+1$.

Our strong assumption that the Y_i meet transversely will specialize to a semistable degeneration in the variational setting, cf. (III.B.1)ff. In contrast, weaker intersection conditions are required below for cycles, chains, and currents on X (or on the Y_I) of "Y-intersection type". One cycle-group which plays a central role below is $Z^n_{\#}(Y) \subset \oplus_i Z^n(Y_i)$, with elements $Z = \{Z_i\}$, the "intersection cycles" on Y. The constituent ($\mathbb{C}$-codim.-n, algebraic) cycles $Z_i \in Z^n(Y_i)$ are required:

(a) to meet all strata $Y_I \subset Y$ properly (i.e., $\mathrm{codim}_{Y_i}(|Z_i| \cap Y_I) \geqslant n + |I| - 1$); and

(b) to "match up" on the $Y_{ij} = Y_i \cap Y_j$ ($Z_i \cdot Y_{ij} = Z_j \cdot Y_{ij}\ \forall i, j$).

(Here the intersection product "$\cdot$" is the standard one defined for properly intersecting cycles/subvarieties, cf. [Fu, Ex. 11.4.4]; *extensions* of this product to various currents are recalled below.) For $Z \in Z^n_{\#}(Y)$, we will define the fundamental class in *co*homology $[Z] \in H^{2n}(Y, \mathbb{Z}(n))$, and denote by $Z^n_{\#}(Y)_{\mathrm{hom}} \subset Z^n_{\#}(Y)$ the subgroup of those Z with $[Z] = 0$.

Using $\mathbb{C}$-coefficients unless otherwise specified, the *intermediate Jacobian* is defined to be $J^n(Y) = F^n H^{2n-1}(Y)\backslash H^{2n-1}(Y)/H^{2n-1}(Y, \mathbb{Z}(n))$. (In the classical case of [singular] curves, $J(Y)$ is termed the generalized Jacobian of Y. Thus, a more consistent terminology would be the generalized intermediate Jacobian; we shall omit the term "generalized".) It is known that there is a canonical identification $J^n(Y) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2n-1}(Y, \mathbb{Z}(n)))$ [Ca1]. In fact, by a well-known result of Deligne, the weights of $H^{2n-1}(Y, \mathbb{Z}(n))$ are in degree (-1) or less, and hence the given extension group is separated, meaning that it is the quotient of a complex vector space by a *discrete* subgroup. The existence of a weight filtration on $J^n(Y)$ (induced from that on $H^{2n-1}(Y)$, cf. III.A.28(b)) follows from [Ca2]. There is another canonical identification

$$J^n(Y) \cong \left(F^{-n+1} H_{2n-1}(Y)\right)^{\vee} \Big/ H^{2n-1}(Y, \mathbb{Z}(n)) \tag{III.A.1}$$

that will be central to the study of limiting Abel-Jacobi mappings.

There are three main purposes to this section (beyond setting the stage for III.B): the first two are

(i) *to define an Abel-Jacobi map* $Z^n_{\#}(Y)_{\mathrm{hom}} \to J^n(Y)$; *and*

(ii) *to interpret this map in "classical" terms — i.e., as integrals over chains — using the identification (III.A.1).*

This will be done via the following programme:

Step 1: construction of currents for the de Rham homology and cohomology of Y (together with the pairing);

Step 2: construction of chains (integral currents) for the Betti cohomology of Y;

Step 3: definition of the *motivic cohomology* groups $H^{2n+m}_{\mathcal{M}}(Y, \mathbb{Z}(n))$ via a complex of higher Chow precycles (from [KL]), and the map

$$Z^n_{\#}(Y) \to H^{2n}_{\mathcal{M}}(Y, \mathbb{Z}(n)); \tag{III.A.2}$$

Step 4: definition of the *Deligne cohomology* groups $H^{2n+m}_{\mathcal{D}}(Y, \mathbb{Z}(n))$ and (suitably adapting [KL]) the Deligne cycle-class map

$$H^{2n}_{\mathcal{M}}(Y, \mathbb{Z}(n)) \xrightarrow{c_{\mathcal{D}}} H^{2n}_{\mathcal{D}}(Y, \mathbb{Z}(n)); \tag{III.A.3}$$

Step 5: definition of the subgroups $H^{2n}_{\mathcal{M}}(Y, \mathbb{Z}(n))_{\text{hom}}$ and $Z^n_{\#}(Y)_{\text{hom}}$, and of $\mathrm{AJ}^n_Y : H^{2n}_{\mathcal{M}}(Y, \mathbb{Z}(n))_{\text{hom}} \to J^n(Y)$ (by restricting $c_{\mathcal{D}}$);

Step 6: description of AJ^n_Y in terms of improper integrals against integral currents; and

Step 7: specialization of this description to the map in (i) above — that is, to the image of $Z^n_{\#}(Y)_{\text{hom}}$ by (III.A.2).

The semisimplicial structure arising from a hyper-resolution of Y induces compatible weight filtrations on the terms of (III.A.3). We can now state our last main goal:

(iii) *to understand the weight-graded pieces of $c_{\mathcal{D}}$ in terms of regulator maps on higher Chow groups (and how this relates to the map in (i)).*

This is done in Step 5 (cf. (III.A.29)ff); also see the running example in Steps 5-7 and Theorem IV.4 (and its proof).

Step 1: Using the (mostly standard) notation

- $|I| = \ell + 1$ if $I = \{i_1, \ldots, i_{\ell+1}\}$
- $Y_I = \cap_{i \in I} Y_i$
- $Y^{[\ell]} = \amalg_{|I|=\ell+1} Y_I$
- $\jmath_{I,j} : Y_{I \cup \{j\}} \hookrightarrow Y_I$ the inclusion (for $j \notin I$)
- $\langle I \rangle_i :=$ position of i in I (in the sense of $\langle \{i_1, \ldots, i_m\} \rangle_{i_k} = k$ for $i_1 < \cdots < i_m$)
- $Y^I := \cup_{j \notin I} Y_{I \cup \{j\}} \subset Y_I$

we shall construct double-complexes of currents whose associated simple complexes compute homology and cohomology of Y (with $\mathbb{C}$-coefficients). A partial regularization of the homology chains is then necessary to pair them. The use of *currents* to represent *co*homology may seem perverse, but is necessary in order to receive the explicit AJ maps from motivic cohomology.

Denote by $A^r(X)$ the smooth degree-r forms on a complex analytic M-manifold X.

As a notational template for what follows, we shall recall the usual double-complex of C^∞ forms for computing cohomology of a SNCD in X from [GS p. 71], [KK pp. 151–153]:

$$A^{\ell,k}_Y(n) := A^{2n+k}(Y^{[\ell]}) = \oplus_{|I|=\ell+1} A^{2n+k}(Y_I) \tag{III.A.4}$$

with differentials $d_0 : A^{\ell,k}_Y(n) \to A^{\ell,k+1}_Y(n)$ resp. $d_1 : A^{\ell,k}_Y(n) \to A^{\ell+1,k}_Y(n)$ given by exterior differential d resp. "intersection differential" $\partial_{\mathfrak{I}} := \sum_{|I|=\ell+1} \sum_{j \notin I} (-1)^{\langle j \rangle_{I \cup \{j\}}} (\jmath_{I,j})^*$. The associated

Mayer-Vietoris spectral sequence degenerates at E_2, and converges to $H^*(A_Y^\bullet(n)) \cong H^{2n+*}(Y)$, where the total complex $A_Y^m(n) := \oplus_{\ell+k=m} A_Y^{\ell,k}(n)$ has differential $d_0 + (-1)^k d_1 = \underline{\underline{d}}$. (In general for our double complexes, the total differential is a doubly-underlined version of whatever d_0 is.) The "natural" weight filtration W_{-j} on $H^*(A_Y^\bullet(n))$ is defined to be the image of all $\underline{\underline{d}}$-cocycles with entries zero above the $(-j)^{\underline{\text{th}}}$ row; shifting this up by $2n$ gives the usual weight filtration on cohomology. Another important bit of general notation for double-complexes is

$$\langle\cdot\rangle_\ell : \; A_Y^{\ell,k}(n) \hookrightarrow A_Y^{\ell+k}(n); \tag{III.A.5}$$

given $\xi \in A_Y^{\ell,k}(n)$, $\langle\xi\rangle_\ell \in \oplus_{\ell'+k'=\ell+k} A_Y^{\ell',k'}(n)$ has by definition $(\ell',k')^{\underline{\text{th}}}$ entry $\left\{\begin{array}{ll} 0, & (\ell',k') \neq (\ell,k) \\ \xi, & (\ell',k') = (\ell,k) \end{array}\right.$.

Now let U be an open set in X. The currents $K \in D^r(U)$ on U are continuous linear functionals on compactly supported forms $\alpha \in A_c^{2M-r}(U)$, with the pairing frequently written $(K(\alpha) =) \int_U K \wedge \alpha$ and $dK \in D^{r+1}(U)$ defined compatibly with integration by parts. The (locally) normal currents $\mathcal{N}^r(U)$ are those K for which K, dK are Radon measures [Ki1, p. 43], and include the currents of integration δ_Γ over piecewise smooth Borel-Moore $(2N-m)$-chains Γ. The complexes $A^\bullet(U) \overset{\simeq}{\hookrightarrow} \mathcal{N}^\bullet(U) \overset{\simeq}{\hookrightarrow} D^\bullet(U)$ compute $H^*(U)$ and sheafify to complexes of fine sheaves $A_X^\bullet \overset{\simeq}{\hookrightarrow} \mathcal{N}_X^\bullet \overset{\simeq}{\hookrightarrow} D_X^\bullet$ with hypercohomology groups $H^*(X)$; in both cases the quasi-isomorphisms are Hodge-filtered.

If $W \overset{\imath}{\subset} X$ is a (complex) codimension c submanifold, then there is a natural Gysin (push-forward) map of complexes $\imath_* : \mathcal{N}^\bullet(W) \to \mathcal{N}^{\bullet+2c}(X)$ (likewise for $D^\bullet$) inducing $H^*(W) \xrightarrow{\imath_*} H^{*+2c}(X)$. (In general, push-forwards of currents are defined simply as the adjoint of pullback on C^∞ forms.) Currents cannot in general be restricted (there is no $\imath^*$); however, there is a (filtered-)quasi-isomorphic subcomplex $\mathcal{N}^\bullet\{W\}(X) \subset \mathcal{N}^\bullet(X)$ of "W-intersection-type" normal currents which are "current-transversal" to W and *do* admit pullback (compatibly with d). Roughly speaking, if W is cut out locally by $y=0$, $K \wedge \frac{dy}{y}$ and $dK \wedge \frac{dy}{y}$ must be locally L^1; in $\mathbb{C}^2$, the current of integration over (say) $y^2 - x^4 = 0$ is current-transversal to the x-axis, so it is a weaker notion than the usual "transversality".

In fact the story we will review is that where W is replaced by a SNCD $Y = \cup Y_i \subset X$ (with $\jmath_i : Y_i \hookrightarrow X$). We shall use the notation $A^\bullet(X, \log\langle Y\rangle)$ for the C^∞ log complex, which computes $H^*(X\backslash Y)$ and which (as global sections of fine sheaves) it suffices to describe locally: if $U \cap Y = \{z_1 \cdot \dots \cdot z_k = 0\}$ then $A^\bullet(U, \log\langle U\cap Y\rangle) = \sum_I A^{\bullet-|I|}(U) \otimes \frac{dz_I}{z_I}$ where $\frac{dz_I}{z_I} = \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_{|I|}}}{z_{i_{|I|}}}$; the exterior derivative induced from d on the smooth part is denoted "d" (quotes included).

The definition of intersection currents is complicated and readers after the main ideas may wish to skim from here to (III.A.9). Let $\psi_I :=$characteristic function of $X\backslash Y_I$, and $\lambda_p^{(I)}$ be a sequence of smooth "cutoff functions" (compactly supported on $X\backslash Y_I$) limiting to ψ_I as in [Ki2, p. 192]. For $K_0 \in \mathcal{N}^r(X\backslash Y_I)$ admitting a simple extension $\widetilde{K_0} \in D^r(X)$ (cf [op.cit.]), we have (for $\alpha \in A_c^{2M-r}(X)$) $\widetilde{K_0}(\alpha) = \lim_{p\to\infty} K_0(\lambda_p^{(I)}\alpha)$. By "multiplication by ψ_I" we shall mean the map $\mathcal{N}^r(X) \to D^r(X)$ [well-]defined by $\psi_I K := (\widetilde{K|_{X\backslash Y_I}})$ (cf. [Ki1, p. 43]). $K \in \mathcal{N}^r(X)$ is of *Y_I-residue-type* if $\psi_I K = K$ and $\psi_I dK \in \mathcal{N}^{r+1}(X)$, and *$Y_I$-transversal* if also $\psi_I dK = dK$; for K of Y_i-residue-type, $\mathrm{Res}_{Y_i}(K) \in \mathcal{N}^{r-1}(Y_i)$ is defined by $2\pi\sqrt{-1}(\jmath_i)_* \mathrm{Res}_{Y_i}(K) := dK - \psi_i dK$. There is no closedness assumption for taking residues.

Now write $s_i(K) := (\widetilde{K|_{X\backslash Y_i} \wedge \frac{dz_i}{z_i}})$, if this exists; a Y_i-transversal K is of *Y_i-intersection-type*, i.e. $K \in \mathcal{N}^r\{Y_i\}(X)$, if $s_i(K)$ and $s_i(dK)$ exist and are normal of Y_i-residue-type. (Note that in this case, $s_i(K)$ is the unique normal current satisfying $K \wedge dz_i = z_i \cdot s_i(K)$.) The intersection $\jmath_i^* : \mathcal{N}^r\{Y_i\}(X) \to \mathcal{N}^r(Y_i)$ is then defined by $\jmath_i^* K = \mathrm{Res}_{Y_i}(s_i(K))$, or $(\jmath_i)_*\jmath_i^* K = d[s_i(K)] - s_i(dK)$. We refer to [KL, Def. 8.5] for the full definition of Y-intersection-type currents $K \in \mathcal{N}^r\{Y\}(X)$. Three main points are that

(a) there are normal currents S_I (resp. $\mathfrak{S}_I$) [e.g., $S_i = s_i(K)$] in $\mathcal{N}^{r+|I|(\text{resp.}+1)}\{\cup_{j\notin I} Y_j\}(X)$ playing the role of $K \wedge \frac{dz_I}{z_I}$ (resp. $dK \wedge \frac{dz_I}{z_I}$),

(b) $\jmath_i^* K = \mathrm{Res}_{Y_i}(S_i)$ defines a map $\jmath_i^* : \mathcal{N}^r\{Y\}(X) \to \mathcal{N}^r\{Y^i\}(Y_i)$, and

(c) $\mathcal{N}^\bullet\{Y\}(X) \hookrightarrow \mathcal{N}^\bullet(X)$ is an $F^\bullet$-filtered quasi-isomorphism [KL, Lemma 8.7].

We will also need the following

Lemma III.A.6. *(i) For $K \in \mathcal{N}^r\{Y\}(X)$, $\jmath_{i,j}^* \jmath_i^* K = \jmath_{j,i}^* \jmath_j^* K$.*

(ii) There is a natural map $\mathcal{N}^r\{Y\}(X) \otimes A^s(Y, \log\langle \cup_{i\in I} Y_i\rangle) \xrightarrow{\wedge} \mathcal{N}^{r+s}\{\cup_{j\notin I} Y_j\}(X)$ inducing a pairing

$$\mathcal{N}^r\{Y\}(X) \otimes A^{2M-r}(X, \log\langle Y\rangle) \xrightarrow{\int\circ\wedge} \mathbb{C},$$

and we have ($j \notin I$) $\mathrm{Res}_{Y_j}(K\wedge\alpha) = \jmath_j^* K \wedge \mathrm{Res}_{Y_j}(\alpha)$.

Proof. (i) By injectivity of the push-forwards it suffices to check

$$\jmath_{i*}\jmath_i^*\jmath_{j*}\jmath_j^* K = \jmath_{j*}\jmath_j^*\jmath_{i*}\jmath_i^* K. \tag{III.A.7}$$

First, we claim that $s_j(S_i) = S_{ij}$, again referring to [KL Def. 8.5]. For 2 normal currents A, B of Y_i-residue-type satisfying $z_i A = z_i B$, we have $A = B$, as $(A-B)(\alpha) = \lim_{p\to\infty}(A-B)(\lambda_p^i \alpha) = \lim_{p\to\infty}[z_i(A-B)]\left(\frac{\lambda_p^i}{z_i}\alpha\right) = \lim_{p\to\infty} 0 = 0$. Since $z_i \cdot (S_i \wedge dz_j) = z_i \cdot (S_{ij} z_j)$, $S_{ij} z_j = S_i \wedge dz_j = \psi_j S_i \wedge dz_j = \widetilde{S_i|_{X\setminus Y_j}} \wedge dz_j = z_j \cdot \left(\widetilde{S_i|_{X\setminus Y_j}} \wedge \frac{dz_j}{z_j}\right) = z_j s_j(S_i) \implies S_{ij} = s_j(S_i)$.

Next, we claim that S_j and $\mathfrak{S}_j$ (cf [op. cit.]) are of Y_i-intersection-type. The required Y_i-transversality follows from [op. cit., 8.5(b)(iii)], and (using arguments similar to the above)

$$s_j(S_i) = S_{ij},\ s_j(dS_i) = -\psi_j dS_{ij},\ s_j(\mathfrak{S}_i) = \mathfrak{S}_{ij},\ s_j(d\mathfrak{S}_i) = \psi_j d\mathfrak{S}_{ij}$$

resp.

$$s_i(S_j) = -S_{ij},\ s_i(dS_j) = -\psi_i dS_{ij},\ s_i(\mathfrak{S}_j) = -\mathfrak{S}_{ij},\ s_i(d\mathfrak{S}_j) = -\psi_i d\mathfrak{S}_{ij}$$

are of Y_i- resp. Y_j-residue-type. Now (III.A.7) becomes

$$(1-\psi_i)d[s_i(dS_j - \mathfrak{S}_j)] = (1-\psi_j)d[s_j(dS_i - \mathfrak{S}_i)];$$

in fact both sides of this are supported on Y_{ij}, and we have

$$(1-\psi_{ij})d[s_i(dS_j) - s_i(\mathfrak{S}_j)] = (1-\psi_{ij})d[s_j(dS_i) - s_j(\mathfrak{S}_i)]. \tag{III.A.8}$$

(Note that $1 - \psi_i - \psi_j + \{\psi_i\psi_j\} = 1 - \psi_{ij}$, in the sense that this holds for the cutoff functions, where $\{\psi_i\psi_j\}$ means to take simultaneous rather than successive limits.) Subtracting and using the formulas above, it remains to check that

$$(1-\psi_{ij})d[2\mathfrak{S}_{ij} - \psi_i dS_{ij} - \psi_j dS_{ij}]$$

is zero. Now dS_{ij} (using [op. cit., 8.5(b)(iv)]) is invariant under multiplication by ψ_{ij}, so that

$$(\psi_i + \psi_j)dS_{ij} = \{\psi_i\psi_j\}dS_{ij} + dS_{ij} = \mathfrak{S}_{ij} + dS_{ij};$$

moreover, $ddS_{ij} = 0$. So (III.A.8) becomes

$$(1-\psi_{ij})d\mathfrak{S}_{ij},$$

where again KL8.5(b)(iv) says $\mathfrak{S}_{ij}$ is Y_{ij}-transversal, in particular $d\mathfrak{S}_{ij} = \psi_{ij} d\mathfrak{S}_{ij}$; done.

(ii) Working locally and writing $\alpha = \sum_I \frac{dz_I}{z_I} \wedge \alpha_I$ with $\alpha_I \in A^{s-|I|}(X)$, we set $K \wedge \alpha := \sum_I S_I \wedge \alpha_I = \sum_{I \ni j} S_I \wedge \alpha_I + \sum_{\substack{I \\ j\notin I}} S_I \wedge \alpha_I$ where the $j \notin I$ term is Y_j-transversal (j is fixed).

Then

$$\jmath_{j*}\operatorname{Res}_{Y_j}(K\wedge\alpha)=\jmath_{j*}\sum_{I\ni j}(-1)^{\langle j\rangle_I}\operatorname{Res}_{Y_j}\left(S_j\wedge\frac{dz_{I\setminus j}}{z_{I\setminus j}}\wedge\alpha_I\right)$$
$$=\jmath_{j*}\left\{\jmath_j^*K\wedge\left(\sum_{I\ni j}(-1)^{\langle j\rangle_I}\frac{dz_{I\setminus j}}{z_{I\setminus j}}\wedge\jmath_j^*\alpha_I\right)\right\};$$

details are left to the reader. □

In view of Lemma III.A.6(i) we may unambiguously define $\jmath_I^*:\mathcal{N}^r\{Y\}(X)\to\mathcal{N}^r\{Y^I\}(Y_I)$ by iteration of pullbacks (in any order).

Now take $M=2n(=\dim(X))$. We are ready to define the intersection-current analogue of (III.A.4). Writing $D^r_{\#}(Y_I):=\mathcal{N}^r\{Y^I\}(Y_I)$ we set

$$D_Y^{\ell,k}(n):=\oplus_{|I|=\ell+1}D_{\#}^{2n+k}(Y_I),\quad d_0=d,\ d_1=\partial_{\mathfrak{I}},\tag{III.A.9}$$

with Hodge filtration $F^pD_Y^{\ell,k}(n)=\oplus F^pD_{\#}^{2n+k}(Y_I)$. That the total (simple) complexes compute $H^{2n+*}(Y)$ and $F^pH^{2n+*}(Y)$ (resp.) follows again from the Mayer-Vietoris s.s. and strict compatibility of all maps with $F^\bullet$.

Ultimately we are going to represent (a lift of) AJ of motivic cohomology of Y by $\underline{\underline{d}}$-cocycles in the total complex of $D_Y^{\ell,k-1}(n)$, and view these as functionals on homology of Y (cf. (III.A.1)). To that end, if we put

$$[F^{-p}]A^Y_{\ell,k}(\log)(-n):=\oplus_{|I|=\ell+1}[F^{2n-p-\ell-1}]A^{2n-2\ell-k-1}\left(Y_I,\log\left\langle Y^I\right\rangle\right)\tag{III.A.10}$$

then by Lemma III.A.6(ii) there are pairings $D_Y^{\ell,k-1}(n)\otimes A^Y_{\ell,k}(\log)(-n)\xrightarrow{\int\circ\wedge}\mathbb{C}$. The catch is that, while we *shall* write $[F^{-p}]A^Y_m(\log)(-n)$ for $\oplus_{\ell+k=m}$ of it, (III.A.10) is *not* a bicomplex (and though one could define at least a vertical differential, we will not).

Instead, we include it in one, as follows. For ($\dim_{\mathbb{C}}X=$)M arbitrary and any $\omega\in A^q(X,\log\langle Y\rangle)$, $\eta\in A^{2M-q}(X)$, the integral $\int_X\omega\wedge\eta$ is absolutely convergent, and this leads to a Hodge-filtered inclusion $A^q(X,\log\langle Y\rangle)\hookrightarrow D^q(Y)$. This is *not* a map of complexes, since $d\omega=$ "d"$\omega+2\pi\sqrt{-1}\sum_i\jmath_{i*}\operatorname{Res}_{Y_i}(\omega)$ (where d is the differential on currents). Now taking $M=2n$, consider the *homological* bicomplex ($F^\bullet$ same as in (III.A.10))

$$D^Y_{\ell,k}(-n):=\oplus_{|I|=\ell+1}D^{2n-2\ell-k-1}(Y_I),\quad d_0=d,\ d_1=Gy\tag{III.A.11}$$

where $Gy=2\pi\sqrt{-1}\sum_{|I|=\ell}\sum_{i\in I}(-1)^{\langle i\rangle_I}(\jmath_{I\setminus\{i\},i})_*$, both d_i *decrease* indices, and $\underline{\underline{d}}:=(-1)^k\{d_0+d_1\}$. Then $[F^*]A^Y_{\ell,k}(\log)(-n)\subset[F^*]D^Y_{\ell,k}(-n)$ (and $[F^*]A^Y_m(\log)(-n)\subset[F^*]D^Y_m(-n)$) are sub*spaces*; since there is no Gysin for C^∞ log forms (and as before "d" and d disagree) we shall leave it at that.

The main point is that the total complex of (III.A.11) computes homology of Y

$$H_*(F^{-p}D^Y_\bullet(-n))\cong F^{-p}H_{(2n-1)+*}(Y,\mathbb{C}),\tag{III.A.12}$$

and we want to be able to partially regularize $\underline{\underline{d}}$-cycles (representing classes in (III.A.12)) so that they pair with representatives of cohomology.

Proposition III.A.13. *(i) Every $K\in\ker(\underline{\underline{d}})\subset F^{-n+1}D_0^Y(-n)$ may, by addition of a ($\underline{\underline{d}}$-)boundary, be moved to lie in the subspace $\ker(\underline{\underline{d}})\cap F^{-n+1}A_0^Y(\log)(-n)$.*

(ii) Writing $\omega=\{\omega^{[\ell]}\}_{\ell\geqslant0}=\{\omega_I\}_{\substack{\ell\geqslant0\\|I|=\ell+1}}\in\ker(\underline{\underline{d}})\cap A_0^Y(\log)(-n)$ and $\kappa=\{\kappa^{[\ell]}\}\in\{\ker(\underline{\underline{d}})\subset D_Y^{-1}(n)\}$ for representatives of classes in $H_{2n-1}(Y)$ and $H^{2n-1}(Y)$ (resp.), the "summing along the

diagonal" formula $\langle\kappa,\omega\rangle := \sum_{\ell\geqslant 0}(-2\pi\sqrt{-1})^{\ell}\sum_{|I|=\ell+1}\int_{Y_I}\kappa_I\wedge\omega_I$ *induces a well-defined pairing.*

Notation here is that e.g. $\kappa^{[\ell]}=\{\kappa_I\}_{|I|=\ell+1}$ is the entry of κ in $D^Y_{\ell,-\ell}(-n)$; also write $\langle\cdot\rangle_\ell$ for the inclusion of $D^Y_{\ell,-\ell}(-n)\hookrightarrow D^Y_0(-n)$. We need the following

Lemma III.A.14. *Let $W=\cup W_\gamma\subset Y_I$ be a SNCD, and let $\Omega_\gamma\in F^{p-1}A^{m-1}(W_\gamma,\log\langle W^\gamma\rangle)$ be such that (considered as a current) $d\sum_\gamma \iota_{\gamma*}\Omega_\gamma=0$. (That is, $\mathrm{Res}_{\gamma_1}\Omega_{\gamma_2}+\mathrm{Res}_{\gamma_2}\Omega_{\gamma_1}=0$ on $W_{\gamma_1\gamma_2}$ $\forall\gamma_1\neq\gamma_2$.) Then $\exists\ \Omega\in F^pA^m(Y_I,\log\langle W\rangle)$ which (viewed as a current) has $d\Omega=2\pi\sqrt{-1}\sum_\gamma\iota_{\gamma*}\Omega_\gamma$ mod $A^{m+1}(Y_I)$.*

Proof. Let $\{B_\beta\}$ be a system of polydisks $\subset Y_I$ covering W, and let $\{\eta_\beta\}$ be C^∞ forms giving a partition of unity on W and falling off to 0 outside W. For some β let $U=B_\beta$, so that $W\cap U=:V=\cup V_j=\cup\{z_j=0\}$ in local holomorphic coordinates.

For each j write $\Omega_j=\sum_{J\ni j}(-1)^{\langle j\rangle_J}\alpha_j^J\wedge\frac{dz_{J\setminus j}}{z_{J\setminus j}}$ for $\alpha_j^J\in A^{0,m-|J|}(V_j)$ (with $\alpha_j^J=0$ for $|J|<p$). Now on V_{jk}, $0=\mathrm{Res}_k\Omega_j+\mathrm{Res}_j\Omega_k=\sum_{J\ni j,k}(-1)^{\langle j\rangle_J+\langle k\rangle_{J\setminus j}}\left(\iota_{jk}^*\alpha_j^J-\iota_{kj}^*\alpha_k^J\right)\wedge\frac{dz_{J\setminus\{j,k\}}}{z_{J\setminus\{j,k\}}}\implies$ $\iota_{jk}^*\alpha_j^J=\iota_{kj}^*\alpha_k^J=:\alpha_{jk}^J$ $\forall$ J and $j,k\in J$. From this it is clear that for $j\in J_0\subset J$, setting $\alpha_{J_0}^J:=\iota_{J_0}^*\alpha_j^J$ is independent of the choice of $j\in J$; and that (writing $\pi_{J_0}:(B_\beta\cong)\Delta^q\twoheadrightarrow\Delta^{q-|J_0|}$ for the projection killing $\underline{z}_{J_0}$) $\alpha^J:=\sum_{\mu\geqslant 0}(-1)^\mu\sum_{|J_0|=\mu+1}\pi_{J_0}^*\alpha_{J_0}^J$ has $\iota_j^*\alpha^J=\alpha_j^J$ $\forall j$.

Setting $\Omega^\beta:=\sum_J\alpha^J\wedge\frac{dz_J}{z_J}$ then solves the local problem, and $\sum\eta_\beta\Omega^\beta$ the global problem. □

Proof of III.A.13. (i) Let $\ell\geqslant 0$. By an "$(\ell+1)$-log-smooth 0-cycle" we mean

$$K=\{K^{[0]},K^{[1]},\ldots,K^{[\ell]};\omega^{[\ell+1]},\ldots,\omega^{[2n-1]}\}\in\ker(\underline{\underline{d}})\subset F^{-n+1}D_0^Y(-n),$$

with all ω's C^∞ log. It is enough to produce $\Gamma^{[\ell]}\in F^{n-\ell}D^{2n-\ell-2}(Y^{[\ell]})$ s.t. $K-\underline{\underline{d}}\left\langle\Gamma^{[\ell]}\right\rangle_\ell$ is ℓ-log-smooth. Since each $dK_I=2\pi\sqrt{-1}\sum_{j\notin I}(-1)^{\langle j\rangle_{I\cup\{j\}}}(\jmath_{I,j})_*\omega_{I\cup\{j\}}$ and $ddK_I=0$, Lemma III.A.14 applies and $\exists\ \omega_I\in F^{n-\ell}A^{2n-\ell-1}(Y_I,\log\langle Y^I\rangle)$ with $d[K_I-\omega_I]\in F^{n-\ell}A^{2n-\ell}(Y_I)$ $\forall|I|=\ell+1$. By acyclicity of $\frac{F^{n-\ell}D^\bullet(Y_I)}{F^{n-\ell}A^\bullet(Y_I)}$, $\exists\ \Gamma_I\in F^{n-\ell}D^{2n-\ell-2}(Y_I)$ and $\Omega_I\in F^{n-\ell}A^{2n-\ell-1}(Y_I)$ s.t. $K_I-\omega_I=d[\Gamma_I]+\Omega_I$; that is,

$$\underline{\underline{d}}\left\langle\{\Gamma_I\}\right\rangle_\ell=\langle\{K_I\}\rangle_\ell-\langle\{\Omega_I+\omega_I\}\rangle_\ell\pm\left\langle Gy(\Gamma^{[\ell]})\right\rangle_{\ell-1},$$

and we are done.

(ii) (0-)log-smooth 0-cycles which yield trivial homology classes can be written as $\underline{\underline{d}}$ of elements of $A^Y_{-1}(-n)$ (no log poles). The well-definedness is then a messy but straightforward exercise; left to reader. □

Here is a useful statement characterizing log-smooth representatives of homology classes.

Observation. Any collection $\{\omega_I\}\in F^{-n+1}A_0^Y(\log)(-n)$ which is $\underline{\underline{d}}$-closed (in $D^Y_\bullet(-n)$) satisfies

$$\omega_J=\pm\,\mathrm{Res}_{J\setminus\{i\}}(\omega_i)\quad\forall i\in J;\tag{III.A.15}$$

that is, the ω_J for $|J|\geqslant 2$ are determined by the ω_i on the irreducible components of Y. In fact, given $\{\omega_i\in A^{2n-1}(Y_i,\log\langle Y^i\rangle)\}$ "d"-closed and satisfying merely $\mathrm{Res}_j\,\omega_i+\mathrm{Res}_i\,\omega_j=0$, (III.A.15) produces such a collection $\{\omega_I\}$.

Proof. Viewed as currents, they must satisfy $d[\omega_I]=2\pi\sqrt{-1}\sum_{j\notin I}(-1)^{\langle j\rangle_{I\cup\{j\}}}(\jmath_{I,j})_*\omega_{I\cup\{j\}}$; clearly this forces "$d$"$\omega_I=0$. So $\sum_{j\notin I}(-1)^{\langle j\rangle_{I\cup\{j\}}}(\jmath_{I,j})_*\mathrm{Res}_j\,\omega_I=\frac{1}{2\pi\sqrt{-1}}\{d\omega_I-\text{“}d\text{”}\omega_I\}$ implies $\mathrm{Res}_j\,\omega_I=\omega_{I\cup\{j\}}$, and (III.A.15) follows from $\mathrm{Res}_{I=\{i_1,\ldots,i_k\}}=\mathrm{Res}_{i_1}\circ\cdots\circ\mathrm{Res}_{i_k}$. Indeed, (III.A.15) is clearly (together with "d"-closedness) a sufficient condition for $\underline{\underline{d}}$-closedness; hence (using the well-known commutativity of "d" and Res) the last statement. □

Remark. For any $\alpha \in A^r(X, \log\langle Y\rangle)$, $\mathrm{Res}_{Y_{ij}}(\mathrm{Res}_{Y_i}(\alpha)) = -\mathrm{Res}_{Y_{ij}}(\mathrm{Res}_{Y_j}(\alpha))$; α need not be closed. [It is enough (by continuity) to check this in a neighborhood of a point $y \in Y_i \cap Y_j$ where $z_i z_j = 0$ defines Y: if $\alpha = \alpha_0 + \alpha_i \wedge \frac{dz_i}{z_i} + \alpha_j \wedge \frac{dz_j}{z_j} + \alpha_{ij} \wedge \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$ ($\alpha_0, \alpha_i, \alpha_j, \alpha_{ij}$ locally smooth) then both sides equal $(-1)^{\deg(\alpha)} \jmath_{ij}^* \alpha_{ij}$.] So putting $\omega_i = \mathrm{Res}_{Y_i}(\alpha)$ for such an α is one way to get $\{\omega_i\}$ with compatible residues as above.

Step 2: Assume that all the integral cohomologies of the $Y^{[\ell]}$ are torsion-free (otherwise the construction here is valid $\otimes\mathbb{Q}$). We shall construct a double-complex out of certain generalized chains on the $Y^{[\ell]}$ computing the integral cohomology of Y. The chains must admit $(\jmath_{I,j})$-pullbacks (i.e., intersections) for this to work, and at first glance piecewise C^∞ chains on Y_I with a Y^I-transversality condition (in the sense of differential topology) would seem to suffice. However, in subsequent steps we will need these chains to include (for example) all algebraic cycles (on Y_I) properly intersecting (all substrata of) the Y^I — so we must admit chains with intersections "in the right codimension but to arbitrary order" still producing chains of the given class. While this can certainly be done for C^∞ chains, it appears to be more natural (especially for the proof of Theorem III.B.5) to use the already extant theory of *intersection integral currents*.

For an open set U as in Step 1, the (locally) integral currents $\mathcal{I}^r(U)$ are those $T \in \mathcal{N}^r(U)$ such that T, dT can be locally approximated (with respect to a natural metric on normal currents) by integration over Lipschitz chains (cf. [Ki3, p. 193] for details). They include C^∞ Borel-Moore[1] $(2M-r)$-chains and, for r even, algebraic cycles (i.e., integration over these; we shall frequently suppress the distinction); and one has $\mathbb{H}^*(X, \mathcal{I}_X^\bullet) \cong H^*(\mathcal{I}^\bullet(X)) \cong H^{BM}_{2M-*}(X,\mathbb{Z}) \cong H^*(X,\mathbb{Z})$. The pairing of $\omega \in A_c^{2M-r}(X)$ and $\Gamma \in \mathcal{I}^r(X)$ will be written in one of several equivalent forms: $\Gamma(\omega) = \int_\Gamma \omega = \int_X \Gamma \wedge \omega$; if they are not of complementary degree then $\Gamma \wedge \omega$ is a normal current. For integral currents we write ∂ in lieu of d.

Referring to the definition of normal currents of intersection type with respect to a SNCD $Y \subset X$ [KL, Def. 8.5], we define the subsheaf $\mathcal{I}_X^r\{Y\} \subset \mathcal{I}_X^r \cap \mathcal{N}_X^r\{Y\}$ by requiring in addition that for $\Gamma \in \mathcal{I}_X^r\{Y\}(U)$

$$\begin{array}{ll} \text{(a)} & \text{[support condition] } \mathrm{supp}(\Gamma) \cap Y_I = \emptyset \ \forall I \text{ with } 2|I| + r > 2M, \\ & \mathrm{supp}(\partial\Gamma) \cap Y_I = \emptyset \ \forall I \text{ with } 2|I| + r + 1 > 2M, \text{ and} \\ \text{(b)} & \text{[intersection condition] } \jmath_I^*\Gamma \text{ belongs to } \mathcal{I}_{Y_I}^r\{Y^I\}(U \cap Y_I) \\ & \text{(as opposed to merely } \mathcal{N}_{Y_I}^r\{Y^I\}(U \cap Y_I)). \end{array} \tag{III.A.16}$$

We will sometimes write $Y_I \cdot \Gamma$ for $\jmath_I^*\Gamma$.

Lemma III.A.17 (Poincaré Lemma). $\mathcal{I}_X^\bullet\{Y\} \overset{\simeq}{\hookrightarrow} \mathcal{I}_X^\bullet$.

Proof. Essentially the same as the proof of [Ki2, Lemma 7.12], with an induction on the depth of the SNCD, and with $\Delta^m\{\cup_{i=1}^m \{z_i = 0\}\}$ (the braces enclose what one must be current-transversal to) replacing $\Delta^m\{\underline{0}\}$. The induction obtains from noticing that a compact-supported ∂-closed integral current of degree $2m$ is 0; while the support of one of lesser degree cannot (by condition (a)) intersect the origin. Hence it is compactly supported on $\Delta^m \setminus \{\underline{0}\}$ and the intersection divisor is now of depth $r-1$. Applying the induction, the current is equivalent (in the intersection complex, modulo image of ∂) to 0, a point, or the sphere S^{2m-1} on $\Delta^m \setminus \{\underline{0}\}$; on Δ^m this is ∂B^r (and B^r is of intersection type). □

[1] i.e. not necessarily compactly supported (they are functionals on compactly supported forms)

The groups $C^r_{\#}(Y_I) = C^r_{\#}(Y_I;\mathbb{Z}) := \mathcal{I}^r\{Y^I\}(Y_I)$ will be referred to as the "intersection cochains"; we also define $C^r_{\#}(X) := \mathcal{I}^r\{Y\}(X)$ and

$$C^r_{\#}(Y) := \ker\{\oplus_i C^r_{\#}(Y_i) \xrightarrow{\partial_{\mathfrak{I}}} \oplus_{i<j} C^r_{\#}(Y_{ij})\}.$$

The latter are collections of (not necessarily ∂-closed) "chains" $\{\Gamma_i\}$ satisfying $\Gamma_i \cdot Y_{ij} = \Gamma_j \cdot Y_{ij}$ $\forall i,j$. From the Mayer-Vietoris spectral sequence and Lemma III.A.17, it follows that

$$B^{\ell,k}_Y(n) := \oplus_{|I|=\ell+1} C^{2n+k}_{\#}(Y_I;\mathbb{Z}(n)), \quad d_0 = \partial,\ d_1 = \partial_{\mathfrak{I}} \tag{III.A.18}$$

has $H^*(B^\bullet_Y(n), \underline{\underline{\partial}}) \cong H^{2n+*}(Y;\mathbb{Z}(n))$ (where $\mathbb{Z}(n) := (2\pi\sqrt{-1})^n\mathbb{Z}$).

We need a technical result on cochains $\Gamma = \{\Gamma^{[0]}, \Gamma^{[1]}, \ldots, \Gamma^{[2n-1]}\}$ in the total complex (the main points of which are (iv),(v),(vi) below). Let $\Delta^N =: U \supset' V = \cup_{i=1}^{k'} V_i = \cup_{i=1}^{k'}\{z_i = 0\} \supset V = \cup_{i=1}^{k} V_i$ for $k \leqslant k' \leqslant N$, and $\mathcal{I}^\ell_{\text{loc}}(q,N,k',k) := \oplus_{\substack{|I| = \ell+1 \\ I \subset \{1,\ldots,k\}}} \Gamma_c(V_I, \mathcal{I}^q_{V_I}\{'V^I\})$ with intersection differential $\partial_{\mathfrak{I}}$ (on $\mathcal{I}^\bullet_{\text{loc}}(q,N,k',k)$).

Lemma III.A.19. *(i)* $\Gamma_c(U, \mathcal{I}^q_U\{'V\}) \twoheadrightarrow \{\ker(\partial_{\mathfrak{I}}) \subset \mathcal{I}^0_{loc}(q,N,k',k)\}$

(ii) $H^i(\mathcal{I}^\bullet_{loc}(q,N,k',k)) = \{0\}$*,* $i \neq 0$

(iii) $H^i(B^{\bullet,q}_Y(n)) = \{0\}$ *for* $i \neq 0$

(iv) $C^{2n+q}_{\#}(X;\mathbb{Z}(n)) \twoheadrightarrow C^{2n+q}_{\#}(Y,\mathbb{Z}(n)) = \{\ker(\partial_{\mathfrak{I}}) \subset B^{0,q}_Y(n)\}$

(v) Any 0*-cocycle* γ *(in* $B^\bullet_Y(n)$*) can be moved into* $\{\ker(\partial) \cap \ker(\partial_{\mathfrak{I}}) \subset B^{0,0}_Y(n)\} = \{\ker(\partial) \subset C^{2n}_{\#}(Y;\mathbb{Z}(n))\}$

(vi) Any (-1)*-cochain* Γ *(in* $B^\bullet_Y(n)$*) bounding on a* 0*-cocycle* γ *in* $B^{0,0}_Y(n)$ *(*$\underline{\underline{\partial}}\Gamma = \gamma$*) can be moved into* $\{\ker(\partial_{\mathfrak{I}}) \subset B^{0,-1}_Y(n)\} = C^{2n-1}_{\#}(Y;\mathbb{Z}(n))$*.*

Proof. The main issues are (i) and (ii). Once these are known, (iii) and (iv) follow from (resp.) (ii) and (i) [by taking a cover of (X,Y) with each neighborhood homeomorphic to a situation of the form (U,V)] as the sheaves of integral currents are fine. (v) and (vi) are immediate consequences of (iii).

(i) We have $\Gamma_i \in \Gamma_c(V_i, \mathcal{I}^q_{V_i}\{'V^i\})$ $(i = 1,\ldots,k)$ with $\jmath_j^*\Gamma_i = \jmath_i^*\Gamma_j$ in each V_{ij}. Suppose that Γ_1 thru Γ_{m-1} are identically zero; then (writing $\boxtimes$ for exterior product) $\tilde{\Gamma}_m := z_m^{-1}([-\epsilon,\epsilon]) \boxtimes \Gamma_m \in \Gamma_c(U, \mathcal{I}^q_U\{'V\})$ has $\jmath_m^*\tilde{\Gamma}_m = \Gamma_m$, and $\jmath_i^*\tilde{\Gamma}_m = z_m^{-1}([-\epsilon,\epsilon]) \boxtimes \jmath_i^*\Gamma_m = z_m^{-1}([-\epsilon,\epsilon]) \boxtimes \jmath_m^*\Gamma_i = 0$ for $i < m$. A simple induction completes the proof.

(ii) We induce on k (with $N \geqslant k' \geqslant k$). Define a map $\mathcal{I}^\bullet_{\text{loc}}(q,N,k',k) \overset{\varepsilon_{\{k\}}}{\twoheadrightarrow} \mathcal{I}^\bullet_{\text{loc}}(q,N,k',k-1)$ of complexes by forgetting all indices I *containing* $\{k\}$. Given $\Gamma^{[\ell]} \in \ker(\partial_{\mathfrak{I}})$, by induction we have $\varepsilon_{\{k\}}(\Gamma^{[\ell]}) = \partial_{\mathfrak{I}}(\xi^{[\ell-1]})$. Writing $\tilde{\xi}^{[\ell-1]}$ for the lift of $\xi^{[\ell-1]}$ obtained by taking $\tilde{\xi}_{I_0} = 0$ $\forall I_0 \ni k$, $(\Gamma^{[\ell]} - \partial_{\mathfrak{I}}\tilde{\xi}^{[\ell-1]})_I = 0$ for [all I with] $k \notin I$. The problem therefore drops to the *sub*complex $\mathcal{I}^\bullet_{\text{loc}}(q,N-1,k'-1,k-1)[-1]$ if $\ell > 1$ (the $[-1]$ replacing ℓ by $\ell-1$, and Δ^N replaced by V_k). Eventually $\ell = 1$, and we are reduced to the "base case" (i) already proved. $\square$

Step 3: This and the next step are mostly summaries of material from [KL, §8.2-3] and [KLM,§5], to which we defer more technical aspects of the story. (Points related to the integral currents introduced here, will be carefully explained.) First, we will define a $4^{\underline{\text{th}}}$ quadrant bicomplex computing motivic cohomology of Y. In general, an algebraic cycle on a variety $\mathbb{X}$ will be said to properly intersect a SNCD Y if it meets all the Y_I (not just the Y_i) properly.

Begin with a version of the affine line: $\square := \mathbb{P}^1\backslash\{1\}$. We call $\square^q$ the "algebraic q-cube" with coordinates $(z_1,\ldots,z_q)$; its *facets* are the irreducible components of $\partial\square^q := \{\underline{z} \in \square^q \mid z_i = 0$ or ∞ for some $i\}$. Let $\mathbb{X}$ be a smooth complete algebraic variety $/\mathbb{C}$, $c^p(X,q)$ the codim.-p cycles on $\mathbb{X}\times\square^q$ meeting $\mathbb{X}\times\partial\square^q$ properly, and $d^p(\mathbb{X},q)$ the facet-pullback ("degenerate") cycles amongst these; then $Z^p(\mathbb{X},q) := \frac{c^p(\mathbb{X},q)}{d^p(\mathbb{X},q)}$. Out of these "higher Chow precycle" groups one constructs a complex $Z^p(\mathbb{X},-\bullet)$ with differential $\partial_{\mathcal{B}}$ ($\mathcal{B}$ =Bloch) obtained by alternating facet pullbacks, and cohomology the higher Chow groups

$$H^*(Z^p(\mathbb{X},-\bullet)) \cong: \mathrm{CH}^p(\mathbb{X},-*) \cong H^{2p+*}_{\mathcal{M}}(\mathbb{X},\mathbb{Z}(p)). \tag{III.A.20}$$

For example, $\mathcal{W}\in Z^p(\mathbb{X},1)$ has $\partial_{\mathcal{B}}\mathcal{W} := \mathcal{W}\cdot(\mathbb{X}\times\{0\}) - \mathcal{W}\cdot(\mathbb{X}\times\{\infty\}) \in Z^p(\mathbb{X},0) = Z^p(\mathbb{X})$; clearly $\mathrm{im}(\partial_{\mathcal{B}})$ is just the rational equivalences, so $\mathrm{CH}^p(\mathbb{X},0) := \frac{Z^p(\mathbb{X},0)}{\partial_{\mathcal{B}}(Z^p(\mathbb{X},1))} = \mathrm{CH}^p(\mathbb{X})$. One can think of $\mathrm{CH}^p(\mathbb{X},1) = \frac{\ker(\partial_{\mathcal{B}})\subset Z^p(\mathbb{X},1)}{\partial_{\mathcal{B}}(Z^p(\mathbb{X},2))}$ as "empty rational equivalences modulo higher rational equivalences".

On a singular variety the second isomorphism of (III.A.20) is no longer correct (CH is motivic Borel-Moore homology). To compute $H^*_{\mathcal{M}}$ we need Levine's "intersection higher Chow precycles" $Z^p(\mathbb{X},q)_Y$ meeting the SNCD $(\mathbb{X}\times\partial\square^q)\cup(Y\times\square^q)$ properly, so that the differential $\partial_{\mathfrak{I}}$ can be constructed. Moreover, to ensure that certain "KLM currents" on $\mathbb{X}$ attached to the precycles are well-defined (which will be used for the AJ map), we require them to meet properly certain real semi-analytic subsets of $\mathbb{X}\times\square^q$. These conditions (cf. [KLM §5]) cut out $Z^p_{\mathbb{R}}(\mathbb{X},-\bullet)\subset Z^p(\mathbb{X},-\bullet)$, and have been refined in [KL, §8.2] to define a subcomplex $Z^p_{\mathbb{R}}(\mathbb{X},-\bullet)_Y \subset Z^p_{\mathbb{R}}(\mathbb{X},-\bullet)\cap Z^p(\mathbb{X},-\bullet)_Y$ on which the cycles and KLM currents both admit pullbacks: e.g., $Z^p_{\mathbb{R}}(\mathbb{X},q)_Y \xrightarrow{\jmath_i^*} Z^p_{\mathbb{R}}(Y_i,q)_{Y^i}$ (also written $\cdot Y_i$). That all these cycle-complexes are quasi-isomorphic is checked in [KL, Lemma 8.14]. Writing for simplicity $Z^p_{\#}(Y_I,q) := Z^p_{\mathbb{R}}(Y_I,q)_{Y^I}$, we now have the

Definition. The *motivic cohomology* of Y is defined by

$$H^{2n+m}_{\mathcal{M}}(Y,\mathbb{Z}(n)) := H^m(Z^\bullet_Y(n),\underline{\underline{\partial_{\mathcal{B}}}}),$$

the total cohomology of the double complex

$$Z^{\ell,k}_Y(n) := \oplus_{|I|=\ell+1} Z^n_{\#}(Y_I,-k), \quad d_0 = \partial_{\mathcal{B}},\ d_1 = \partial_{\mathfrak{I}}. \tag{III.A.21}$$

Set $Z^n_{\#}(Y) := \ker\left\{\oplus_i Z^n_{\#}(Y_i) \xrightarrow{\partial_{\mathfrak{I}}} \oplus_{i<j} Z^n_{\#}(Y_{ij})\right\} = \left\{\ker(\underline{\underline{\partial_{\mathcal{B}}}})\cap Z^{0,0}_Y(n)\right\}$.

Of course, (III.A.21) is just as valid with $\mathbb{X}$ replaced by an open analytic subset X still containing Y, and we can take $Z^n_{\#}(X)$ to consist of complex *analytic* cycles of codimension n meeting Y (i.e. all the Y_I) properly. We then have the key composition

$$Z^n_{\#}(X) \xrightarrow{\iota^*} Z^n_{\#}(Y) \xrightarrow{[\cdot]_{\mathcal{M}}} H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n)), \tag{III.A.22}$$

where $[\cdot]_{\mathcal{M}}$ is $\left\{\ker(\underline{\underline{\partial_{\mathcal{B}}}})\cap Z^{0,0}_Y(n)\right\} \overset{\langle\cdot\rangle_0}{\hookrightarrow} \left\{\ker(\underline{\underline{\partial_{\mathcal{B}}}})\subset Z^0_Y(n)\right\} \twoheadrightarrow H^0(Z^\bullet_Y(n))$. That $\iota^* = \oplus \jmath_i^*$ sends $\mathcal{Z}\in Z^n_{\#}(X)$ to $Z^n_{\#}(Y)$ encodes the fact that $(\mathcal{Z}\cdot Y_i)\cdot Y_j = (\mathcal{Z}\cdot Y_j)\cdot Y_i$. The same reasoning applied to objects from Step 2 leads to

$$\left\{\begin{array}{c}\ker(\partial)\subset\\ C^{2n}_{\#}(X;\mathbb{Z}(n))\end{array}\right\} \xrightarrow{\iota^*} \left\{\begin{array}{c}\ker(\partial)\subset\\ C^{2n}_{\#}(Y;\mathbb{Z}(n))\end{array}\right\} \xrightarrow{[\cdot]} H^{2n}(Y,\mathbb{Z}(n)). \tag{III.A.23}$$

Lemma III.A.24. *Mapping algebraic cycles to the currents of integration over them (viz., $\mathcal{Z}\mapsto(2\pi\sqrt{-1})^n\delta_{\mathcal{Z}}$) sends the first two terms of (III.A.22) to the first two terms of (III.A.23), producing a commutative square.*

Proof. That this map "intertwines intersection" (i.e., $\delta_{\mathcal{Z}\cdot Y_I} = \jmath_I^*\delta_{\mathcal{Z}}$) is just [Ki2, Prop. 7.8] applied inductively. That (for $\mathcal{Z}\in Z^n_{\#}(X)$) $\delta_{\mathcal{Z}}$ is an SNCD-intersection current does need to be checked; one must bound the integrals of $C^\infty \log\langle Y\rangle$ forms over $|\mathcal{Z}|$. This is done by a (branched) finite-degree projection from $|\mathcal{Z}|$ to a polycylinder of the same dimension, followed by a polar integral argument (essentially $\left|\prod_i \int_{\Delta_\epsilon} \frac{dz_i\wedge d\bar{z}_i}{z_i}\right| < \infty$); this is standard. $\square$

The next step will provide a map extending this commutative square to the right-hand terms of (III.A.22-23).

Recalling the natural weight filtration on total cohomology of (cohomological) bicomplexes from Step 1, we clearly have $W_0 H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n)) = H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))$ (since $Z^{\bullet,\bullet}_Y(n)$ is $4^{\underline{\text{th}}}$ quadrant) and $W_{-w-1}H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n)) = \{0\}$ ($w =$depth).

Example. ($w=1$) $Y = Y_1 \cup Y_2$ and $Z_i \in Z^n_{\#}(Y_i)$ ($i=1,2$) with $Z_1\cdot Y_{12} = Z_2 \cdot Y_{12}$, so that $Z \in Z^n_{\#}(Y)$. If $Z_i \equiv_{\text{rat}} 0\ \forall i$, i.e. $\exists W_i \in Z^n_{\#}(Y_i,1)$ with $\partial_{\mathcal{B}} W_i = Z_i$, then (in $Z^{\bullet}_Y(n)$) $\langle\{Z_i\}\rangle_0 - \underline{\underline{\partial_{\mathcal{B}}}}\langle\{W_i\}\rangle_0 = \langle W_2\cdot Y_{12} - W_1 \cdot Y_{12}\rangle_1$. That is, $[Z]_{\mathcal{M}}$ is represented by the difference of the restrictions of the rational equivalences to Y_{12}; and so $[Z]_{\mathcal{M}} \in W_{-1}H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n)) \cong \frac{\mathrm{CH}^n(Y_{12},1)}{\partial_{\mathfrak{I}}\left(\oplus_{i=1}^2 \mathrm{CH}^2(Y_i,1)\right)}$.

Step 4: Our warm-up act here is to map from (III.A.21) to (III.A.18). There is the evident morphism of complexes $Z^{\bullet,0}_Y(n) \to B^{\bullet,0}_Y(n)$ induced (as in Lemma III.A.24) by $Z \mapsto (2\pi\sqrt{-1})^n\delta_Z$. More generally, let $\mathbb{R}^- =$negative reals on $\mathbb{P}^1$, oriented (as a real analytic chain) so as to have $\partial\mathbb{R}^- = \{0\}-\{\infty\}$. Viewing $\mathcal{W} \in Z^{\ell,k}_Y(n)$ as a correspondence $\mathcal{W} \subset Y^{[\ell]} \times (\mathbb{P}^1)^{-k}$, we may use it to push $(2\pi\sqrt{-1})^n$ times the chain $\underbrace{\mathbb{R}^-\times\cdots\times\mathbb{R}^-}_{(-k)\text{ times}} =: T_{(-k)}$ down to $Y^{[\ell]}$ to obtain a map $Z^{\ell,k}_Y(n) \to B^{\ell,k}_Y(n)$ commuting with the two differentials (cf. [KLM, eqns. (5.4-5)] for d_0; "formally obvious" for d_1, but see the Remark). This induces the fundamental class map $H^{2n+*}_{\mathcal{M}}(Y,\mathbb{Z}(n)) \xrightarrow{cl} H^{2n+*}(Y,\mathbb{Z}(n))$ promised (for $*=0$) in Step 3; the image of $[Z]_{\mathcal{M}}$ (cf. (III.A.22)) will by abuse of notation be denoted $[Z]$.

Remark. A real analytic ($\mathbb{R}^{an}$) r-chain is any $\Gamma \in \mathcal{I}^{2M-r}(X)$ representable as a locally finite sum of [integrals over] oriented $\mathbb{R}^{an}$ semianalytic sets; Γ,Γ' meet properly if the real analytic dimensions of the intersections of supports of $\Gamma, \partial\Gamma, \Gamma', \partial\Gamma'$ are not larger than expected. If Γ and Y (i.e. all Y_I) meet properly, then the bounding argument of Lemma III.A.24 goes through using [Ha, Cor. 2.9], making $\Gamma \in \mathcal{I}^{2M-r}\{Y\}(X)$; in fact, by [op. cit., Thm. 4.3] (and the coincidence of $\jmath_I^*\Gamma$ with the corresponding Federer slice), $\jmath_I^*\Gamma$ is again $\mathbb{R}^{an}$. More generally, Hardt has defined [op. cit., §5] a proper intersection theory for $\mathbb{R}^{an}$ chains which includes such pullbacks (to $\mathbb{C}$-submanifolds) as well as intersections of $\mathbb{C}$-algebraic cycles. In our setting, the $\mathbb{R}^{an}$ intersection conditions of [KL, Def. 8.2] on $\mathcal{W}$ ensure that $\mathcal{W}\cdot(Y_I \times T_{-k})$ meets $(Y_J \times \square^{-k})$ ($J \supsetneq I$) properly, making it (and hence its pushforward to Y_I) of intersection type. Compatibility of our map with d_1 then follows from associativity and commutativity of Hardt's product.

In order to augment our fundamental class map $[\cdot]$ to a Deligne cycle-class map, we must introduce our last bicomplex:

$$K^{\ell,k}_Y(n) := B^{\ell,k}_Y(n) \oplus F^n D^{\ell,k}_Y(n) \oplus D^{\ell,k-1}_Y(n), \quad d_0 = D,\ d_1 = \partial_{\mathfrak{I}}, \tag{III.A.25}$$

where D is the "cone differential" sending a triple of currents $(\tau^{[\ell]}, \Omega^{[\ell]}, R^{[\ell]}) \in K^{\ell,k}_Y(n)$ to $(-\partial\tau^{[\ell]}, -d\Omega^{[\ell]}, dR^{[\ell]} - \Omega^{[\ell]} + \tau^{[\ell]}) \in K^{\ell,k+1}_Y(n)$; $H^*(K^*_Y(n), \underline{\underline{D}}) =: H^{2n+*}_{\mathcal{D}}(Y,\mathbb{Z}(n))$ is the *Deligne cohomology.* Following [KL, §8] we shall now define a map of bicomplexes $Z^{\ell,k}_Y(n) \xrightarrow{\tilde{c}^{\ell,k}_{\mathcal{D}}} K^{\ell,k}_Y(n)$ by associating to a higher Chow precycle in $Z^n_{\#}(Y_I,-k)$ its triple of KLM currents on Y_I,

$$\mathcal{W} \longmapsto (-2\pi\sqrt{-1})^{n+k}\left((2\pi\sqrt{-1})^{-k}T_{\mathcal{W}},\, \Omega_{\mathcal{W}},\, R_{\mathcal{W}}\right). \tag{III.A.26}$$

(More precisely, the $\oplus_{|I|=\ell+1}$ of (III.A.26) gives $\tilde{c}^{\ell,k}_{\mathcal{D}}$; note that $T_{\mathcal{W}}$ has been constructed above.)

Start with the currents on $(\mathbb{P}^1)^{-k}$: $T_{-k} := (\mathbb{R}^-)^{\times(-k)} \in \mathcal{I}^k$; $\Omega_{-k} := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{-k}}{z_{-k}} \in F^{-k}\mathcal{N}^{-k}$; and $R_{-k} = R(z_1,\ldots,z_{-k}) := \sum_{j=1}^{-k} ((-1)^k 2\pi\sqrt{-1})^{j-1} \log^-(z_j) \frac{dz_{j+1}}{z_{j+1}} \wedge \cdots \wedge \frac{dz_{-k}}{z_{-k}} \cdot \delta_{(z_1,\ldots,z_{j-1})\in\mathbb{R}^{j-1}} \in \mathcal{N}^{-k-1}$. Here $\log^- z$ is the 0-current on $\mathbb{P}^1$ given by the principal branch of $\log z$ on $\mathbb{P}^1 \backslash \overline{\mathbb{R}^-}$, and satisfies $d[\log^- z] = \frac{dz}{z} - (2\pi\sqrt{-1})\delta_{\mathbb{R}^-}$; one has also $\frac{1}{2\pi\sqrt{-1}} d\left[\frac{dz}{z}\right] = \delta_{\{0\}} - \delta_{\{\infty\}}$. These identities lead to the more general formula[2] (cf. [KLM, §5.3]) on $(\mathbb{P}^1)^{-k}$

$$d[R_{-k}] = \Omega_{-k} - (2\pi\sqrt{-1})^{-k}\delta_{T_{-k}} + \{\text{Res-terms supported on } \overline{\partial\square^{-k}}\}. \tag{III.A.27}$$

To obtain the KLM currents we again use $\mathcal{W}$ as a correspondence $\subset Y_I \times (\mathbb{P}^1)^{-k}$ to transform $T_{-k}, \Omega_{-k}, R_{-k}$. According to [KL, Prop. 8.17],

$$\pi_*^{Y_I}\left(\delta_{\mathcal{W}} \cdot \pi_\square^* \{T_{-k}, \Omega_{-k}, \text{resp. } R_{-k}\}\right) =: \{T_{\mathcal{W}}, \Omega_{\mathcal{W}}, \text{resp. } R_{\mathcal{W}}\}$$

are defined, belong to $\left\{C_\#^{2n+k}(Y_I),\, F^n D_\#^{2n+k}(Y_I), \text{ resp. } D_\#^{2n+k-1}(Y_I)\right\}$, and cause (III.A.26) to intertwine d_0 and d_1. For d_0, this is a consequence of the relation $d[R_{\mathcal{W}}] = \Omega_{\mathcal{W}} - (2\pi\sqrt{-1})^{-k}\delta_{T_{\mathcal{W}}} - 2\pi\sqrt{-1}R_{\partial_{\mathcal{B}}\mathcal{W}}$ implied by (III.A.27). The end result is the map of total cohomologies

$$c_{\mathcal{D}} : H_{\mathcal{M}}^{2n+*}(Y, \mathbb{Z}(n)) \to H_{\mathcal{D}}^{2n+*}(Y, \mathbb{Z}(n)),$$

compatible with natural weight filtrations (arising from the double complexes).

Example. Given a codim.-$(n-1)$ subvariety $W \subset Y_I$ with desingularization $\tilde{W} \xrightarrow{\iota_W} Y_I$ and morphism $f : \tilde{W} \to \mathbb{P}^1$, $(\iota_W)_*$ of the "graph" of f in $\tilde{W} \times \mathbb{P}^1$ yields a cycle $\mathcal{W} \in Z^n(Y_I \times \mathbb{P}^1)$ (also written (W,f)). If this lies in $Z_\#^n(Y_I, 1)$, then from $T_1 = \mathbb{R}^-$, $\Omega_1 = \frac{dz}{z}$, $R_1 = \log^- z$ one has $T_{\mathcal{W}} = (\iota_W)_* T_f$, $\Omega_{\mathcal{W}} = (\iota_W)_* \frac{df}{f}$, $R_{\mathcal{W}} = (\iota_W)_* \log^- f$. In the simple case where $\tilde{W} = W$, the proper intersection conditions can be described in terms of codimensions of (open subsets of) subvarieties of W: for each $J \supsetneq I$, let $r := |J| - |I|$; then $\operatorname{codim}_{\mathbb{R}}(f^{-1}(\mathbb{R}^-) \cap \iota_W^{-1}(Y_J)) \geqslant 2r+1$; $\operatorname{codim}_{\mathbb{C}}(f^{-1}\{0,\infty\} \cap \iota_W^{-1}(Y_J)) \geqslant r+1$; $f \equiv 1$ on any components of $\iota_W^{-1}(Y_J)$ of $\mathbb{C}$-codim $> r$; and finally, if $f \equiv \alpha$ (constant) on W, then $\alpha \notin \overline{\mathbb{R}^-}$.

Step 5: To extract (for $* = 0$) the AJ map from $c_{\mathcal{D}}$, observe that the identification (abbreviating $K_Y^\bullet(n)$ by $K^\bullet$, etc.) $K^\bullet \cong \operatorname{Cone}\{B^\bullet \oplus F^n D^\bullet \to D^\bullet\}[-1]$ produces a long-exact sequence

$$\to H^{-1}(B^\bullet \oplus F^n D^\bullet) \to H^{-1}(D^\bullet) \to H^0(K^\bullet) \to H^0(B^\bullet \oplus F^n D^\bullet) \to H^0(D^\bullet) \to .$$

In its short-exact truncation

$$\frac{H^0(D^\bullet)}{H^0(B^\bullet) \oplus H^0(F^n D^\bullet)} \xrightarrow{\alpha} H^0(K^\bullet)$$
$$\xrightarrow{\beta} \ker\left\{H^0(B^\bullet) \oplus H^0(F^n D^\bullet) \to H^0(D^\bullet)\right\},$$

α,β are induced by $R \mapsto (0,0,R)$ and $(T,\Omega,R) \mapsto (T,\Omega)$. Of course, this s.e.s. is nothing but

$$0 \to J^n(Y) \xrightarrow{\alpha} H_{\mathcal{D}}^{2n}(Y, \mathbb{Z}(n)) \xrightarrow{\beta} \mathrm{Hg}^n(Y) \to 0$$

where $J^n(Y) = \frac{H^{2n-1}(Y)}{F^n H^{2n-1}(Y) + H^{2n-1}(Y,\mathbb{Z}(n))} \cong \operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2n-1}(Y, \mathbb{Z}(n)))$ and $\mathrm{Hg}^n(Y) = F^n H^{2n}(Y) \cap H^{2n}(Y, \mathbb{Z}(n)) = \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), H^{2n}(Y, \mathbb{Z}(n)))$.

Pullback along the desingularization $Y^{[0]} \overset{\sigma}{\twoheadrightarrow} Y$ induces a morphism of MHS hence $\sigma^* : \mathrm{Hg}^n(Y) \to \oplus_i \mathrm{Hg}^n(Y_i)$. Since weights of $H^{2n}(Y, \mathbb{Z}(n))$ are $\leqslant 0$ and $\mathrm{Gr}_0^W H^{2n}(Y, \mathbb{Z}(n)) \cong \ker\left\{\oplus_i H^{2n}(Y_i, \mathbb{Z}(n)) \xrightarrow{\partial_{\mathfrak{I}}} \oplus_{i<j} H^{2n}(Y_{ij}, \mathbb{Z}(n))\right\}$, σ^* is injective. This (and similar arguments) justify the

[2] In general we abuse notation by writing δ_T for integral currents T when they are added to more general normal currents.

Lemma-Definition III.A.28. *(a)* $H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))_{\mathrm{hom}} := \ker(cl) = \ker(\beta\circ c_{\mathcal{D}}) = \ker(\sigma^*\circ\beta\circ c_{\mathcal{D}})$*, with the* $W_\bullet$*-filtration restricted from* $H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))$*.*

(b) The Abel-Jacobi mapping $\mathrm{AJ}^n_Y : H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))_{\mathrm{hom}} \to J^n(Y)$ *induced by* $c_{\mathcal{D}}$ *is compatible with weight filtrations, where* $W_{-j}J^n(Y) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), W_{-j-1}H^{2n-1}(Y,\mathbb{Z}(n)))$ *is the one restricted from* $H_{\mathcal{D}}$*. (Note that we have* $\mathrm{Gr}^W_{-j}J^n(Y) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \mathrm{Gr}^W_{-j-1}H^{2n-1}(Y,\mathbb{Z}(n)))$*; in particular,* $\mathrm{Gr}^W_0 J^n(Y) \subset \oplus_i J^n(Y_i)$*.)*

(c) For $Z = \{Z_i\} \in Z^n_{\#}(Y)$ *(cf. (III.A.21)ff), the equivalent conditions* $[Z]_{\mathcal{M}} \in H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))_{\mathrm{hom}}$ $\iff$ $[Z] = 0 \in \mathrm{Hg}^n(Y)$ $\iff$ $[Z_i] = 0 \in \mathrm{Hg}^n(Y_i)\,(\forall i)$ *define* $Z^n_{\#}(Y)_{\mathrm{hom}}$*. For such* Z*,* $\mathrm{AJ}^n_Y([Z]_{\mathcal{M}})$ *is notated* $\mathrm{AJ}^n_Y(Z)$*, and* $\mathrm{Gr}^W_0\mathrm{AJ}^n_Y(Z) = \{\mathrm{AJ}^n_{Y_i}(Z_i)\}$*.*

(d) Taking $Z^n_{\#}(X)_{\mathrm{hom}} := \ker\left\{Z^n_{\#}(X) \to H^{2n}(C^\bullet_{\#}(X;\mathbb{Z}(n)))\right\}$ *(cf. Lemmas III.A.17 and III.A.24),* $\iota^*(Z^n_{\#}(X)_{\mathrm{hom}}) \subset Z^n_{\#}(Y)_{\mathrm{hom}}$*.*

There are now two crucial observations. First, viewed as a map of complexes (in k) for each fixed $\ell \geqslant 1$, $\tilde{c}^{\ell,k}_{\mathcal{D}}$ is the KLM map on $Y^{[\ell]}$ inducing

$$\mathrm{CH}^n(Y^{[\ell]},\ell) \xrightarrow{\mathrm{AJ}^{n,\ell}_{Y^{[\ell]}}} J^{n,\ell}(Y^{[\ell]}) := \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(0), H^{2n-\ell-1}(Y^{[\ell]},\mathbb{Z}(n))\right). \tag{III.A.29}$$

Therefore, $\mathrm{Gr}^W_{-\ell}\mathrm{AJ}^n_Y$ *is induced from* $\mathrm{AJ}^{n,\ell}_{Y^{[\ell]}}$ *by taking (compatible) subquotients (or quotients, if* $\ell = w$*) on both sides of (III.A.29).* Furthermore:

Proposition III.A.30. *For* $Z = \{Z_i\} \in Z^n_{\#}(Y_i)$ *with all* $Z_i \equiv_{\text{rat}} 0$*,* $[Z]_{\mathcal{M}} \in W_{-1}H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n)) = W_{-1}H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))_{\mathrm{hom}}$*.*

Proof. What might seem surprising is that $[Z]_{\mathcal{M}} \neq 0$; this is one way (motivic) cohomology differs from homology. Thinking of Z (alg. cycle on $Y^{[0]}$) as a degree-0 $\underline{\underline{\partial_{\mathcal{B}}}}$-cocycle in $Z^\bullet_Y(n)$ with support in $Z^{0,0}_Y(n)$, the $\equiv_{\text{rat}}$-condition says $\exists\mathcal{W}(=\{\mathcal{W}_i\}) \in Z^{0,-1}_Y(n)$ with $\partial_{\mathcal{B}}\mathcal{W} = Z$. Hence, $Z - \underline{\underline{\partial_{\mathcal{B}}}}\mathcal{W} = \partial_{\mathfrak{I}}\mathcal{W} \in Z^{1,-1}_Y(n)$ represents $[Z]_{\mathcal{M}}$. □

Example. We put the observations to work. Let Z be as in Propostion III.A.30 on $Y = Y_1 \cup Y_2$ ($w = 1$), so that $\partial_{\mathcal{B}}\mathcal{W}_i = Z_i$ (for $\mathcal{W}_i \in Z^n_{\#}(Y_i,1)$); then $\partial_{\mathfrak{I}}\mathcal{W} = \mathcal{W}_1\cdot Y_{12} - \mathcal{W}_2\cdot Y_{12} \in Z^n(Y_{12},1)$ $\left(= Z^{1,-1}_Y(n)\right)$ is $\partial_{\mathcal{B}}$-closed. Its image under the composition $\mathrm{CH}^n(Y_{12},1) \xrightarrow{AJ^{n,1}} J^{n,1}(Y_{12}) \twoheadrightarrow \frac{J^{n,1}(Y_{12})}{\mathrm{im}(\oplus_i J^{n,1}(Y_i))} \cong W_{-1}J^n(Y) \hookrightarrow J^n(Y)$, is $\mathrm{AJ}^n_Y(Z)$.

Remark. In the example, we used that the weight filtration is induced by the Mayer-Vietoris s.e.s.

$$0 \to \frac{H^{2n-2}(Y_{12})}{\oplus H^{2n-2}(Y_i)} \to H^{2n-1}(Y) \to \ker\left\{\begin{array}{c}\oplus H^{2n-1}(Y_i) \to \\ H^{2n-1}(Y_{12})\end{array}\right\} \to 0, \tag{III.A.31}$$

which is preserved by $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \text{—})$ (i.e., passes to Jacobians) because after $\otimes\mathbb{Z}(n)$ all weights are negative.

Step 6: We are now ready to put everything together. Recall that $J^n(Y) \cong \frac{(F^{-n+1}H_{2n-1}(Y))^\vee}{\mathrm{im}\{H^{2n-1}(Y,\mathbb{Z}(n))\}}$, and let $Z \in H^{2n}_{\mathcal{M}}(Y,\mathbb{Z}(n))_{\mathrm{hom}}$ be represented by a collection $\{Z^{[\ell]}\}_{\ell\geqslant 0} = \{Z_I\}_{\substack{\ell\geqslant 0\\ |I|=\ell+1}} \in \oplus_{\ell\geqslant 0}Z^{\ell,-\ell}_Y(n) = Z^0_Y(n)$ in $\ker(\underline{\underline{\partial_{\mathcal{B}}}})$. (This need *not* come from $Z^n_{\#}(Y)$.) The map $\tilde{c}^{\bullet,\bullet}_{\mathcal{D}}$ of double complexes (cf. (III.A.26)) gives

$$\left\{(-2\pi\sqrt{-1})^{n-\ell}\left((2\pi\sqrt{-1})^{\ell}T_{Z^{[\ell]}}, \Omega_{Z^{[\ell]}}, R_{Z^{[\ell]}}\right)\right\}_{\ell\geqslant 0} \in \ker(\underline{\underline{D}}) \subset K^0_Y(n). \tag{III.A.32}$$

Using this, the goal is to represent a lift $\widetilde{\mathrm{AJ}^n_Y(Z)} \in H^{2n-1}(Y,\mathbb{C})$ of $\mathrm{AJ}^n_Y(Z)$ as an explicit functional on C^∞-log representatives $\omega = \{\omega^{[\ell]}\}_{\ell\geqslant 0} = \{\omega_I\}_{\substack{\ell \geqslant 0 \\ |I| = \ell+1}}$ (cf. (III.A.13ff) of classes in $F^{-n+1}H_{2n-1}(Y)$.

The hypothesis that Z be cohomologically trivial means that β(III.A.32) $=$

$$\left\{(-2\pi\sqrt{-1})^{n-\ell}\left((2\pi\sqrt{-1})^{\ell}T_{Z^{[\ell]}},\Omega_{Z^{[\ell]}}\right)\right\}_{\ell\geqslant 0} \in B^0_Y(n)\oplus F^nD^0_Y(n) \tag{III.A.33}$$

is a total coboundary in $B^\bullet \oplus F^nD^\bullet$ — say, $\underline{\underline{\partial}} \oplus \underline{\underline{d}}$ of

$$\left\{(-2\pi\sqrt{-1})^{n-\ell}\left((2\pi\sqrt{-1})^{\ell}\Gamma^{[\ell]},\Xi^{[\ell]}\right)\right\}_{\ell\geqslant 0} \in B^{-1}_Y(n)\oplus F^nD^{-1}_Y(n).$$

Adding $\underline{\underline{D}}$((III.A.33), 0) to (III.A.32), we are left with α of

$$\left\{(-2\pi\sqrt{-1})^{n-\ell}\left(R_{Z^{[\ell]}}+\Xi^{[\ell]}-(2\pi\sqrt{-1})^{\ell}\delta_{\Gamma^{[\ell]}}\right)\right\}_{\ell\geqslant 0} \in D^{-1}_Y(n),$$

which is therefore $\underline{\underline{d}}$-closed and defines $\widetilde{\mathrm{AJ}^n_Y(Z)}$. Noting that $\int_{Y^{[\ell]}}\Xi^{[\ell]}\wedge\omega^{[\ell]} = 0$ by type ($\Xi^{[\ell]} \in F^nD^{2n-\ell-1}_{\#}(Y^{[\ell]})$, $\omega^{[\ell]} \in F^{n-\ell}A^{2n-\ell-1}(Y^{[\ell]},\log\langle Y^\bullet\rangle)$, $\dim Y^{[\ell]} = 2n-\ell-1$), the pairing defined in III.A.13(ii) gives

$$\boxed{\left\langle \widetilde{\mathrm{AJ}^n_Y(Z)},\omega\right\rangle = (-2\pi\sqrt{-1})^n\sum_{\ell\geqslant 0}\left(\int_{Y^{[\ell]}}R_{Z^{[\ell]}}\wedge\omega^{[\ell]} - (2\pi\sqrt{-1})^{\ell}\int_{\Gamma^{[\ell]}}\omega^{[\ell]}\right).} \tag{III.A.34}$$

While the integrals in (III.A.34) are "improper", they are well-defined by Lemma III.A.6(ii). We leave it as an algebra exercise to show that the functional produced either by a $\underline{\underline{\partial_{\mathcal{B}}}}$-coboundary Z or or by a different choice of (III.A.33), lies in the "periods" $\mathrm{im}\,\{H^{2n-1}(Y,\mathbb{Z}(n))\}$.

Example. $Y = Y_1\cup Y_2$, $\{Z_i \equiv_{\mathrm{rat}} 0\}$ and $\{\mathcal{W}_i\}$ as in the example of Step 5. Ignoring twists, the Hodge-filtered dual sequence to (III.A.31) is

$$0 \leftarrow \ker\left\{\ F^{n-1}H^{2n-2}(Y_{12}) \overset{\mathrm{Gy}}{\to} \oplus_i F^nH^{2n}(Y_i)\ \right\} \overset{\mathrm{Res}}{\longleftarrow} F^{-n+1}H_{2n-1}(Y)$$
$$\longleftarrow \frac{\oplus_i F^nH^{2n-1}(Y_i)}{\mathrm{Gy}\left(F^{n-1}H^{2n-3}(Y_{12})\right)} \leftarrow 0; \tag{III.A.35}$$

we shall write $\widetilde{\mathrm{AJ}^n_Y(Z)}$ as a functional on the left-hand term (co)im(Res) $\subset F^{n-1}H^{2n-2}(Y_{12})$ — that is, on the ω_{12} in $\omega = \{\{\omega_1,\omega_2\},\omega_{12}\}$. Now $\partial_{\mathfrak{I}}\mathcal{W}$ is $\partial_{\mathcal{B}}$-closed, so $\tau := T_{\partial_{\mathfrak{I}}\mathcal{W}}$ has class in $\mathrm{Hom}_{\mathrm{MHS}}\left(\mathbb{Z}(0),H^{2n-2}(Y_{12},\mathbb{Z}(n))\right)$ which (in view of our non-torsion assumption on the $\{H^*(Y_I)\}$) is zero. Write $\tau = \partial\mu$, μ a $(2n-2)$-chain on Y_{12}. Since $[Z]_{\mathcal{M}}$ is represented by $\langle\partial_{\mathfrak{I}}\mathcal{W}\rangle_1$, and $\partial_{\mathfrak{I}}\mathcal{W} = \sum_\lambda(W_\lambda,f_\lambda)$ as in the example of Step 4 (with $\partial_{\mathfrak{I}}\mathcal{W}$ replacing $\mathcal{W}$ and $I = \{1,2\}$), (III.A.34) becomes

$$\frac{1}{(-2\pi\sqrt{-1})^n}\left\langle\widetilde{\mathrm{AJ}^n_Y(Z)},\omega\right\rangle = \sum_\lambda\int_{\widetilde{W_\lambda}}(\log^- f_\lambda)\iota^*_{W_\lambda}\omega_{12}\ -\ 2\pi\sqrt{-1}\int_\mu\omega_{12}. \tag{III.A.36}$$

Step 7: Now consider an algebraic cycle $Z = \{Z_i\} \in Z^n_{\#}(Y)_{\mathrm{hom}} \subset Z^{0,0}_Y(n)$; this has trivial cohomology class $[Z] \in H^{2n}(Y,\mathbb{Z}(n))$ and satisfies $Z_i\cdot Y_{ij} = Z_j\cdot Y_{ij}$. Since $[Z]_{\mathcal{M}}$ is represented by $\langle Z\rangle_0$ (via the inclusion $Z^{0,0}_Y(n)\subset Z^0_Y(n)$), all $Z^{[\ell]}$ for $\ell\geqslant 1$ are 0 in Step 6. Moreover, $\tilde{c}^{0,0}_{\mathcal{D}}(Z)$ is $\left\{(-2\pi\sqrt{-1})^n\left(\delta_{Z_i},\delta_{Z_i},0\right)\right\}_i$ and (III.A.32) is $\left\langle\tilde{c}^{0,0}_{\mathcal{D}}(Z)\right\rangle_0$. By Lemma III.A.24 (and the remark after), $[Z]=0 \implies \left\langle\left\{(2\pi\sqrt{-1})^n\delta_{Z_i}\right\}\right\rangle_0$ is $\underline{\underline{\partial}}$ [in $B^\bullet_Y(n)$] of $(2\pi\sqrt{-1})^n$ times some $\Gamma = \{\Gamma^{[\ell]}\}_{\ell\geqslant 0}$; by Lemma III.A.19(vi) we can arrange that all $\Gamma^{[\ell]} = 0$ for $\ell\geqslant 1$. (That is, $\Gamma(=\Gamma^{[0]}) = \{\Gamma_i\}$ with

$Z_i = \partial \Gamma_i$ on each Y_i and $\Gamma_i \cdot Y_{ij} = \Gamma_j \cdot Y_{ij}$.) Hence, (III.A.34) specializes to

$$\boxed{\frac{1}{(-2\pi\sqrt{-1})^n}\left\langle \widetilde{\mathrm{AJ}^n_Y(Z)}, \omega \right\rangle = \int_{\Gamma^{[0]}} \omega^{[0]} = \sum_i \int_{\Gamma_i} \omega_i,} \tag{III.A.37}$$

where the $\int_{\Gamma_i} \omega_i$ are absolutely convergent by the Remark of Step 4.

In the event that Z is the pullback of a cycle from X, from III.A.28(d) follows the

Lemma III.A.38. *For $\mathcal{Z} \in Z^n_{\#}(X)_{\mathrm{hom}}$,*

(a) $\exists\, \tilde{\Gamma} \in C^{2n-1}_{\#}(X)$ bounding on $\delta_{\mathcal{Z}} (\in C^{2n}_{\#}(X))$, and $\iota^* \tilde{\Gamma} = \{\Gamma_i\} := \{\tilde{\Gamma} \cdot Y_i\} \in C^{2n-1}_{\#}(Y)$ bounds on $Z := \iota^* \mathcal{Z} \in Z^n_{\#}(Y)_{\mathrm{hom}}$ in the sense just described; moreover,

(b) applying (III.A.37), we have $\frac{1}{(-2\pi\sqrt{-1})^n}\left\langle \widetilde{\mathrm{AJ}^n_Y(\iota^*\mathcal{Z})}, \omega \right\rangle = \sum_i \int_{\tilde{\Gamma}\cdot Y_i} \omega_i$.

Example. Continuing from Step 6 ($Y = Y_1 \cup Y_2$, $Z \in Z^n_{\#}(Y)$, both $Z_i \equiv_{\mathrm{rat}} 0$), Propostion III.A.30 $\Longrightarrow$ $[Z] = 0$; the point is now that $[Z]_{\mathcal{M}}$ is represented *both* by $\langle \partial_{\mathfrak{I}} \mathcal{W} \rangle_1$ *and* $\langle Z \rangle_0$. Hence (using well-definedness of the pairing in III.A.13(ii)) the RHS's of (III.A.36) and (III.A.37), taken over a *basis* for $F^{-n+1}H_{2n-1}(Y)$, agree modulo periods.

It is instructive to work explicitly through this for $n = 1$, where (say) Y_1 and Y_2 are curves meeting in 3 nodes $\{p_\alpha\}$. In fact, we can choose $\{\Gamma_i\}$ and $\{\mathcal{W}_i\}$ so that the two RHS's are *equal* for individual $[\omega] \in F^{-n+1}H_{2n-1}(Y)$. Note that for $\omega = \{\{\omega_1, \omega_2\}, \omega_{12}\}$ and $\alpha = 1, 2, 3$ we have $\mathrm{Res}_{p_\alpha} \omega_1 = -\mathrm{Res}_{p_\alpha} \omega_2 = \omega_{12}(p_\alpha)$, so that $d[\omega_i] = (-1)^{i-1} \sum_\alpha \omega_{12}(p_\alpha) \delta_{\{p_\alpha\}}$. Now $Z_i = (f_i)$ for $(f_i) \in \mathbb{C}(Y_i)^*$, and we may *choose* $\Gamma_i = T_{f_i}$ so that (using $\frac{df_i}{f_i} \wedge \omega_i = 0$ by type) $\sum_i \int_{\Gamma_i} \omega_i = \sum_i \int_{Y_i} \delta_{\Gamma_i} \wedge \omega_i = \frac{1}{2\pi\sqrt{-1}} \sum_i \int_{Y_i} d[\log^- f_i] \wedge \omega_i = \frac{1}{2\pi\sqrt{-1}} \sum_i \int_{Y_i} (\log^- f_i) d[\omega_i] = \sum_\alpha \sum_i (-1)^{i-1} (\log^- f_i(p_\alpha)) \omega_{12}(p_\alpha) \equiv \sum_\alpha \log^- \left(\frac{f_1(p_\alpha)}{f_2(p_\alpha)}\right) \omega_{12}(p_\alpha) \bmod \mathbb{Z}(1)$, as desired.

Taking $\omega_i^{\beta=1,2}$ to have $\mathrm{Res}_{p_\alpha} \omega_i^\beta = \begin{cases} (-1)^{i-1}\delta_{\alpha\beta}, & \alpha = 1,2 \\ (-1)^i, & \alpha = 3 \end{cases}$, $\frac{\omega^1}{2\pi\sqrt{-1}}$, $\frac{\omega^2}{2\pi\sqrt{-1}}$ give a basis for the LHS of (III.A.35) (viewed as coim(Res)). Evaluating against these induces an isomorphism $W_{-1}J^1(Y) \to (\mathbb{C}/\mathbb{Z}(1))^{\oplus 2}$, sending $\mathrm{AJ}^1_Y(Z) \mapsto \left(\log\left(\frac{f_1(p_1)/f_2(p_1)}{f_1(p_3)/f_2(p_3)}\right), \log\left(\frac{f_1(p_2)/f_2(p_2)}{f_1(p_3)/f_2(p_3)}\right)\right)$.

Discussion. We will conclude this section by discussing *the wrong way to define* AJ_Y *for a normal crossing variety* Y.

First we recall that for Y *smooth* and connected of complex dimension $2n-1$ and $Z \in Z^n(Y)_{\mathrm{hom}}$, we have from the exact homology sequence of the pair $(X, |Z|)$, with $\mathbb{Z}$ coefficients,

$$0 \to H_{2n-1}(Y) \to H_{2n-1}(Y, |Z|) \xrightarrow{\partial} \mathbb{Z}(n-1) \to 0$$

where $\mathbb{Z}(n-1) \subseteq \mathrm{Ker}\{H_{2n-2}(|Z|, \mathbb{Z}(n-1)) \to H_{2n-2}(Y, \mathbb{Z}(n-1))\}$ is generated by the class of Z. We may think of the chain Γ with $\partial\Gamma = Z$ as giving the element of

$$\mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(n-1), H_{2n-1}(Y)\right) =: J_{n-1}(Y) \tag{III.A.39}$$

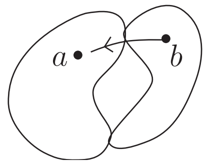
corresponding to the above short-exact sequence. Since Y is smooth and connected this group is canonically isomorphic to $\mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(0), H^{2n-1}(Y, \mathbb{Z}(n))\right) =: J^n(Y)$ and under this isomorphism

$$\Gamma \text{ corresponds to } \mathrm{AJ}_Y(Z)\,.$$

Our point here is that this does not work in the non-smooth case:

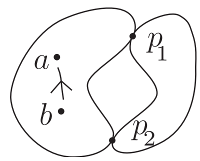
> *For $Y \subset X$ an SNCD, using Γ with $\partial\Gamma = Z$ as an element in (III.A.39) is the wrong way to define $\mathrm{AJ}_Y(Z)$.*

This is illustrated by the simple example



$(Y_i \cong \mathbb{P}^1)$

where $Z = a - b$ (and the Γ is *not* in $C^1_{\#}(Y)$). Not only does one not expect to be able to define $\mathrm{AJ}_Y(Z)$ in this case (as $[Z] \neq 0$ in $H^2(Y)$), for this Y one has $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H_1(Y)) = 0$. Consequently, also the homological definition does not give anything in the following equally simple example where $Z = a - b$



.

Here we know that we should have

$$\mathrm{AJ}_Y(Z) = f(p_1)/f(p_2) \in \mathbb{C}^*$$

for f a function with $(f) = a - b$.

A more general statement is this: For a $(2n-1)$-dimensional SNCD Y one has a natural morphism of MHS $H^{2n-1}(Y) \to H_{2n-1}(Y)(-2n+1)$ inducing

$$J^n(Y) \to J_{n-1}(Y)\ .$$

It is easy to see that this map factors through $\underset{i}{\oplus} J^n(Y_i)$, which shows that none of the "non-classical" information contained in $\mathrm{AJ}_Y(Z)$ is left on the RHS.

III.B Limits of Abel-Jacobi mappings in 1-parameter families

We assume given the standard situation

$$\mathcal{X} \xrightarrow{\pi} S \tag{III.B.1}$$

of a semistable degeneration (SSD): that is,

- S is a disk with parameter s vanishing at $s_0 \in S$, and $S^* = S \backslash \{s_0\}$ is the punctured disk;
- $\mathcal{X}$ is a (smooth) complex analytic manifold of dimension $2n$, with $\mathcal{X}^* := \pi^{-1}(S^*)$;
- π is proper, connected, projective, and holomorphic, with smooth restriction to $\mathcal{X}^* \to S^*$ (so that by GAGA fibres X_s, $s \in S^*$, are smooth projective algebraic varieties of dimension $2n-1$); and
- the fibre X_{s_0} is a SNCD (with components X_i all having multiplicity one). Write $X_{s_0} = \cup X_i \overset{\iota}{\hookrightarrow} \mathcal{X}$.

We are then in the situation of §III.A with $X = \mathcal{X}$, $Y = X_{s_0}$, $Y_i = X_i$. Denote by $Z^n_\pi(\mathcal{X})$ the group of codimension-n algebraic cycles "in general position", i.e. that intersect all the fibres X_s and strata X_I properly; clearly $Z^n_\pi(\mathcal{X}) \subset Z^n_{\#}(\mathcal{X})$. Note that any $\mathcal{Z} \in Z^n_{\#}(\mathcal{X})$ restricts to a general position cycle over some smaller disk.

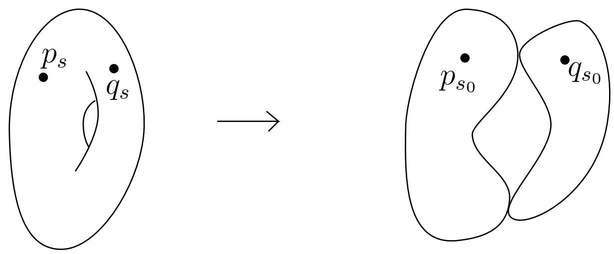
For $\mathcal{Z} \in Z^n_\pi(\mathcal{X})$, setting $\mathcal{Z}^* := \mathcal{Z} \cap \mathcal{X}^*$ and $Z_s := \mathcal{Z} \cdot X_s$ ($s \in S^*$), we consider the conditions

$$\begin{cases} \text{(i)} & \text{the fundamental class } [\mathcal{Z}] = 0 \text{ in } H^{2n}(\mathcal{X};\mathbb{Z}); \\ \text{(ii)} & \text{the fundamental class } [\mathcal{Z}^*] = 0 \text{ in } H^{2n}(\mathcal{X}^*;\mathbb{Z}); \\ \text{(iii)} & \text{the fundamental class } [Z_s] = 0 \text{ in } H^{2n}(X_s;\mathbb{Z}). \end{cases} \tag{III.B.2}$$

We note that

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}$$

and all of the implications are strict, as shown by simple examples, such as the degeneration (type I_2 in Kodaira's list)



Then $Z_s = p_s - q_s$ satisfies (iii) but not (ii), and $Z_s = 2(p_s - q_s)$ satisfies (ii) but not (i). Here, we take $\mathcal{Z} = \cup_s Z_s$.

We also note that in case (iii), the fundamental class in homology

$$[Z_{s_0}]_{\text{hom}} \in H_{2n-2}(X_{s_0}, \mathbb{Z})$$

is zero. The fundamental class in *co*homology $[Z_{s_0}] \in H^{2n}(X_{s_0}, \mathbb{Z})$, as defined in the preceeding section, is non-zero. This again illustrates the necessity for using cycles, chains and currents of intersection type so as to obtain *co*homology classes.

Because of our SSD assumption, the monodromy $T : H^{2n-1}(X_s, \mathbb{Z}) \to H^{2n-1}(X_s, \mathbb{Z})$, $s \neq s_0$, is unipotent, and we shall denote by $(\mathcal{H}_e, \mathcal{F}_e^p, \mathcal{H}_{\mathbb{Z},e}, \nabla)$ the canonically extended VHS associated to the $(2n-1)^{\text{st}}$ cohomology along the fibres of the family (III.B.1). We note that $\mathcal{H}_{\mathcal{Z},e} = j_*(R^{2n-1}_{\pi_*}\mathbb{Z}_{\mathcal{X}^*})$ where $j : S^* \hookrightarrow S$ is the inclusion. As in section II.A we set $\hat{\mathcal{J}}_e = \mathcal{F}_e^n \backslash \mathcal{H}_e / \mathcal{H}_{\mathbb{Z},e}$ and recall the subsheaf $\mathcal{J}_e \subset \hat{\mathcal{J}}_e$ defined by the sections of $\mathbf{J}_e \to S$ as in section II.A. Here, as before we are omitting reference to the transversality condition satisfied by normal functions. For a cycle $\mathcal{Z} \in Z^n_\pi(\mathcal{X})$ satisfying (iii) above the corresponding normal function $\nu_{\mathcal{Z}}$ is defined over S^* by

$$\nu_{\mathcal{Z}}(s) = \mathrm{AJ}_{X_s}(Z_s)\ . \tag{III.B.3}$$

More precisely, there is a fibre space of complex Lie groups $\mathbf{J} \to S^*$ with fibres

$$J_s = F^n H^{2n-1}(X_s) \backslash H^{2n-1}(X_s) / H^{2n-1}(X_s, \mathbb{Z})$$

and where the restriction $\mathcal{J}$ of $\mathcal{J}_e$ to S^* is $\mathcal{J} = \mathcal{O}_{S^*}(\mathbf{J})$, the sheaf of holomorphic sections of $\mathbf{J} \to S^*$. That the RHS of (III.B.3) gives such a section is well-known (cf. [Ki1, intro.].

In case (i) above Steve Zucker [Z1], [Z2] has proved that $\nu_{\mathcal{Z}}$ extends to a section of $\hat{\mathcal{J}}_e$, and this has been extended to case (ii) by El Zein and Zucker [EZ]. Below we shall prove the

Proposition III.B.4. *In case (iii) in (III.B.2) $\nu_{\mathcal{Z}}$ gives an admissible normal function in the sense of section II.A.*

It then follows from Theorem II.B.9 that $\tilde{\mathbf{J}}_e \to S$ graphs $\nu_{\mathcal{Z}}$, which in particular will imply the results of Zucker and El Zein-Zucker with the refinement that in cases (i) and (ii) ν_ζ gives a section of $\mathcal{J}_e \subset \hat{\mathcal{J}}_e$.

Remark. The discrepancy between (ii) and (iii) is just that between normal functions (on S) and ANF's, hence is detected by the torsion group G. This is visible geometrically: in the notation of the proof of Thm. (III.C.3) below,

$$\frac{\ker\left\{H^{2n}(\mathcal{X},\mathbb{Z}) \to H^{2n}(X_s,\mathbb{Z})\right\}}{\operatorname{im}\left\{H^{2n}(\mathcal{X},\mathbb{Z}) \to H^{2n}(\mathcal{X}^*,\mathbb{Z})\right\}} \cong \frac{\ker(\nu_{\mathbb{Z}})}{\operatorname{im}(\alpha_{\mathbb{Z}})} \cong \frac{\ker(\nu_{\mathbb{Z}})}{\ker(\mu_{\mathbb{Z}})} \cong G\ .$$

In cases (i) and (ii) the "value" $\nu_{\mathcal{Z}}(s_0) \in J_{e,s_0}$ is defined and in this section we want to relate it to generalized "periods" on X_{s_0}. In fact, it is preferable here to work without Propostion III.B.4 and the results of part II, because the existence of a well-defined value $\nu_{\mathcal{Z}}(s_0) \in \hat{J}_{e,s_0}$ follows from [EZ], and the corollary to our main result (III.B.6) below yields a second proof that it lies in J_{e,s_0} (in the geometric case). To explain this point, we consider the short-exact sheaf sequence on S

$$\mathcal{H}_{\mathbb{Z},e} \hookrightarrow \mathcal{F}^n_e \backslash \mathcal{H}_e \twoheadrightarrow \hat{\mathcal{J}}_e$$

where $j_*(\mathcal{H}_{\mathbb{Z}}) = \mathcal{H}_{\mathbb{Z},e}$, and let $\nu^*_{\mathcal{Z}} \in \Gamma(S^*,\mathcal{J})$ be associated to $\mathcal{Z}$ in case (ii). By the main theorem of [EZ], $\nu^*_{\mathcal{Z}}$ lifts to a section $\nu_{\mathcal{Z}} \in \Gamma(S,\hat{\mathcal{J}}_e)$; shrinking S if necessary we may lift further to $\tilde{\nu}_{\mathcal{Z}} \in \Gamma(S,\mathcal{F}^n_e\backslash\mathcal{H}_e)$. This is a (holomorphic) section of a vector bundle and may therefore be evaluated at s_0; writing $\mathrm{pr} : \mathcal{F}^n_{e,s_0}\backslash\mathcal{H}_{e,s_0} \twoheadrightarrow \hat{J}_{e,s_0}$ we can *define*

$$\lim_{s\to s_0} \nu^*_{\mathcal{Z}}(s) := \mathrm{pr}(\tilde{\nu}_{\mathcal{Z}}(s_0))\ .$$

It is straightforward to check that this is well-defined, the point being that "pr" exactly kills the invariant cycles.

Now we come to the interpretation as periods on X_{s_0}. The map $Z^n_\pi(\mathcal{X}) \to Z^n_{\#}(X_{s_0})$ given by intersecting with the $\{X_i\}$, sends $\mathcal{Z}$ to an intersection cycle $Z_{s_0} = \{Z_i\}$ on X_{s_0}; $\langle Z_{s_0}\rangle_0$ (cf. (III.A.5) and (III.A.22)ff) then yields a motivic cocycle with class $[Z_{s_0}]_{\mathcal{M}}$. In case (i) in (III.B.2), from Lemma-Definition III.A.28

$$[Z] = 0 \implies [Z_{s_0}] = 0 \implies [Z_{s_0}]_{\mathcal{M}} \in H^{2n}_{\mathcal{M}}(X_{s_0},\mathbb{Z}(n))_{\mathrm{hom}},$$

and $\mathrm{AJ}_{X_{s_0}}(Z_{s_0}) \in J^n(X_{s_0})$ is defined.

Recall the Clemens-Schmid (C-S) complex (cf. [Mo1], [KK, §5])

$$\to H^{2n-1}(X_{s_0}) \xrightarrow{\rho} H^{2n-1}_{\lim}(X_s) \xrightarrow{T-I} H^{2n-1}_{\lim}(X_s) \xrightarrow{\sigma} H_{2n-1}(X_{s_0})(-2n+1) \to$$

which is defined (but not exact) integrally, with the Clemens retraction map (cf. [Cl1]) $\mathcal{X} \xrightarrow{r} X_{s_0}$ inducing $\rho := r^*$ and $\sigma := r_* \circ (\text{Poincare duality})$. Taken $\otimes\mathbb{Q}$, C-S becomes an *exact sequence of MHS*, and one may replace $T - I$ by N; with this change it is even self-dual. Now since ρ is a MHS morphism, there is an induced map $AJ(\rho) : J(X_{s_0}) \to \hat{J}_{e,s_o}$, factoring through J_{e,s_0} since $\operatorname{im}(\rho) \subseteq \ker(N)$. The main result of this section is the

Theorem III.B.5. *With the above notations, in case (i) in (III.B.2)*

$$\nu_{\mathcal{Z}}(s_0) = AJ(\rho)\left(\mathrm{AJ}_{X_{s_0}}(Z_{s_0})\right).$$

We shall write this as

$$\boxed{\lim_{s\to s_0} \mathrm{AJ}_{X_s}(Z_s) = AJ(\rho)\left(\mathrm{AJ}_{X_{s_0}}(Z_{s_0})\right).} \tag{III.B.6}$$

Corollary. *We have*

$$\nu_{\mathcal{Z}}(s_0) \in J_{e,s_0} = F^n Ker N_{\mathbb{C}} \backslash Ker N_{\mathbb{C}} / \mathcal{H}_{\mathbb{Z},e;s_0}.$$

Of course, this corollary follows from Theorem II.A.9 and Proposition III.B.4. However, (III.B.5–III.B.6) together with §III.A will give considerably more, binding limits of AJ maps together with regulators on higher algebraic K-theory on the substrata of X_{s_0}. To this end, if we put a weight filtration on J_{e,s_0} using the monodromy weight filtration, viz. $W_{-j}J_{e,s_0} := \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(0), W_{-j-1}\ker(T-I)\right)$, then (since ρ is a morphism of MHS) $J(\rho)$ is compatible with $W_\bullet$.

We shall need a lemma for Theorem III.B.5's proof (and for subsequent use); in the following the same notations $\omega(s)$, $\lim_{s\to s_0}\omega(s)$, and $\omega(s_0)$ are used for classes and their representatives.

Lemma-Definition III.B.7. *(i) A (holomorphic) section $\omega(s)\in\Gamma(S,\mathcal{F}^n_e)$ may be represented by a (d_{rel}-)closed C^∞ relative $\log\langle X_{s_0}\rangle$ $(2n-1)$-form on $\mathcal{X}$.*

(ii) Its restriction to the fiber $\mathcal{F}^n_{e,s_0} = F^nH^{2n-1}_{lim}(X_s,\mathbb{C})$ over s_0 of the corresponding vector bundle shall be denoted $\lim_{s\to s_0}\omega(s)$, and corresponds to restricting the representative. (Note that $\Gamma(S,\mathcal{F}^n_e)\twoheadrightarrow\mathcal{F}^n_{e,s_0}$.)

(iii) Pullback of the representative to the components of X_{s_0} defines $\{\omega(s_0)_i\}$ with compatible residues as in the Observation (end of Step 1); taking residues as in (III.A.15) completes this to $\{\omega(s_0)_I\} =: \omega(s_0)\in F^{-n+1}H_{2n-1}(X_{s_0})$.

(iv) Pullback is factored by restriction, with $\sigma(\lim_{s\to s_0}\omega(s)) = \omega(s_0)$.

Proof. (i) and (ii) follow (resp.) from $\mathcal{F}^n_e \cong \mathcal{R}^{2n-1}\pi_*F^nA^\bullet_{\mathcal{X}/S}(\log\langle X_{s_0}\rangle)$ [Z3, Corollary (p. 130)] and $\mathbb{H}^{2n-1}(F^nA^n_{\mathcal{X}/S}(\log\langle X_{s_0}\rangle)\otimes\mathcal{O}_{X_{s_0}}) \cong F^nH^{2n-1}_{\lim}(X_s,\mathbb{C})$ [op. cit., (24)(i) and preceding discussion]. To see (iii), lift $\omega(s)$ to a (relatively closed) C^∞ $\log\langle X_{s_0}\rangle$ form on $\mathcal{X}$, notice that $\omega(s_0)_i = \mathrm{Res}_{X_i}(\omega\wedge\pi^*\frac{ds}{s})$, and use the Remark at the end of III.A Step 1. (iv) is immediate from the description on the level of representatives. □

Proof of Theorem III.B.5. We recall the identifications

$$\begin{cases} \hat{J}_{e,s_0}\cong\left(F^nH^{2n-1}_{\lim}(X_s)\right)^\vee/H^{2n-1}(X_s,\mathbb{Z}(n)) \\ J(X_{s_0})\cong\left(F^{-n+1}H_{2n-1}(X_{s_0})\right)^\vee/H^{2n-1}(X_{s_0},\mathbb{Z}(n)) \end{cases}.$$

They are related by the commutative diagram

$$\begin{tikzcd} F^n_{e,s_0}\otimes\left(F^n_{e,s_0}\setminus H_{e,s_0}\right) \arrow[r, "Q(\cdot,\cdot)"] \arrow[d, "\sigma"] & \mathbb{C} \arrow[d, equal] \\ F^{-n+1}H_{2n-1}(X_{s_0})\otimes\left(F^nH^{2n-1}(X_{s_0})\setminus H^{2n-1}(X_{s_0})\right) \arrow[u, "\rho"'] \arrow[r, "\langle\cdot,\cdot\rangle"] & \mathbb{C}, \end{tikzcd}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between homology and cohomology.

Below we shall choose lifts $\widetilde{\mathrm{AJ}_{X_{s_0}}(Z_{s_0})}\in\left(F^{-n+1}H_{2n-1}(X_{s_0})\right)^\vee$ (of $\mathrm{AJ}_{X_{s_0}}(Z_{s_0})$) and $\tilde{\nu}_{\mathcal{Z}}\in\Gamma(S,(\mathcal{F}^n_e)^\vee)$ (as above, of $\nu_{\mathcal{Z}}$); the latter specializes to a lift $\tilde{\nu}_{\mathcal{Z}}(s_0)\in\left(F^{n-1}H^{2n-1}_{\lim}(X_s)\right)^\vee$ of $\lim_{s\to s_0}\mathrm{AJ}_{X_s}(Z_s)$. Taking an arbitrary ω as in [the proof of] III.B.7 (which yields a section of $\mathcal{F}^n_e$) and pairing with $\tilde{\nu}_{\mathcal{Z}}$, yields a holomorphic function $\tilde{\nu}_{\mathcal{Z},\omega}$ over S. If we can show ($\forall$ such ω)

$$\left[\widetilde{\mathrm{AJ}_{X_{s_0}}(Z_{s_0})}\right](\omega(s_0)) = \tilde{\nu}_{\mathcal{Z},\omega}(s_0) \tag{III.B.8}$$

then we are done: using the diagram + III.B.7(iv) on the LHS and the notation III.B.7(ii) for the RHS, (III.B.8) becomes

$$\left[\rho\left(\widetilde{\mathrm{AJ}_{X_{s_0}}(Z_{s_0})}\right)\right](\lim_{s\to s_0}\omega(s)) = [\tilde{\nu}_{\mathcal{Z}}(s_0)](\lim_{s\to s_0}\omega(s)).$$

That is, $\rho\left(\widetilde{\mathrm{AJ}_{X_{s_0}}(Z_{s_0})}\right)$ and $\tilde{\nu}_{\mathcal{Z}}(s_0)$ are the same functional, and going modulo periods gives (III.B.6).

Obviously, for (III.B.8) to hold, our chosen lifts must be "compatible" in some sense. We have assumed that $\mathcal{Z}\in Z^n_\pi(\mathcal{X})_{\mathrm{hom}}$; fix once and for all a $\tilde{\Gamma}\in C^{2n-1}_{\#}(\mathcal{X})$ as in Lemma III.A.38(a). For open $U \subset S$ let $\tilde{A}(U) \subset F^n A^{2n-1}\left(\pi^{-1}(U), \log\left\langle X_{s_0} \cap \pi^{-1}(U)\right\rangle\right)$ consist of relatively closed forms producing (by fiberwise restriction) holomorphic sections of $\mathcal{F}_e^n$. Then $\omega \mapsto \pi_*(\tilde{\Gamma} \wedge \omega)$ induces a sheaf map $\tilde{A} \to \mathcal{N}_S^0$; that this is $\mathcal{O}_S$-linear and factors through $[\tilde{A} \twoheadrightarrow]\mathcal{F}_e^n \to \mathcal{O}_S[\subset \mathcal{N}_S^0]$ is easy and done in [Ki1, prop. (3.2.1)]. In this way $\tilde{\Gamma}$ gives a holomorphic section of $(\mathcal{F}_e^n)^\vee$. Moreover, using [Ki3, Thm. 2.3.4] it is of intersection type with respect to (in addition to X_{s_0}) X_s for a.e. $s \in S^*$. By [Ki1, Prop. 3.2.2(2)], for a.e. $s \in S^*$ $\pi^*(\tilde{\Gamma} \wedge \omega)(s) = \int_{\Gamma_s} \iota^*_{X_s}\omega$ (where $\Gamma_s := \tilde{\Gamma} \cdot X_s$ satisfies $\partial\Gamma_s = Z_s$). A holomorphic section of $(\mathcal{F}_e^n)^\vee$ a.e. lifting (III.B.3) lifts (III.B.3) everywhere, so we may take $\tilde{\nu}_{\mathcal{Z}} := (2\pi\sqrt{-1})^n \pi_*(\tilde{\Gamma} \wedge (\cdot))$, $\tilde{\nu}_{\mathcal{Z},\omega} := (2\pi\sqrt{-1})^n \pi_*(\tilde{\Gamma} \wedge \omega)$.

The second lift is much easier to construct; using the *same* $\tilde{\Gamma}$, and taking $\Gamma_i := \tilde{\Gamma} \cdot X_i$, we are in the situation of (II.A.38)(b) so that $\omega(s_0) \mapsto (2\pi\sqrt{-1})^n \sum_i \int_{\Gamma_i} \omega(s_0)_i$ (cf. III.B.7(iii)) gives $\widetilde{\mathrm{AJ}_{X_{s_0}}}(Z_{s_0})$. Now we simply compute: writing T for $\tilde{\Gamma} \wedge \omega \wedge \pi^* \frac{ds}{s}$ (which is a normal current of Y_i-residue-type $\forall i$ and Y_J-transversal $\forall |J| > 1$) and ψ_0 for the characteristic function of S^* (see Step 1), for "arbitrary" ω

$$
\begin{aligned}
\frac{1}{(2\pi\sqrt{-1})^n}\tilde{\nu}_{\mathcal{Z},\omega}(s_0) &= \left(\pi_*(\tilde{\Gamma} \wedge \omega)\right)(s_0) = \mathrm{Res}_{s_0}\left(\pi_*(\tilde{\Gamma} \wedge \omega) \cdot \frac{ds}{s}\right) \\
&= \mathrm{Res}_{s_0}(\pi_* T) = \frac{1}{2\pi\sqrt{-1}}(1-\psi_0)d[\pi_* T] \\
&= \frac{1}{2\pi\sqrt{-1}}\pi_*\left\{(1-\pi^*\psi_0)d[T]\right\} = \frac{1}{2\pi\sqrt{-1}}\pi_*\left\{\sum_i (1-\psi_i)d[T]\right\} \\
&= \pi_*\left\{\sum_i \jmath_{i*}\mathrm{Res}_{X_i}(T)\right\} = \sum_i \int_{X_i} \mathrm{Res}_{X_i}\left(\tilde{\Gamma} \wedge [\omega \wedge \pi^* \frac{ds}{s}]\right),
\end{aligned}
$$

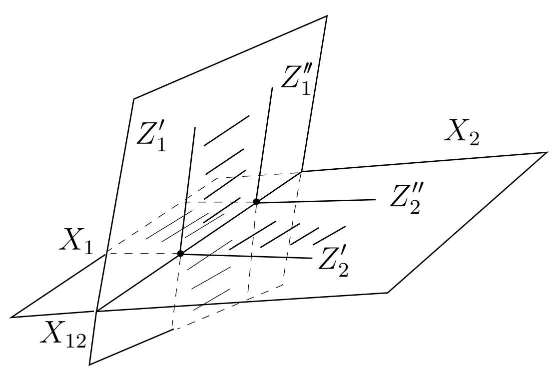
which by Lemma III.A.6(ii)

$$
= \sum_i \int_{X_i} (\jmath_i^*\tilde{\Gamma}) \wedge \mathrm{Res}_{X_i}\left(\omega \wedge \pi^*\frac{ds}{s}\right) = \sum_i \int_{\Gamma_i} \omega(s_0)_i. \quad \square
$$

Remark. The essence of the proof is little more than "$\lim_{s\to s_0}\int_{\Gamma_s}\omega(s) = \int_{\Gamma_{s_0}}\omega(s_0) = \sum_i \int_{\Gamma_i}\omega(s_0)_i$", with added analytic and interpretative complications. Chief amongst these is that $\tilde{\Gamma} \in C^{2n-1}_{\#}(\mathcal{X})$, $\Gamma_{s_0} \in C^{2n-1}_{\#}(X_{s_0})$ (cf. Step 2 and Lemma III.A.38) so that everything converges.

Example. (i) For $n = 1$, this means $[\mathrm{supp}]\Gamma_{s_0}$ does not pass through the "nodes" $X^{[1]} \subset X_{s_0}$.

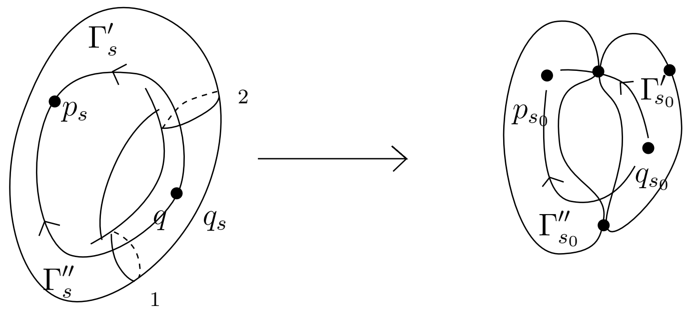
(ii) For $n = 2$ with $X = X_1 \cup X_2$, consider the case where $\mathcal{Z} = \mathcal{Z}' - \mathcal{Z}''$ ($|\mathcal{Z}'| \cap |\mathcal{Z}''| = \emptyset$) intersects X_1, X_2, X_{12} transversely (in the sense of differential topology), hence to first order. Then the intersection integral current $\tilde{\Gamma}$ ($\partial\tilde{\Gamma} = \mathcal{Z}$) can be chosen to be [integration over] a C^∞ chain which meets the X_I transversely. In the local, schematic illustration



$Z_{s_0} = \{Z_1' - Z_1'', Z_2' - Z_2''\}$ while $\Gamma_{s_0} = \{\Gamma_1, \Gamma_2\}$ consists of the two smooth shaded regions. The latter meet the surface X_{12} (transversely) in real codimension 2 (on $|\Gamma_i|$), so that $\Gamma_1 \cdot X_{12} = \Gamma_2 \cdot X_{12}$ is a 1-chain with boundary the 0-cycle $(\mathcal{Z}' - \mathcal{Z}'') \cdot X_{12}$. If $X_{12} \subset X_1$ is locally given by $z = 0$, then the 3-form $\omega(s_0)$ (cf. III.B.7) restricted to X_1 is locally $\omega(s_0)_1 = \alpha \wedge \frac{dz}{z} + \beta$ (α, β C^∞). We can arrange for local *real* C^∞ coordinates on (say) Γ_1 of the form (u, v, w), $u + iv = z$, so that $\Gamma_1 \cdot X_{12} = \{w \in [a, b]\} \cap \{u = v = 0\}$. Hence, in agreement with the general results of III.A (cf. Step 7), $\int_{\Gamma_1} \omega(s_0)_1$ is bounded by an integral of the form $\int \frac{|f| du \wedge dv \wedge dw}{\sqrt{u^2+v^2}}$ (f C^∞) and is absolutely convergent. It is then essentially by continuity of integration that $\lim_{s \to s_0} \int_{\Gamma_s} \omega(s) = \sum_i \int_{\Gamma_i} \omega(s_0)_i$.

We now turn our attention to case (ii) in (III.B.2). Here the normal function $\nu^*_{\mathcal{Z}}$ has *no* singularity and extends to a section $\nu_{\mathcal{Z}}$ of $\mathcal{J}_e$ over S. However, excluding case (i) the geometric interpretation of $\nu_{\mathcal{Z}}(s_0)$ *cannot* be RHS(III.B.6) because $[Z_{s_0}] \neq 0$ in $H^{2n}(X_{s_0})$. The following illustrates the problem with trying to salvage "$\left\langle \tilde{\nu}_{\mathcal{Z}}(s_0), \frac{\lim_{s \to s_0} \omega(s)}{(2\pi\sqrt{-1})^n} \right\rangle = \sum_i \int_{\Gamma_i} \omega(s_0)_i$" independently of (III.B.6).

Example. An elliptic fibration of type I_2 in Kodaira's notation map be pictured as



where the cycles $\delta_1 \equiv_{\text{hom}} \delta_2$ vanish as $s \to s_0$. The monodromy is $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. For the cycle $Z_s = 2(p_s - q_s)$ we have $Z_s = \partial\Gamma_s$, with $\Gamma_s = \Gamma'_s + \Gamma''_s$ multivalued. (Of course, $\mathcal{Z} = \cup_{s \in S} Z_s$ as usual; but there is no global $\tilde{\Gamma}$.) The Dehn twists produced by monodromy in Γ'_s, Γ''_s are opposite in orientation, however, which makes $\int_{\Gamma_s}$ single-valued as a section of $(\mathcal{F}^1)^\vee$ over S^*. So for ω as in III.B.7 ($\omega(s) \in H^0(\Omega^1_{X_s})$; $\omega(s_0) \in H^0(\omega_{X_{s_0}})$ with residues ± 1 at the nodes), $\int_{\Gamma'_s} \omega(s)$ and $\int_{\Gamma''_s} \omega(s)$ diverge logarithmically as $s \to s_0$ but their sum does not. Hence, $\int_{\Gamma_s} \omega(s)$ is nothing but $\left\langle \tilde{\nu}_{\mathcal{Z}}(s), \frac{\omega(s)}{2\pi\sqrt{-1}} \right\rangle$ with $\lim_{s \to s_0} \int_{\Gamma_s} \omega(s) = \left\langle \tilde{\nu}_{\mathcal{Z}}(s_0), \frac{\lim_{s \to s_0} \omega(s)}{2\pi\sqrt{-1}} \right\rangle$; the limit may also be regarded as a principal value of the sum of divergent integrals $\int_{\Gamma_{s_0} \cap X_1} \omega(s_0) + \int_{\Gamma_{s_0} \cap X_2} \omega(s_0)$, but this P.V. cannot be read off from geometric data on X_{s_0}. Again, the problem (and the reason for the divergences) is that Γ_{s_0} is *not* in $C^1_{\#}(X_{s_0})$, and cannot be, since $[Z_{s_0}] \neq 0$.

To determine the class in $H^2(X_{s_0}) \cong H^2(\mathcal{X})$ of a (non-general-position) cycle in $Z^1(\mathcal{X})$, it suffices to compute intersection numbers (via $H^2(\mathcal{X}, \mathbb{Z}) \times H^2_c(\mathcal{X}, \mathbb{Z}) \to H^4_c(\mathcal{X}, \mathbb{Z}) \cong \mathbb{Z}$) with X_1, X_2. For $\hat{\mathcal{Z}}_0 := 3\mathcal{Z} + 2X_1 - 2X_2$ these are zero (using $X_i \cdot X_i = -2$), so $[\hat{\mathcal{Z}}_0] = 0$. Moving $\hat{\mathcal{Z}}_0$ into general position via a rational equivalence on $\mathcal{X}$ (shrinking $\mathcal{X}$ if necessary) therefore produces $\hat{\mathcal{Z}} \in Z^1_{\#}(\mathcal{X})_{\text{hom}}$ (so that $\hat{Z}_i := \hat{\mathcal{Z}} \cdot X_i$ are degree-zero 0-cycles). Hence we are back in case (III.B.2)(i), and can choose $\hat{\Gamma}$ ($\partial\hat{\Gamma} = \hat{\mathcal{Z}}$) so that $\lim_{s \to s_0} \int_{\hat{\Gamma}_s} \omega(s) = \sum_i \int_{\hat{\Gamma}_i} \omega(s_0)$. The key point (to be proved in general below, with a slight change in the choice of Γ_s) is that because $\hat{\mathcal{Z}}^* \equiv_{\text{rat}} 3\mathcal{Z}^*$ on $\mathcal{X}^*$, $\hat{\Gamma}$ may be chosen in such a way that $\int_{\hat{\Gamma}_s} \omega(s) = 3 \int_{\Gamma_s} \omega(s)$ *on the nose*: $\tilde{\nu}_{\hat{\mathcal{Z}}} = 3\tilde{\nu}_{\mathcal{Z}}$ on S^*. Up to torsion, we have therefore found a "geometric interpretation on X_{s_0}" of $\lim_{s \to s_0} \int_{\Gamma_s} \omega(s)$. (We should note that this does not work for case (iii), viz. $Z_s = p_s - q_s$, since in this case $\int_{\Gamma_s} \omega(s)$ itself is log-divergent.)

The example just concluded suggests that the discrepancy between (i) and (ii) (like that between (ii) and (iii)) is torsion from the standpoint of normal functions. The next result implies that this is true at least for $n = 1, 2$; to state it, we shall say (for $\mathcal{Z}, \mathcal{Z}' \in Z^n(\mathcal{X})$) that $\mathcal{Z}'$ is a *modification* of $\mathcal{Z}$ iff $\mathrm{supp}(\mathcal{Z}' - \mathcal{Z}) \subset X_{s_0}$.

Proposition III.B.9. *Let $\mathcal{Z} \in Z^n_\pi(\mathcal{X})$, and assume the HC if $n > 2$. Referring to (III.B.2), in case (iii) there exists a cohomologically trivial modification $\hat{\mathcal{Z}}_0$ of a multiple $m\mathcal{Z}$ ($m \in \mathbb{Z}$). Moving this in a rational equivalence to $\hat{\mathcal{Z}} \in Z^n_\pi(\mathcal{X})_{\mathrm{hom}}$, we have $m\nu_{\mathcal{Z}} = \nu_{\hat{\mathcal{Z}}}$. In case (ii), for any lift $\tilde{\nu}_{\mathcal{Z}}$ we may arrange a lift $\tilde{\nu}_{\hat{\mathcal{Z}}}$ so that $m\tilde{\nu}_{\mathcal{Z}} = \tilde{\nu}_{\hat{\mathcal{Z}}}$.*

Remark. In case (ii) the proof will construct $\tilde{\nu}_{\hat{\mathcal{Z}}}$ as $\pi_*(\hat{\Gamma}\wedge(\cdot))$, leading to the geometric interpretation $m\tilde{\nu}_{\mathcal{Z}}(s_0) = (2\pi\sqrt{-1})^n \sum_i \int_{\hat{\Gamma}_i}(\cdot)$.

Proof. We first note that in case (iii), $\nu_{\mathcal{Z}}$ is meant as a section of the Néron model via Propostion III.B.4, though the equality $m\nu_{\mathcal{Z}} = \nu_{\hat{\mathcal{Z}}}$ is as sections of $\mathcal{J}_e$. This and $m\tilde{\nu}_{\mathcal{Z}} = \tilde{\nu}_{\hat{\mathcal{Z}}}$, of course, need only be verified on S^*.

In case (iii), consider the portion of C-S ($\mathbb{Z}$-coeffs., exact$\otimes\mathbb{Q}$)

$$\begin{array}{ccc} \to H_{2n}(X_{s_0})(-2n) \xrightarrow{\mu} & H^{2n}(X_{s_0}) & \xrightarrow{\rho} H^{2n}_{\lim}(X_s) \to \\ & \| \wr & \\ & H^{2n}(\mathcal{X})\,; & \end{array}$$

since $\rho([\mathcal{Z}]) = 0$, $[\mathcal{Z}] \in \mathrm{im}(\mu_{\mathbb{Q}})$. Now μ is a morphism of MHS, so by strictness

$$[\mathcal{Z}] \in \mathrm{im}\{W_{-2n}H_{2n}(X_{s_0}, \mathbb{Q})\} = \mathrm{im}\{\oplus_i H_{2n}(X_i, \mathbb{Q}) \to H_{2n}(X_{s_0}, \mathbb{Q})\}.$$

Again by strictness, this time relative to the Hodge filtrations, $[\mathcal{Z}] \in \mathrm{im}\{\oplus_i \mathrm{Hg}^{n-1}(X_i)_{\mathbb{Q}} \xrightarrow{\mu'} H^{2n}(X_{s_0}, \mathbb{Q})\}$. Now assume the HC in codim $(n-1)$ (OK for $n \leqslant 2$; requires working rationally); we then have over $\mathbb{Q}$

$$\begin{tikzcd} \oplus_i Z^{n-1}(X_i) \arrow[r, two heads, "\mathrm{HC}"] & \oplus_i \mathrm{Hg}^{n-1}(X_i) \arrow[r, "\iota_*"] \arrow[rr, bend right, "\mu'"'] & H^{2n}(\mathcal{X}) \arrow[r, "\iota^*"] & H^{2n}(X_{s_0}). \end{tikzcd}$$

This shows that we can find $m \in \mathbb{Z}$ and $V_i \in Z^n(X_i)$ such that $\hat{\mathcal{Z}}_0 := m\mathcal{Z} - \sum V_i$ has $[\hat{\mathcal{Z}}_0] = 0$ on $\mathcal{X}$.

Using $Z^n_{\#}(\mathcal{X}, -\bullet) \overset{\simeq}{\hookrightarrow} Z^n(\mathcal{X}, -\bullet)$ and shrinking S if necessary, $\exists\ \mathcal{W} \in Z^n(\mathcal{X}, 1)$ with pullback to all X_s, $s \in S^*$, and $\partial_{\mathcal{B}}\mathcal{W} = \hat{\mathcal{Z}} - \hat{\mathcal{Z}}_0$ (for $\hat{\mathcal{Z}} \in Z^n_\pi(\mathcal{X})_{\mathrm{hom}}$). Consequently $T_{\mathcal{W}} \in \mathcal{I}^{2n-1}(\mathcal{X})$ is of intersection type w.r.t. all smooth fibres, and $\partial T_{\mathcal{W}} = \hat{\mathcal{Z}} - \hat{\mathcal{Z}}_0$ [viewed as chains]. On $\mathcal{X}^*$, $\hat{\mathcal{Z}}_0 = m\mathcal{Z}$; hence for $s \in S^*$ (writing $W_s := \mathcal{W} \cdot X_s$) $\partial_{\mathcal{B}} W_s = \hat{Z}_s - mZ_s$, so that (via (III.B.3)) $\nu^*_{\hat{\mathcal{Z}}} = m\nu^*_{\mathcal{Z}}$.

In case (ii), $\mathcal{Z} = \partial\Gamma$ for $\Gamma \in \mathcal{I}^{2n-1}(\mathcal{X}^*)$, which provides a lift $\tilde{\nu}^*_{\mathcal{Z}} := \pi_*(\Gamma \wedge (\cdot))$ (ignoring $(2\pi\sqrt{-1})^n$'s) as in the proof of Theorem III.B.5, but only over S^*. We obtain two *a priori* different lifts of $\nu_{\hat{\mathcal{Z}}}$: by writing $\hat{\mathcal{Z}} = \partial\hat{\Gamma}$ ($\hat{\Gamma} \in C^{2n-1}_{\#}(\mathcal{X})$), $\tilde{\nu}_{\hat{\mathcal{Z}}} := \pi_*(\hat{\Gamma} \wedge (\cdot))$ (over S); and by observing $\partial(T_{\mathcal{W}}|_{\mathcal{X}^*} + m\Gamma) = \hat{\mathcal{Z}}|_{\mathcal{X}^*} - \hat{\mathcal{Z}}_0|_{\mathcal{X}^*} + m\mathcal{Z} = \hat{\mathcal{Z}}|_{\mathcal{X}^*}$, ${}'\tilde{\nu}^*_{\hat{\mathcal{Z}}} := \pi_*\left((T_{\mathcal{W}} + m\Gamma) \wedge (\cdot)\right)$ (over S^*). These can only differ by an invariant cycle, and we can modify $\hat{\Gamma}$ to make this difference zero. But for $s \in S^*$, writing $\mathbb{P}^1 \overset{\pi_\square}{\leftarrow} \widetilde{\widetilde{W}}_s \overset{\pi_X}{\to} X_s$, $\pi_*(T_{\mathcal{W}} \wedge \omega)(s) = \int_{T_{W_s}} \omega(s) = \int_{\pi_{X*}\pi^*_\square \mathbb{R}^-} \omega(s) = \int_{\mathbb{R}^-} \pi_{\square*}\pi^*_X\omega(s) = 0$ since the integrand is a closed $(1,0)$-current on $\mathbb{P}^1$ hence (by a standard $\bar{\partial}$-regularity result) in $\Omega^1(\mathbb{P}^1) = \{0\}$. So $\tilde{\nu}^*_{\hat{\mathcal{Z}}} ={}' \tilde{\nu}^*_{\hat{\mathcal{Z}}} = m\tilde{\nu}^*_{\mathcal{Z}}$, done. $\square$

It remains to prove Propostion III.B.4, the logarithmic growth part of which is done in [SZ, Prop. 5.28]. Let $\mathcal{Z} \in Z^n_\pi(\mathcal{X})$ be as in (III.B.2)(iii); shrinking S if necessary we may assume that for any acyclic $U \subset S^*$, $\mathcal{Z} \cap \pi^{-1}(U)$ is homeomorphic to $Z_s \times U$ ($s \in U$). The map $H_{2n-1}(X_s, |Z_s|; \mathbb{Z}) \to (F^n H^{2n-1}(X_s))^\vee$ ($s \in S^*$) sending $\Gamma_s \mapsto \int_{\Gamma_s}$ identifies the action of monodromy on a lift $\tilde{\nu}^*_{\mathcal{Z}}(s)$ of the normal function and that of $\tilde{T} \in \mathrm{End}\left(H^{2n-1}(X_s \backslash |Z_s|, \mathbb{Z})\right)$ (which lifts $T \in End\left(H^{2n-1}(X_s, \mathbb{Z})\right)$) on $[\Gamma_s]$. Writing $\mathsf{j} : H^{2n-1}(X_s, \mathbb{Z}) \to H^{2n-1}(X_s \backslash |Z_s|, \mathbb{Z})$ we must show the

Claim. $(\tilde{T}-I)\tilde{\nu}_{\mathcal{Z}}(s) \in \operatorname{im}(\mathsf{j} \circ (T_{\mathbb{Q}}-I)_{\mathbb{Z}})$.

Proof. Working integrally and ignoring MHS, the composition

$$H^{2n-1}(X_s) \overset{\cong}{\to} H^{2n}(\partial\mathcal{X}, X_s) \overset{\gamma}{\to} H^{2n}(\partial\mathcal{X}) \overset{\cong}{\leftarrow} H^{2n}(\mathcal{X}^*) \overset{\mathrm{Res}}{\to} H^{2n+1}_{X_{s_0}}(\mathcal{X})$$

identifies with σ in the C-S complex above (Theorem III.B.5). Here "*Res*" is the connecting homomorphism of the localization sequence for $\mathcal{X}$, X_{s_0}. Define $\tilde{\sigma}$ by the dotted arrow in

$$\begin{tikzcd}
{} \arrow[r] & H^{2n-1}(X_s \setminus |Z_s|) \arrow[r] \arrow[dr, "\tilde{T}-I"'] & H^{2n}(\partial\mathcal{X} \setminus |\partial\mathcal{Z}|, X_s \setminus |Z_s|) \arrow[r, "\tilde{\gamma}"] & H^{2n}(\partial\mathcal{X} \setminus |\partial\mathcal{Z}|) \arrow[r] & {} \\
& & H^{2n-1}(X_s \setminus |Z_s|) \arrow[u, "\cong"] \arrow[dr, dashed] & H^{2n}(\mathcal{X}^* \setminus |\mathcal{Z}^*|) \arrow[u, "\cong"'] \arrow[d, "\widetilde{\mathrm{Res}}"'] & \\
& & & H^{2n+1}_{X_{s_0} \setminus |Z_{s_0}|}(\mathcal{X} \setminus |\mathcal{Z}|); &
\end{tikzcd}$$

here the top row is exact, and so $\tilde{\sigma} \circ (\tilde{T}-I) = 0$. By construction, the following diagram commutes:

$$\begin{tikzcd}
H^{2n}_{|Z_s|}(X_s) \arrow[r, "\text{zero}", "\text{on } Z_s"'] & H^{2n}_{|Z_s|}(X_s) & \\
H^{2n-1}(X_s \setminus |Z_s|) \arrow[u, "\partial"] \arrow[r, "\tilde{T}-I"] & H^{2n-1}(X_s \setminus |Z_s|) \arrow[u, "\partial"] \arrow[r, "\tilde{\sigma}"] & H^{2n+1}_{X_{s_0} \setminus |Z_{s_0}|}(\mathcal{X} \setminus |\mathcal{Z}|) \\
H^{2n-1}(X_s) \arrow[u] \arrow[r, "T-I"] & H^{2n-1}(X_s) \arrow[u, "\mathsf{j}"] \arrow[r, "\sigma"] & H^{2n+1}_{X_{s_0}}(\mathcal{X}) \arrow[u, "\mathsf{j}_0"] \\
& & \underbrace{H^{2n+1}_{|Z_{s_0}|}(\mathcal{X})}_{0} \arrow[u]
\end{tikzcd}$$

where the columns and the bottom row are exact $\otimes\mathbb{Q}$ (at the middle term). We caution that the middle row is (in case (iii)) *not* exact $\otimes\mathbb{Q}$ (even for $n=1$; cf. [As]). In any case, clearly j_0 is injective, hence $(\tilde{T}-I)\Gamma_s \in \mathsf{j}(\ker \sigma_{\mathbb{Z}}) = \mathsf{j}(\operatorname{im}(T_{\mathbb{Q}}-I)_{\mathbb{Z}})$. □

Remark. A referee has pointed out that Propostion III.B.4 also follows from the Theorem of [SZ] that the direct image of the constant sheaf by a nonproper map is an admissible variation of MHS, since this implies that the VMHS associated to a cycle is admissible (using the stability of admissibility under subquotients). Since the monodromy property for lifts of normal functions is crucial for us and the proof in [op. cit., §5] is rather general and technical, we chose to include the direct proof above.

III.C Identification of the extension group G in the Néron model in the geometric case

For use in the computation of examples we shall give, in the geometric case, a description of G that results by intertwining two Clemens-Schmid sequences.

Recall that we are working with a semistable degeneration $\mathcal{X} \to S$ over a disk ($\dim \mathcal{X} = 2n$), so the singular fiber $X_{s_0} = \cup X_i$ is an SNCD. Throughout this subsection, we assume that the $H^*(X_s, \mathbb{Z})$ ($s \in S^*$) are torsion-free, as is the case when the X_s are smooth projective hypersurfaces. It is convenient to augment $\mathcal{X}$ to a "complete total space" $\mathbb{X}$, i.e. a smooth projective variety containing $\mathcal{X}$ as an (analytic) open subset, and write

$$\begin{tikzcd} X_{s_0} \arrow[r, hook, "\jmath"] \arrow[dr, hook, "\imath"'] & \mathbb{X} \\ & \mathcal{X} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

for the inclusion.

We will have to work with abelian groups of finite rank which contain torsion. Let $W_{\mathbb{Z}}$ be one such, $W_{\mathbb{Z}}^{\mathrm{tor}}$ its torsion subgroup, and $W_{\mathbb{Q}} := W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For a subgroup $U_{\mathbb{Z}} \subset W_{\mathbb{Z}}$, we set

$$(U_{\mathbb{Q}})_{\mathbb{Z}} = U_{\mathbb{Q}} \cap W_{\mathbb{Z}} = \mathrm{Ker}(W_{\mathbb{Z}} \to W_{\mathbb{Q}}/U_{\mathbb{Q}}) \ .$$

If $\varphi_{\mathbb{Z}} : W_{\mathbb{Z}} \to V_{\mathbb{Z}}$ is a morphism, one has

$$\begin{aligned} (\mathrm{Im}\, \varphi_{\mathbb{Q}})_{\mathbb{Z}} &= \mathrm{Ker}\left(V_{\mathbb{Z}} \to V_{\mathbb{Q}}/\phi(W_{\mathbb{Q}})\right) \ , \\ (\mathrm{Ker}\, \varphi_{\mathbb{Q}})_{\mathbb{Z}} &= \mathrm{Ker}(W_{\mathbb{Z}} \to V_{\mathbb{Q}}) \ . \end{aligned}$$

Moreover,

$$\textit{if } V_{\mathbb{Z}} \textit{ is torsion-free then } (\mathrm{Ker}\, \varphi_{\mathbb{Q}})_{\mathbb{Z}} = \mathrm{Ker}\, \varphi_{\mathbb{Z}} \ . \tag{III.C.1}$$

Our first result relates the group of components of the Néron model to the failure of the composition, with $\mathbb{Z}$-coefficients,

$$H_{2n}(X_{s_0}) \xrightarrow[\jmath_{\mathbb{Z}}^*]{} H_{2n}(\mathbb{X}) \cong H^{2n}(\mathbb{X}) \xrightarrow[\jmath_*^{\mathbb{Z}}]{} H^{2n}(X_{s_0}) \tag{III.C.2}$$

to be "integrally strict."

Theorem III.C.3. *Assume that $H_{2n-1}(X_{s_0}, \mathbb{Z})$ is torsion-free. Then we have the identification*

$$G \cong \frac{(\mathrm{Im}\, \jmath_{\mathbb{Q}}^* \circ \jmath_*^{\mathbb{Q}})_{\mathbb{Z}}}{\mathrm{Im}\, \jmath_{\mathbb{Z}}^* \circ \jmath_*^{\mathbb{Z}}} \ .$$

Proof. We intertwine the integrally long-exact cohomology sequences (bold arrows) of the pairs $(\mathcal{X}, \partial\mathcal{X})$ and $(\partial\mathcal{X}, X_s)$, where we assume that $X_s \subset \partial\mathcal{X}$,

$$\begin{tikzcd}[column sep=small] & \arrow[r, dashed, "\psi"] & H^{2n}(\mathcal{X}, \partial\mathcal{X}) \arrow[r, "\mu"] & H^{2n}(\mathcal{X}) \arrow[rr, dashed, "\nu"] \arrow[dr, "\alpha"'] & & H^{2n}(X_s) \arrow[r, "T-I"] & {} \\ & & & & H^{2n}(\partial\mathcal{X}) \arrow[ur, "\beta"'] \arrow[dr, "\delta"] & & \\ H^{2n-1}(\mathcal{X}) \arrow[r, dashed, "\nu'"] & H^{2n-1}(X_s) \arrow[rr, "(T-I)'"] & & H^{2n}(\partial\mathcal{X}, X_s) \arrow[ur, "\gamma"] \arrow[rr, dashed, "\psi'"] & & H^{2n+1}(\mathcal{X}, \partial\mathcal{X}) \arrow[r, "\mu'"] & H^{2n+1}(\mathcal{X}) \ . \end{tikzcd}$$

Here "$T-I$" is meant under the identification of $H^*(\partial\mathcal{X}, X_s)$ with $H^{*-1}(X_s)$. Since $X_{s_0} \overset{\imath}{\hookrightarrow} \mathcal{X}$ is a homotopy equivalence, we have *integrally*

$$H^*(\mathcal{X}) \cong H^*(X_{s_0}), \ \ H^*(\mathcal{X}, \partial\mathcal{X}) \cong H_{4n-*}(\mathcal{X}) \cong H_{4n-*}(X_{s_0}) \ ;$$

ν and ψ become respecively r^* and r_* under these identifications. So if we tensor with $\mathbb{Q}$, the rows become Clemens-Schmid sequences; they are therefore *rationally* exact. Using this and duality of r^* and r_*, we have

$$(\mathrm{Ker}(T-I)'_{\mathbb{Q}})^{\perp} = (\mathrm{Im}\, \nu'_{\mathbb{Q}})^{\perp} = (\mathrm{Im}(\psi'_{\mathbb{Q}})^{\vee})^{\perp} = \mathrm{Ker}\, \psi'_{\mathbb{Q}} \tag{III.C.4}$$

where all the groups are considered to lie inside $H^{2n-1}(X_s, \mathbb{Q})$. Since $H^{2n+1}(\mathcal{X}, \partial\mathcal{X}) \cong H_{2n-1}(X_{s_0})$ is torsion-free, (III.C.4) and (III.C.1) combine to give

$$\left(\mathrm{Ker}(T-I)'_{\mathbb{Q}}\right)^{\perp}_{\mathbb{Z}} = (\mathrm{Ker}\, \psi'_{\mathbb{Q}})_{\mathbb{Z}} = \mathrm{Ker}\, \psi'_{\mathbb{Z}} \ .$$

Again from (III.C.1) and $\mathbb{Q}$-exactness of the rows,

$$\operatorname{Ker}\nu_{\mathbb{Z}} = (\operatorname{Ker}\nu_{\mathbb{Q}})_{\mathbb{Z}} = (\operatorname{Im}\mu_{\mathbb{Q}})_{\mathbb{Z}} .$$

Now writing

$$\beta\oplus\delta : H^{2n}(\partial\mathfrak{X}) \to H^{2n}(X_s)\oplus H^{2n+1}(\mathfrak{X},\partial\mathfrak{X})$$

we have

$$\frac{\operatorname{Ker}\psi'_{\mathbb{Z}}}{\operatorname{Im}(T_{\mathbb{Z}}-I)'} \underset{\cong}{\xrightarrow{\gamma}} \operatorname{Ker}(\beta_{\mathbb{Z}}\oplus\delta_{\mathbb{Z}}) \underset{\cong}{\xleftarrow{\alpha}} \frac{\operatorname{Ker}\nu_{\mathbb{Z}}}{\operatorname{Im}\mu_{\mathbb{Z}}} .$$

Putting everything together,

$$G := \frac{\left(\operatorname{Ker}(T-I)'_{\mathbb{Q}}\right)^{\perp}_{\mathbb{Z}}}{\operatorname{Im}(T-I)'_{\mathbb{Z}}} = \frac{\operatorname{Ker}\psi'_{\mathbb{Z}}}{\operatorname{Im}(T-I)'_{\mathbb{Z}}} \cong \frac{\operatorname{Ker}\nu_{\mathbb{Z}}}{\operatorname{Im}\mu_{\mathbb{Z}}} = \frac{(\operatorname{Im}\mu_{\mathbb{Q}})_{\mathbb{Z}}}{\operatorname{Im}\mu_{\mathbb{Z}}} .$$

Finally, using the identifications we can rewrite $\mu_{\mathbb{Z}}$ as the composition

$$H_{2n}(X_{s_0}) \underset{\imath_*}{\xrightarrow{\cong}} H_{2n}(\mathfrak{X}) \cong H^{2n}(\mathfrak{X},\partial\mathfrak{X}) \to H^{2n}(\mathfrak{X}) \underset{\imath^*}{\xrightarrow{\cong}} H^{2n}(X_{s_0})$$

which is equivalent to (III.C.2). $\square$

We want to use Theorem III.C.3 to reduce the computation of G to an intersection computation. For this we write

$$\tilde{\jmath} : \tilde{X}_{s_0} = \coprod_i X_i \to \mathbb{X} ,$$

$$\tilde{\jmath}^*_{\mathbb{Z}}\circ\tilde{\jmath}^{\mathbb{Z}}_* : \bigoplus_i H_{2n}(X_i,Z) \to \bigoplus_i H^{2n}(X_i,\mathbb{Z}) \cong \bigoplus_i H_{2n-2}(X_i,\mathbb{Z}) , \qquad \text{(III.C.5)}$$

and $X^{[l]} = \coprod_{|I|=l+1} X_I$ as usual. As motivation for why this *should* be the map to consider, suppose we were just after the rank of $\jmath^*\circ\jmath_*$ and could ignore the integral structure. Writing it as

$$H_{2n}(X_{s_0},\mathbb{Q}(-n)) \xrightarrow{\jmath_*} H_{2n}(\mathbb{X},\mathbb{Q}(-n)) \cong H^{2n}(\mathbb{X},\mathbb{Q}(n)) \xrightarrow{\jmath^*} H^{2n}(X_{s_0},\mathbb{Q}(n))$$

and applying strictness of W_0 and W_1, we see that the composition

$$\bigoplus_i H_{2n}(X_i,\mathbb{Q}(-n)) \twoheadrightarrow W_0H_{2n}(X_{s_0},\mathbb{Q}(-n)) \xrightarrow{\overline{\jmath^*\circ\jmath_*}} \operatorname{Gr}^W_0 H^{2n}(X_{s_0},\mathbb{Q}(n)) \hookrightarrow \bigoplus_i H^{2n}(X_i,\mathbb{Q}(n))$$

has the same rank. But a weight argument like this is inadequate for the next result.

Theorem III.C.6. *Assume the (co)homologies of the $X^{[l]}$ are all torsion-free, that the action of T on $H^{2n}(X_s)$ (equiv. $H^{2n-2}(X_s)$) is the identity, and that $n\leqslant 2$. Then*

$$G \cong \frac{(\operatorname{Im}\tilde{\jmath}^*_{\mathbb{Q}}\circ\tilde{\jmath}^{\mathbb{Q}}_*)_{\mathbb{Z}}}{\operatorname{Im}\tilde{\jmath}^*_{\mathbb{Z}}\circ\tilde{\jmath}^{\mathbb{Z}}_*} .$$

Proof. The double-complexes of C^∞ chains (resp. intersection integral currents) in the left- (resp. right)-half plane

$$B_{l,k}(n) := C_{2n+2l-k}\left(X^{[-l]};\mathbb{Z}\right)$$
$$B^{l,k}(n) := C^{\#}_{2n-2l-k-2}\left(X^{[l]};\mathbb{Z}\right) := C^{2n+k}_{\#}(X^{[l]};\mathbb{Z})$$

compute homology (resp. cohomology) of X_{s_0}:

$$H_{2n+*}(X_{s_0},\mathbb{Z}) \cong H_*(B_{-\bullet}(n));\;\; H^{2n+*}(X_{s_0},\mathbb{Z}) \cong H^*(B^\bullet(n)) .$$

(Differentials $\alpha_0=\partial$ and $d_1=\mathrm{Gy}$ (resp. $\partial_{\mathfrak{I}}$ in both $B_{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ go up and to the right.) Using $B_\bullet(n)$ one easily checks that the torsion assumption above implies the one in Theorem III.C.3.

Consider the maps

$$\bigoplus_i H_{2n}(X_i,\mathbb{Z}) \xrightarrow{p_{\mathbb{Z}}} H_{2n}(X_{s_0},\mathbb{Z}) \ ,$$
$$H^{2n}(X_{s_0},\mathbb{Z}) \xrightarrow{q_{\mathbb{Z}}} \bigoplus_i H^{2n}(X_i,\mathbb{Z}) \ ;$$

we clearly have $q_{\mathbb{Z}} \circ (\jmath_{\mathbb{Z}}^* \circ \jmath_*^{\mathbb{Z}}) \circ p_{\mathbb{Z}}$. If we can show

(a) $p_{\mathbb{Z}}$ is surjective,

(b) $q_{\mathbb{Z}}$ is injective, and

(c) $\operatorname{Im} q_{\mathbb{Z}} = (\operatorname{Im} q_{\mathbb{Q}})_{\mathbb{Z}}$,

then elementary linear algebra demonstrates the equality of the quotients in the statements of Theorems III.C.3 and III.C.6, establishing the latter result.

For (a), we must show that any 0-cycle in $B_{-\bullet}(n)$ can be moved (by $\underline{\underline{\partial}}$-coboundary) into the $(0,0)$-place $(= \bigoplus_i C_{2n}(X_i,\mathbb{Z}))$. Using the fact that $T-I=0$ on $H^{2n-2}(X_s)$, Clemens-Schmid becomes

$$0 \to H^{2n-2}_{\lim}(X_s,\mathbb{Q}(n-1)) \hookrightarrow H_{2n}(X_{s_0},\mathbb{Q}(-n)) \to H^{2n}(X_{s_0},\mathbb{Q}(n)) \to \ ,$$

with the first term pure of weight 0. It follows that $H_{2n}(X_{s_0},\mathbb{Q}(-n))$ is pure of weight 0, which means (a) can be done rationally; working inductively in the double complex the anti-torsion assumptions ensure that it can be done integrally.

The nontorsion assumptions also show that $H^{2n}(X_{s_0},\mathbb{Z})$ is torsion-free, so (b) would follow from injectivity of $q_{\mathbb{Q}}$. The latter is a consequence of the monodromy assumptions and Clemens-Schmid, which together show $H^{2n}(X_{s_0},\mathbb{Q}(n))$ is pure of weight 0.

For (c), we need to demonstrate that the right-hand-inclusion in

$$(\operatorname{Im} q_{\mathbb{Q}})_{\mathbb{Z}} = \{(\operatorname{Ker} \partial_{\jmath}^{\mathbb{Q}})_{\mathbb{Z}} \subset \bigoplus_i H_{2n-2}(X_i,\mathbb{Z})\} \supseteq \operatorname{Im} q_{\mathbb{Z}}$$

is an equality, by finding a 0-cocycle in $B^{\bullet}(n)$ mapping to a given class in the left-hand side. This is where the condition $n \leqslant 2$, where we will do $n=2$, must be used. Now $B^{\bullet}(2)$ is

$$\begin{array}{ccccc}
\uparrow & & & & \\
C_1^{\#}(X^{[0]}) & & 0 & & \\
\uparrow\partial & & & & \\
\boxed{C_2^{\#}(X^{[0]})} & \xrightarrow{\partial_{\jmath}} & C_0^{\#}(X^{[1]}) & & 0 \\
\uparrow & & \uparrow\partial & & \\
C_3^{\#}(X^{[0]}) & \longrightarrow & \boxed{C_1^{\#}(X^{[1]})} & & 0 \\
 & & \uparrow & & \\
 & & C_2^{\#}(X^{[1]}) & \longrightarrow & C_0^{\#}(X^{[2]}) \\
 & & & & \uparrow \ .
\end{array}$$

A class $[\Gamma] \in H_2(X^{[0]},\mathbb{Z})$ may be represented by an integral ∂-cycle $\Gamma \in C_2^{\#}(X^{[0]};\mathbb{Z})$. If $[\Gamma] \in (\operatorname{Ker} \partial_{\jmath}^{\mathbb{Q}})_{\mathbb{Z}}$ then we can write $\partial_{\jmath}(\Gamma) = \partial\Gamma_1$ for $\Gamma_1 \in C_1^{\#}(X^{[1]};\mathbb{Q})$. But then clearly we can choose $\Gamma_1 \in C_1^{\#}(X^{[1]};\mathbb{Z})$ since an integral 0-cycle which is rationally a boundary must be one integrally. Obviously $\partial_{\jmath}(\Gamma_1)=0$, so $(\Gamma;\Gamma_1)$ gives an integral cycle in the boxed terms in the double-complex, one which maps to Γ. □

Remark. Here we just point out that the assumptions of Theorem III.C.6 are not unrealistic.

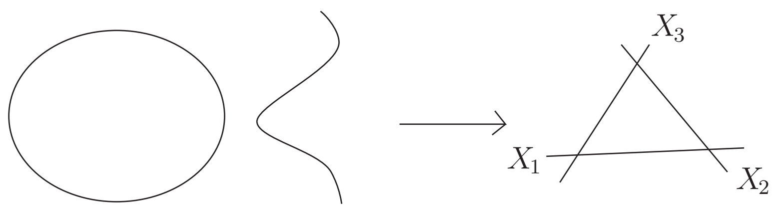
(i) If $H^{2n}(X_s,\mathbb{Q})$ (equiv. H^{2n-2}) is spanned by algebraic classes (e.g., smooth projective hypersurfaces), then the monodromy T is finite. Since we are assuming unipotent monodromy, clearly then $T=I$.

(ii) The process of semistable reduction involves blow-ups; this is not necessarily a problem from the standpoint of the nontorsion assumptions on the $H^*(X^{[l]})$. If a variety (like $\mathbb{P}^m$ or a smooth hypersurface) with torsion-free (co)homology is blown up at a smooth center having no torsion in its own (co)homology, the result will have torsion-free cohomology.

(iii) The double complex argument fails when $n > 2$.

Conclusion. In the geometric case this reduces the computation of G to an intersection calculation. More specifically, the map (III.C.5) is obtained by putting cycles in the image of $\bigoplus_i H_{2n}(X_i, \mathbb{Z})$ in general position in $\mathbb{X}$ (or, if you want, in $(\mathfrak{X}, \partial\mathfrak{X})$), and then intersecting them with the $X_j \subset X_{s_0}$.

Example. For a pencil of cubics degnerating to a triangle



the mapping $\bigoplus_i H_2(X_i) \to \bigoplus_i H_0(X_i)$ has matrix

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} .$$

Here we have used that $X_i^2 = -2$, which for example follows from symmetry and

$$0 = (X_1 + X_2 + X_3)^2 = X_1^2 + X_2^2 + X_3^2 + 6 .$$

Adding the third column to two times the middle column and adding the first two columns to the third reduces the matrix to

$$\begin{pmatrix} -2 & 3 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Since

$$\mathbb{Z}/\mathbb{Z}\left\langle (-2,1), (3,-3) \right\rangle \cong \mathbb{Z}/3\mathbb{Z} ,$$

it follows that

$$G \cong \mathbb{Z}/3\mathbb{Z} .$$

An example with $n = 2$ will be worked out below.

IV. Examples

We shall consider the much studied example (cf., for example, [Sch] and [COGP])

$$s\left(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5\right) + x_1x_2x_3x_4x_5 = 0 , \tag{IV.1}$$

which we shall refer to as the Fermat/pentahedron (F/P) pencil. We are interested in (IV.1) in a neighborhood S of $s = 0$; as usual we set $s_0 = \{0\}$. As written, the degeneration is not semistable; X_{s_0} will denote the fiber over s_0 after semistable reduction, whereas P denotes $\{x_1x_2x_3x_4x_5 = 0\}$. We shall prove the

Theorem IV.2. *(i) The group G of components of the Néron model is*

$$G \cong (\mathbb{Z}/5\mathbb{Z})^3 .$$

(ii) Recalling our notation [ ] *for the map* $\tilde{J}_{e,s_0} \to G$*, if* L'_s *and* L''_s *are lines in* X_s *such that the specializations* L'_{s_0} *and* L''_{s_0} *are in different faces of* P*, then*

$$\left[\lim_{s \to s_0} \mathrm{AJ}_{X_s}(L'_s - L''_s)\right] \neq 0 \, .$$

Moreover, all of G *is generated by the image of such limits.*

If L'_{s_0} and L''_{s_0} both lie in the same face $\mathbb{P}^3_j$ of P, then

$$\mathrm{Gr}_0^W \left(\lim_{s \to s_0} \mathrm{AJ}_{X_s}(L'_s - L''_s) \right) = \left(\mathrm{Gr}_0^W \left(\mathrm{AJ}(\rho) \right) \left(\mathrm{AJ}_{C_{ij}}(p'_i - p''_i) \right) \right. \tag{IV.3}$$

where $p'_i = L'_{s_0} \cap C_{ij}$, $p''_i = L''_{s_0} \cap C_{ij}$ for $i < j$, and C_{ij} are Fermat quintic curves in the faces, to be described below. We were not able to determine if the RHS is of finite order m in $J(C_{ij})$. If so, then for some non-zero integer m we would have

$$\lim_{s \to s_0} \mathrm{AJ}_{X_s}(m(L'_s - L''_s)) \in W_{-1}(J_{e,s_0})$$

and this would be interpreted as an Abel-Jacobi image of a class in $\mathrm{CH}^2(-,1)$ of a surface $X^{[1]}$, to be defined below. Since we do not know this, in order to illustrate in this example how higher Chow groups do arise it is of interest to find a cycle W_{s_0} in $W_k H^4_{\mathcal{M}}(X_{s_0}, \mathbb{Z}(2))$, $k \leqq -2$, that deforms to a family of cycles $W_s \in Z^2(X_s)$.

Theorem IV.4. *There exists a cycle* $W_{s_0} \in Z^2_{\#}(X_{s_0})_{\mathrm{hom}}$ *with the properties:*

(i) W_{s_0} *deforms to* $W_s \in Z^2(X_s)_{\mathrm{hom}}$ *for* $s \in S$*;*

(ii) $\displaystyle\lim_{s \to 0} \int_{\Gamma_s} \omega(s) \in \mathbb{C}/\mathbb{Z}(2)$ *is non-zero, for* $\omega(s) \in F^3_{e,s}$ *as in (III.B.10).*

(iii) $[W_{s_0}]_{\mathcal{M}} \in W_{-3}(H^4_{\mathcal{M}}(X_{s_0}, \mathbb{Z}(2))$.

The value in (ii) may be viewed as a projected image of $\mathrm{AJ}_{X_{s_0}}([W_{s_0}]_{\mathcal{M}})$*. From this we may infer the nontriviality of* $[W_{s_0}]_{\mathcal{M}}$ *as well as* $\mathrm{AJ}_{X_s}(W_s)$ *for general* $s \neq s_0$*, since* $J(X_{s_0}) \xrightarrow[\cong]{J(\rho)} J_{e,s_0}$*.*

Remark. We recall that in this section S is a disc — we are not claiming that W_{s_0} deforms to the whole pencil.

The class of W_{s_0} lives naturally in $\mathrm{CH}^2(\mathbb{C}, 3)$ and the limit in (ii) will be interpreted as a regulator. The forms $\omega(s)$ are generators of $H^0(\Omega^3_{X_s})$ for $s \neq s_0$, and $\omega(s_0) \in H^0(\omega_{X_{s_0}})$ will be normalized in the proof of (ii).

As previously noted, the point of this example is to illustrate how regulators on the higher $\mathrm{CH}^2(X_I, |I| - 1)$ may appear as limits of the usual Abel-Jacobi maps on $\mathrm{CH}^2(X_s)_{\mathrm{hom}}$.

IV.A Semi-stable reduction

Let

$$\begin{array}{ccc} \mathcal{X}_1 & \subset & S \times \mathbb{P}^4 \\ \downarrow & & \\ S & & \end{array}$$

be the variety (IV.1). We want to apply semi-stable reduction (SSR) to obtain

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}_1 \\ \downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

so that our general theory can be applied to $\mathcal{X} \xrightarrow{\pi} S$. The issue here is that $\mathcal{X}_1$ is singular where the base locus of the pencil (IV.1) meets the singularities $\mathbb{P}^2_{ij}$ of P, and these singularities must be resolved. After working through this process we shall arrive at the

Theorem IV.A.1. *Let $W_k(J_{e,s_0})$ denote the filtration induced by the monodromy weight filtration on the intermediate Jacobian $J_{e,s_0} = F^0_{e,s_0} \operatorname{Ker} N_{\mathbb{C}} \backslash \operatorname{Ker} N_{\mathbb{C}}/(\mathcal{H}_{\mathbb{Z}})_{s_0}$. Denote by C the Fermat quintic curve $x^5 + y^5 = 1$. Then the graded pieces of $W_k(J_{e,s_0})$ are given by*

$$\begin{cases} \mathrm{Gr}_0 \cong \overset{10}{\oplus} J(C)(-1) \\ \mathrm{Gr}_{-1} = 0 \\ \mathrm{Gr}_{-2} \cong \overset{40}{\oplus} (\mathbb{C}/\mathbb{Z}(1)) \\ \mathrm{Gr}_{-3} \cong \mathbb{C}/\mathbb{Z}(2)\ . \end{cases}$$

We shall use the following notations:

- $P = \underset{i}{\cup} \mathbb{P}^3_i$;
- $S = \underset{i}{\cup} S_i$ where the union of the Fermat surfaces $S_i = F \cap \mathbb{P}^3_i$ constitutes the base locus of (IV.1);
- $\mathbb{P}^2_{ij} = \mathbb{P}^3_i \cap \mathbb{P}^3_j$, $\mathbb{P}^2_{ijk} = \mathbb{P}^3_i \cap \mathbb{P}^3_j \cap \mathbb{P}^3_k$ etc.;
- $C_{ij} = \mathbb{P}^3_j \cap S_i$ are Fermat quintic curves, the union of whose images in P constitutes the singular part of the base locus in P;
- $X_{s_0} = \underset{i}{\cup} X_i$ where $X_i \to \mathbb{P}^3_i$ is obtained by a sequence of blow-ups.

Then for the X_i we have:

(i) $X_1 = \mathbb{P}^3_1$;

(ii) $X_2 =$ blow-up of $\mathbb{P}^3_2$ along C_{12} with $B_{12} \cong \mathbb{P}\left(\mathcal{O}_C(1) \oplus \mathcal{O}_C(5)\right)$ the resulting $\mathbb{P}^1$-bundle over C_{12};

(iii) $X_3 =$ blowup of $\mathbb{P}^3_3$ along C_{13} to get B_{13}, followed by the blow up along the proper transform of C_{23} to get B_{23}. The second blow up yields a blow up of B_{13} at the 5 points where C_{23} intersects it; denote this by $\tilde{B}_{13}$;

(iv) $X_4 =$ blow up of $\mathbb{P}^3_4$ along C_{14} to get B_{14}, then blow up C_{24} to get B_{24}, which yields a $\tilde{B}^{(0)}_{14}$ as with $\tilde{B}_{13}$ above. Then blowing up along C_{34} gives B_{34}, $\tilde{B}^{(1)}_{14}$ and $\tilde{B}^{(0)}_{24}$;

(v) Finally, blow up $\mathbb{P}^3_5$ as follows to obtain X_5:

$$\begin{aligned} &C_{15} \rightsquigarrow B_{15} \\ &C_{25} \rightsquigarrow \tilde{B}^{(0)}_{15}, B_{25} \\ &C_{35} \rightsquigarrow \tilde{B}^{(1)}_{15}, \tilde{B}^{(0)}_{25}, B_{35} \\ &C_{45} \rightsquigarrow \tilde{B}^{(2)}_{15}, \tilde{B}^{(1)}_{25}, \tilde{B}^{(0)}_{35}, B_{45}\ . \end{aligned}$$

Note that this process is asymmetric, since each C_{ij} only gets blown up in one of the faces of P containing it. These $\mathbb{P}^1$ bundles then get further blown up at points of intersection as in (iii) above.

The motivation for this seemingly arbitrary procedure is as follows: if we take the proper transform of $\mathcal{X}_1$ under blow-up of $\mathbb{P}^4$ along S_1, S_2, S_3, S_4, S_5 (in that order), then the $\{X_s\}_{s \neq s_0}$ remain unchanged while P is replaced by X_{s_0} as just described. One checks in local coordinates that the resulting $\mathcal{X}$ is smooth and $X_{s_0} = \pi^{-1}(s_0)$ is a reduced SNCD in $\mathcal{X}$.

The picture of the E_1-term of the spectral sequence associated to $B^{l,k}_{x_{s_0}}(2)$ (see §III.A), where the columns are the cohomology groups, is

$$
\begin{array}{ccccccc}
\coprod_i X_i & & \coprod_{i<j} X_{ij} & & \coprod_{i<j<k} X_{ijk} & & \coprod_{i<j<k<l} X_{ijkl} \\
5 & & & & & & \\
0 & & & & & & \\
15 & \longrightarrow & 10 & & & & \\
120 & & 0 & & & & \\
15 & \longrightarrow & 60 & \longrightarrow & 10 & & \\
0 & & 0 & & 0 & & \\
5 & \longrightarrow & 10 & \longrightarrow & 10 & \longrightarrow & 5
\end{array}
$$

This gives for the E_2 term, where the circled part is $H^3(X_{s_0})$:

$$
\begin{array}{cccc}
5 & & & \\
0 & & & \\
5 & 0 & & \\
120 & 0 & & \\
5 & 40 & 0 & \\
0 & 0 & 0 & \\
1 & 0 & 0 & 1
\end{array}
\qquad \} W_{-3} \quad \} W_{-2} \quad \} W_{-1} \quad \} W_0
$$

To prove the theorem we need to show that the map $J(X_{s_0}) \xrightarrow{J(\rho)} J_{e,s_0}$ is an isomorphism, or equivalently that ρ in the Clemens-Schmid sequence

$$H_5(X_{s_0}) \to H^3(X_{s_0}) \xrightarrow{\rho} H^3_{\mathrm{lim}}(X_s)$$

is injective. This follows from the above picture of E_2, which gives $H_5(X_{s_0}) = 0$.

IV.B Proof of Theorem IV.2

We shall use Theorem III.C.6, together with the calculations in section IV.A above, to show that for the F/P pencil

$$G \cong (\mathbb{Z}/5\mathbb{Z})^3 .$$

So far as we know, this is the first explicit computation of a Néron model in a non-classical — i.e. $N^2 \neq 0$ — situation.

We shall follow the notations from section IV.A, together with the following: First recall that $\mathcal{X}$ is obtained by *sequentially* blowing up $\mathbb{P}^4$ along S_1, S_2, S_3, S_4, S_5; hence the previously noted asymmetry of X_{s_0}. We will use the notation (i)–(v) from the description of X_{s_0} in §IV.A, as well as the following:

- We denote by $P_1 \subset X_1$ a generic 2-plane, and by $\tilde{P}_i$, $i \geqq 2$, the blow up of a generic $P_i \cong \mathbb{P}^2 \subset \mathbb{P}^3_i$ along the points which arise from the blowing up process described in IV.A;
- L_i is the proper transform in X_i of general line in $\mathbb{P}^3_i$; and finally
- F_{ij} is a general $\mathbb{P}^1$ fibre of $\tilde{B}_{ij} \to C_{ij}$.

We want to compute the map, with $\mathbb{Z}$-coefficients

$$\bigoplus_i H_4(X_i) \xrightarrow{\mu} \bigoplus_j H_2(X_j) ,$$

obtained by putting a cycle on X_i in general position in $\mathfrak{X}$ and then intersecting with the X_j. Obviously the tricky maps are the

$$H_4(X_i) \to H_2(X_i)\ . \tag{IV.B.1}$$

We shall use the following integral bases for these homology groups:

$$\begin{aligned}
H_4(X_1) &= \langle P_1 \rangle & H_2(X_1) &= \langle L_1 \rangle \\
H_4(X_2) &= \left\langle \tilde{P}_2, B_{12} \right\rangle & H_2(X_2) &= \langle L_2, F_{12} \rangle \\
H_4(X_3) &= \left\langle \tilde{P}_3, \tilde{B}_{13}, B_{23} \right\rangle & H_2(X_3) &= \langle L_3, F_{13}, F_{23} \rangle \\
H_4(X_4) &= \left\langle \tilde{P}_4, \tilde{B}_{14}, \tilde{B}_{24}, B_{34} \right\rangle & H_2(X_4) &= \langle L_4, F_{14}, F_{24}, F_{34} \rangle \\
H_4(X_5) &= \left\langle \tilde{P}_5, \tilde{B}_{15}, \tilde{B}_{25}, \tilde{B}_{35}, B_{45} \right\rangle & H_4(X_5) &= \langle L_5, F_{15}, F_{25}, F_{35}, F_{45} \rangle\ .
\end{aligned}$$

Then the matrix of the map (IV.B.1) expressed in the above bases is

		X_1	X_2		X_3			X_4				X_5				
		P_1	$\tilde{P}_2$	B_{12}	$\tilde{P}_3$	$\tilde{B}_{13}$	B_{23}	$\tilde{P}_4$	$\tilde{B}_{14}$	$\tilde{B}_{24}$	B_{34}	$\tilde{P}_5$	$\tilde{B}_{15}$	$\tilde{B}_{25}$	$\tilde{B}_{35}$	B_{45}
X_1	L_1	-4	1	5	1	5	0	1	5	0	0	1	5	0	0	0
X_2	L_2	1	-4	-5	1	0	5	1	0	5	0	1	0	5	0	0
	F_{12}	-5	5	10	0	5	-5	0	5	-5	0	0	5	-5	0	0
X_3	L_3	1	1	0	-4	-5	-5	1	0	0	5	1	0	0	5	0
	F_{13}	-5	0	5	5	10	5	0	5	0	-5	0	5	0	-5	0
	F_{23}	0	-5	-5	5	5	10	0	0	5	-5	0	0	5	-5	0
X_4	L_4	1	1	0	1	0	0	-4	-5	-5	-5	1	0	0	0	5
	F_{14}	-5	0	5	0	5	0	5	10	5	5	0	5	0	0	-5
	F_{24}	0	-5	-5	0	0	5	5	5	10	5	0	0	5	0	-5
	F_{34}	0	0	0	-5	-5	-5	5	5	5	10	0	0	0	5	-5
X_5	L_5	1	1	0	1	0	0	1	0	0	0	-4	-5	-5	-5	-5
	F_{15}	-5	0	5	0	5	0	0	5	0	0	5	10	5	5	5
	F_{25}	0	-5	-5	0	0	5	0	0	5	0	5	5	10	5	5
	F_{35}	0	0	0	-5	-5	-5	0	0	0	5	5	5	5	10	5
	F_{45}	0	0	0	0	0	0	-5	-5	-5	-5	5	5	5	5	10

An integral column reduction, analogous to one in the simple example in Section III.C of cubics degenerating to a triangle, gives

	$P_1 + B_{12} + B_{23}$	$B_{12} + B_{23}$	B_{23}	B_{34}	
X_1	1	5	0	0	L_1
X_2	1	0	5	0	L_2
	0	5	-5	0	F_{12}
X_3	-4	-5	-5	5	L_3
	5	10	5	-5	F_{13}
	5	5	10	-5	F_{23}
X_4	1	0	0	-5	L_4
	0	5	0	5	F_{14}
	0	0	5	5	F_{24}
	-5	-5	-5	10	F_{34}
X_5	1	0	0	0	L_5
	0	5	0	0	F_{15}
	0	0	5	0	F_{25}
	-5	-5	-5	5	F_{35}
	0	0	0	-5	F_{45}

The circled rows also make it clear that the column vectors give an integral basis for $\mathrm{Im}(\mu_{\mathbb{Z}})$, and the vectors

$$\left\{ \begin{array}{c} \mu(P_1 + B_{12} + B_{23}) \\ \frac{\mu(B_{12}+B_{23})}{5} \\ \frac{\mu(B_{23})}{5} \\ \frac{\mu(B_{34})}{5} \end{array} \right. \tag{IV.B.2}$$

are a basis for $\mathrm{Im}(\mu_{\mathbb{Q}}) \cap \mathbb{Z}^{15} = \mathrm{Ker}\{\nu_{\mathbb{Z}} : H^4(X_{s_0}) \to H^4(X_s)\}$. Thus $G \cong (\mathbb{Z}/5\mathbb{Z})^3$ with generators

	$\frac{\mu(B_{12}+B_{23})}{5}$	$\frac{\mu(B_{23})}{5}$	$\frac{\mu(B_{34})}{5}$	
L_1	1	0	0	
L_2	0	1	0	
	1	-1	0	F_{12}
L_3	-1	-1	1	
	2	1	-1	F_{13}
	1	1	-1	F_{23}
L_4	0	0	-1	
	1	0	1	F_{14}
	0	1	1	F_{24}
	-1	-1	2	F_{34}
L_5	0	0	0	
	1	0	0	F_{15}
	0	1	0	F_{25}
	-1	-1	1	F_{35}
	0	0	-1	F_{45}

This completes the proof of (i) in Theorem IV.2. □

We now turn to the proof of (ii) in that theorem. A family of lines on $\mathcal{X}$ is a cycle $\mathcal{L}' \in Z^2_{\#}(\mathcal{X})$ with linear restrictions $\mathcal{L}' \cdot X_s$, $s \neq s_0$. Its limit is

$$L'_{s_0} := \mathcal{L}' \cdot X_{s_0} \in Z^2_{\pi}(X_{s_0}) .$$

The components L'_i of L'_{s_0} obviously will then satisfy matching conditions along the X_{ij}. So retaining our above notation L_i for the cohomology class of the proper transform of a generic line on $\mathbb{P}^3_i$, we cannot expect, for example, $L_1 - L_j$ ($j \geqq 2$) to give the class in $\bigoplus_i H_2(X_i)$ of a "limit of differences of lines". Such a difference would not satisfy matching, would not give a cohomology class on X_{s_0} and would not deform. Rather, $L_1 - L_j$ ($j \geqslant 2$) must be tweaked by F_{ij}'s to cancel face intersections.

With this understood, and referring to (IV.B.2), there are *unique* cohomology classes in $\mathrm{Ker}\,\rho = \mathrm{Ker}\,\nu_{\mathbb{Z}}$ representing such limits and coming from cycles homologous to zero on X_s for $s \neq s_0$

$$\frac{1}{5}\mu(B_{12}) = L_1 - L_2 + 2F_{12} + F_{13} - F_{23} + F_{14} - F_{24} + F_{15} - F_{25}$$

$$\frac{1}{5}\mu(B_{12}+B_{23}) = L_1 - L_3 + F_{12} + 2F_{13} + F_{23} + F_{14} - F_{34} + F_{15} - F_{35}$$

$$\frac{1}{5}\mu(B_{12}+B_{23}+B_{34}) = L_1 - L_4 + F_{12} + F_{13} + 2F_{14} + F_{24} + F_{34} + F_{15} - F_{45}$$

$$-\mu(P_1) - \frac{1}{5}\mu(3B_{12}+2B_{23}+B_{24}) = L_1 - L_5 + F_{12} + F_{13} + 2F_{15} + F_{25} + F_{35} + F_{45} .$$

These are all non-zero under the projection $\mathrm{Ker}\,\rho \to G$, and map onto generators for G under different choices of lines. □

Remark. An interesting issue is when a line $L \subset \mathbb{P}^3_1$ is the limit of actual lines $L_s \subset X_{s_0}$, not just at the cohomology level. The condition, due to Sheldon Katz [Kat2] is that *L meet all the quintic curves C_{1i}*, $i \geqq 2$. The reason for the necessary condition is that, for $s \neq s_0$, L_s will meet each component S_i of the base locus, and therefore in the limit will meet $S_i \cap \mathbb{P}^3_i = C_{ji}$.

IV.C Proof of Theorem IV.4

Step one: Recall our notation

$$X_{s_0} = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$$

where X_5 is the facet of the pentahedron that has all four Fermat curves $C_{i_5} \subset \mathbb{P}^3_i \cap \mathbb{P}^3_5 \qquad 1 \leqq i \leqq 4$ blown up. We want to construct a cycle

$$Z_{s_0} = (0,0,0,0,Z_5) \tag{IV.C.1}$$

in the kernel of

$$\underset{i}{\oplus} Z^2_{\#}(X_i) \to \underset{i<j}{\oplus} Z^2_{\#}(X_{ij}) \tag{IV.C.2}$$

and which deforms to X_s for s near s_0, to give a cycle $\mathcal{Z} \in Z^2_\pi(\mathcal{X})$ with $\mathcal{Z} \cdot X_{s_0} = Z_{s_0}$.

We denote by $X_5 \xrightarrow{\sigma} \mathbb{P}^3_5$ the projection, and by $\tilde{S}_5 \subset X_5$ the proper transform of the Fermat quintic surface $S_5 \subset \mathbb{P}^3_5$.

Definition. A cycle $Z_5 \in Z^2(X_5)_{\text{hom}}$ is "good" if it does not meet the intersections $X_{j_5} = X_j \cap X_5$ for $1 \leqq j \leqq 4$, and if its intersections with each $\tilde{B}_{j5}$ are contained in $\tilde{B}_{j5} \cap \tilde{S}_5$.

Lemma IV.C.3. *Z_5 is good if, and only if, the irreducible components of $\sigma(Z_5)$:*

(i) *do not meet $\mathbb{P}^2_{j_5} \backslash C_{j_5}$*

(ii) *are tangent to S_5 where they meet C_{j_5}, and*

(iii) *are tangent to S_5 to 2^{nd} order where they meet the $P^\alpha_{ijk\nu}$*

Lemma IV.C.4. *If Z_5 is good, then the cycle Z_{s_0} given by (IV.C.1) deforms to $Z_s \in Z^2(X_s)$ for s near s_0. Moreover, it is in the kernel of the map (IV.C.2).*

The proof of Lemma IV.C.3 is by explicit computation in local coordinates; Lemma IV.C.4 is checked by methods similar to those in [Kat1].

The double complex that computes $H^4_{\mathcal{M}}(X_{s_0}, \mathbb{Z}(2))$ is

$$\begin{array}{ccccccc}
\underset{i}{\oplus} Z^2_{\#}(X_i) & \xrightarrow{\partial_{\mathcal{I}}} & \underset{i<j}{\oplus} Z^2_{\#}(X_{ij}) & & & & \\
\uparrow & & \uparrow & & & & \\
 & \longrightarrow & \underset{i<j}{\oplus} Z^2_{\#}(X_{ij},1) & \to & & & \\
 & & \uparrow & & \uparrow & & \\
 & & & \longrightarrow & \underset{i<j<k}{\oplus} Z^2_{\#}(X_{ijk,2}) & \longrightarrow & \\
 & & & & \uparrow & & \uparrow \\
 & & & & & \longrightarrow & \underset{i<j<k<l}{\oplus} Z^2_{\#}(X_{ijkl},3) \\
 & & & & & & \uparrow
\end{array} \tag{IV.C.5}$$

The idea in step one is to construct a cycle in $\oplus Z^2_{\#}(X_{ijkl},3)$ that satisfies a related but different notion of "goodness", and then to move this cycle by a $\underline{\underline{\partial_{\mathcal{B}}}}$-coboundary in (IV.C.5) to $\text{Ker}(\partial_{\mathcal{I}}) \subseteq$

$\bigoplus\limits_i Z^2_{\#}(X_i)$ where it will be a $(0,0,0,0,Z_5)$ with the resulting cycle Z_5 being good as in the definition above. By construction

$$[Z_{s_0}]_{\mathcal{M}} \in W_{-3}H^4_{\mathcal{M}}(X_{s_0}, \mathbb{Z}(2)) \ .$$

DEFINITION. $f(u), g(u) \in \mathbb{C}(\mathbb{P}^1)^*$ is a good pair if

(i) $f(0) = 1 = g(\infty)$

(ii) $f|_{|(g)|} = 1 = g|_{|(f)|}$

(iii) the poles of f are of order 2, except at $u = 1$ where order 3 is permitted.

The last condition will only be relevant to the deformability of the Z_5 to be constructed; (i) and (ii) are essential for moving it in (IV.C.5).

Example.

$$f(u) = \frac{(1+u)^3}{(1-u)^3}, \quad g(u) = \frac{(1-\sqrt{-3}/u)^3}{(1+\sqrt{-3}/u)^3} \ .$$

Setting $\square = (\mathbb{P}^1 \backslash \{1\})$, the graph of the pair f, g in $\mathbb{P}^1 \times \square \times \square$ gives an element

$$W_{\mathrm{rel}} = \{f(u), g(u)\} \in \mathrm{CH}^2((\mathbb{P}^1, \{0,\infty\}), 2)$$

and $\{(u, f(u), g(u)) : u \in \mathbb{P}^1\} \cap \square^3$ gives an element $W_{f,g} \in \mathrm{CH}^2(\mathbb{C}, 3)$. For the above example it is in the image of

$$\mathrm{CH}^2(\mathbb{Q}(\sqrt{-3}), 3) \to \mathrm{CH}^2(\mathbb{C}, 3) \ .$$

The situation may be sumarized by the following commutative diagram, which will be further explained and justified below:

$$\begin{tikzcd}
\textit{Good pairs} \arrow[r] & \textit{Good cycles} \arrow[r] & H^4_{\mathcal{M}}(X_{s_0}(\mathbb{Z}(2))_{\mathrm{hom}} \arrow[r, "\mathrm{AJ}_{X_{s_0}}"] & J(X_{s_0}) \arrow[dd, "\left\langle \bullet, \frac{\omega_{s_0}}{(2\pi\sqrt{-1})^3} \right\rangle"] \\
\cup & Z_{f,g} & & \\
(f,g) \arrow[ur] \arrow[rr, "W_{f,g}"'] \arrow[drr, "\mathcal{N}_{f,g}"'] & & \mathrm{CH}^2(\mathbb{C},3) \arrow[uu, hook, "\alpha"'] \arrow[r, "\mathrm{AJ}_{\mathbb{C}}(3)"] & \mathbb{C}/\mathbb{Z}(2) \arrow[d, "\mathrm{Im}"] \\
& & \mathbb{Z}[\mathbb{C}^*] \arrow[r, "D_2"] & \mathbb{R}
\end{tikzcd} \tag{IV.C.6}$$

The maps

$$\text{Good pairs} \longrightarrow \text{Good cycles}$$

will be constructed below, by successively moving $W_{f,g}$ step by step to $Z_{f,g}$ in (IV.C.5). For the other maps:

- $\mathcal{N}_{f,g} = \sum \nu_\alpha(f)\nu_\beta(g) \left(\frac{\beta}{\alpha}\right)$

where the sum is over $(\alpha, \beta) \in |(f)| \times |(g)|$;

- $D_2(z) = \mathrm{Im}(Li_2(z)) + \arg(1-z)\log|z|$

is the Bloch-Wigner function, extended in the above diagram by linearity to $\mathbb{Z}[\mathbb{C}^*]$;

- $\omega(s) \in H^0(\Omega^3_{X_s})$ for $s \neq s_0$ tends in the limit to the standard form

$$\omega(s_0) = \mathrm{Res}_P \left(\frac{dx \wedge dy \wedge dz \wedge dw}{xyzw} \right) \in H^0(\omega_{X_{s_0}})$$

when x, y, z, w are affine coordinates and the notation means that we take the residue on the pentahedron P and pull back to X_{s_0}; and

- writing $Z_s = \partial\Gamma_s$ we will have for $\left\langle \bullet, \frac{\omega(s_0)}{(2\pi\sqrt{-1})^3} \right\rangle$

$$\lim_{s\to s_0} \tfrac{1}{2\pi\sqrt{-1}} \int_{\Gamma_s} \omega(s) = \tfrac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{s_0}} \omega(s_0) \in \mathbb{C}/\mathbb{Z}(2)\ ; \tag{IV.C.7}$$

the reason for the $\frac{1}{2\pi\sqrt{-1}} \leftrightarrow \frac{1}{(2\pi\sqrt{-1})^3}$ discrepancy is the $(2\pi\sqrt{-1})^2$ in (III.A.34). That this gives the map claimed and that the upper right-hand square of (IV.C.6) commutes will be checked in Proposition IV.C.9 below. To be more explicit, if

$$f(u) = \prod_j \left(1 - \frac{u}{\alpha_j}\right)^{m_j}, g(u) = \prod_k \left(1 - \frac{\beta_k}{u}\right)^{n_k}$$

where $\sum_j m_j = \sum_k n_k = 0$ then

$$\mathcal{N}_{f,g} = \sum_{j,k} m_j n_k \left(\frac{\beta_k}{\alpha_j}\right) \in \mathbb{Z}[\mathbb{C}^*]\ .$$

In the diagram (IV.C.6), the map $\mathrm{AJ}_{\mathbb{C}}(3) : \mathrm{CH}^2(\mathbb{C}, 3) \to \mathbb{C}/\mathbb{Z}(2)$ is the regulator given by

$$W \mapsto \frac{1}{2\pi\sqrt{-1}} \int_W R(x, y, z)\ .$$

Setting $\mathrm{AJ}_{\mathbb{C}}(3)(W_{f,g}) =: R_{f,g}$, one has (essentially by work of Bloch [Bl1]) $\mathrm{Im}\, R_{f,g} = D_2(\mathcal{N}_{f,g})$, and this is encoded in (IV.C.6). The point is that if $\mathrm{Im}\, R_{f,g} \neq 0$ then the cycle

$$W_{f,g} \neq 0 \ \text{ in } \ K_3^{\mathrm{ind}}(\mathbb{C})\ .$$

For the example above $\mathcal{N}_{f,g} = 18(-\sqrt{-3}) - 18(\sqrt{-3})$ and since $D_2(\bar{z}) = -D_2(z)$,

$$D_2(\mathcal{N}_{f,g}) = -36 D_2(\sqrt{-3}) \neq 0\ .$$

Step two: The next step is to move $W = W_{f,g}$ to the upper left position in (IV.C.5) to obtain our desired cycle $Z_{s_0} = Z_{f,g} \in \bigoplus_i Z^2_{\#}(X_i)$. The end result is given by (IV.C.8) below.

The first step will use the following piece of the diagram (IV.C.5)

$$\begin{array}{ccc} Z^2_{\#}(\mathbb{P}^1_x, 2) & & \\ \uparrow{\scriptstyle \partial_B} & & \\ Z^2_{\#}(\mathbb{P}^1_x, 3) & \xrightarrow{i_0 - i_\infty} & Z^2_{\#}(\{0\} \cup \{\infty\}, 3) \end{array}$$

where we have set

$$\begin{cases} \mathbb{P}^1_x = X_{345} \\ \{0\} = X_{2345},\ \ \{\infty\} = X_{1345}\ , \end{cases}$$

Consider

$$A = \left[x; 1 - \frac{1-u}{1+x^5}, f(u), g(u)\right] \in Z^2(\mathbb{P}^1_x, 3)\ .$$

Then

$$\begin{cases} i_0(A) = \left(1 - \frac{1-u}{1+0}, f(u), g(u)\right) = (u, f(u), g(u)) \\ i_\infty(A) = (1, f(u), g(u)) = 0 \end{cases}$$

where the second follows from $1 \notin \square$. Note that $(u, f(u), g(u))$ is $W_{f,g}$. As for the vertical differential $\partial_B(A)$, letting ζ be a primitive 10^{th} root of unity, we obtain the cycle

$$B := \left[x; f(-x^5), g(-x^5)\right] - \sum_{l=0}^{4} \left[\zeta^{2l+1}; f(u), g(u)\right]_{u\in\mathbb{P}^1}$$

in $\mathbb{P}^1_x \times \square^2$. Here, the second terms are graphs of 1-cycles lying over the designated points $x = \zeta^{2l+1}$ of $\mathbb{P}^1_x$.

The next move is in the piece

$$\begin{array}{l} Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,y}, 1) \\ \quad \uparrow \partial_B \\ Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,y}, 2) \longrightarrow Z^2_{\#}(\mathbb{P}^1_x, 2) \oplus Z^2_{\#}(\mathbb{P}^1_y, 2) \oplus Z^2_{\#}(\mathbb{P}^1_{x/y}, 2) \end{array}$$

of (IV.C.5) where $\tilde{\mathbb{P}}^2_{xy} = X_{45}$, and $\mathbb{P}^1_{x/y}$ is the $\mathbb{P}^1$ "at ∞". Here we consider the cycle

$$C := \left[x, y; \frac{1 - f(-x^5)(1+x^5)}{1 + x^5 + y^5}, g(-x^5)\right] - \sum_{l=0}^{4} \pi^*_{\zeta^{2l+1}} \left[\zeta^{2l+1}; f(u), g(u)\right]_{u\in\mathbb{P}^1}$$

in $Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,y}, 2)$. The horizontal map is the sum of the usual restrictions with signs, and sends C to B in $Z^2_{\#}(\mathbb{P}^1_x, 2)$ and 0 in the other summands (using $g(\infty) = 1 = f(0)$ and $1 \notin \square$). On the other hand, ∂_B sends C to

$$D := [\{(1 + x^5)f(-x^5) + y^5 = 0\}; g(-x^5)] - [\{1 + x^5 + y^5 = 0\}; g(-x^5)]$$

in $\tilde{P}^2_{x,y} \times \square$. The components of D are supported over the bracketed curves, by which we mean, "take the proper transform in $\tilde{\mathbb{P}}^2_{x,y}$ of this curve in $\mathbb{P}^2_{x,y}$."

The final move takes place in the diagram

$$\begin{array}{l} Z^2_{\#}(\tilde{\mathbb{P}}^3_{x,y,z}) \\ \quad \uparrow \\ Z^2_{\#}(\tilde{\mathbb{P}}^3_{x,y,z}, 1) \longrightarrow Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,y}, 1) \oplus Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,z}, 1) \oplus Z^2_{\#}(\tilde{\mathbb{P}}^2_{y,z}, 1) \oplus Z^2_{\#}(\tilde{\mathbb{P}}^2_{\infty}, 1) \end{array}$$

where $\tilde{\mathbb{P}}^3_{x,y,z} = X_5$. In $Z^2_{\#}(\mathbb{P}^3_{x,y,z}, 1)$ we consider the cycle

$$\begin{aligned} E = &\left[\{(x, y) : (1 + x^5)f(-x^5) + y^5 = 0\}, z; \left(1 - \frac{1 - g(-x^5)(1 + x^5 + y^5)}{1 + x^5 + y^5 + z^5}\right)\right] \\ &-\pi^* \left[\{(x, y) : 1 + x^5 + y^5 = 0\}; g(-x^5)\right] \end{aligned}$$

where $B_{45} \xrightarrow{\pi} C_{45}$ is the projection and the cycle $\pi^*[\cdot\cdot]$ lives in $\tilde{\mathbb{P}}^3_{x,y,z} \times \square$. The horizontal map (restrictions to faces) sends E to D in $Z^2_{\#}(\tilde{\mathbb{P}}^2_{x,y}, 1)$ and to 0 in the remaining summands. For example, for the restriction of the first term of E to $\tilde{\mathbb{P}}^2_{x,z}$, $y = 0 \Rightarrow (1 + x^5)f(-x^5) = 0 \Rightarrow -x^5 = 1$ or $f(-x^5) = 0 \Rightarrow -x^5 = 1$ or $g(-x^5) = 1$ (since f, g is a good pair). Either of these circumstances will make the big function in parentheses equal to 1 ($\notin \square$).

Finally, our desired cycle is

$$\begin{aligned} (Z_5)_{f,g} := \partial_B(D) = &\{(x, y, z) : (1 + x^5)f(-x^5) + y^5 = 0, (1 + x^5 + y^5)g(-x^5) + z^5 = 0\} \\ &-\{(x, y, z) : (1 + x^5)f(-x^5) + y^5 = 0, 1 + x^5 + y^5 + z^5 = 0\}\ . \end{aligned} \tag{IV.C.8}$$

Note that the 2^{nd} term lies in $S_5 = \{1 + x^5 + y^5 + z^5 = 0\}$; the first does not.

Conclusion. The cycle Z_{s_0} given by (IV.C.1) and (IV.C.8) satisfies (i) in Theorem IV.4.

Step 3: It remains to prove (ii) in Theorem IV.4. This will follow from (IV.C.7) together with the

Proposition IV.C.9. *We have*

$$\mathrm{Im}\left(\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_{s_0}}\omega(s_0)\right)=D_2(\mathcal{N}_{f,g})\neq 0\ .$$

Proof. We first give $\omega(s_0)$ explicitly; we need a collection of compatible log forms on substrata of X_{s_0} in the discussion around (III.A.31). These are obtained by pulling back

$$\hat{\Omega}:=d\log(x_2/x_1)\wedge\cdots\wedge d\log(x_5/x_1)\in\Omega^3(\mathcal{X}_1^*,\log\langle P\rangle)$$

along the semi-stable reduction $\mathcal{X}\to\mathcal{X}_1$ to

$$\Omega\in\Omega^3(\mathcal{X}^*,\log\langle X_{s_0}\rangle)\ ,$$

and taking iterated residues along the X_I. If we write x,y,z for affine coordinates on $\mathbb{P}^3_5$, then for example $\omega_5,\omega_{45},\omega_{345},\omega_{2345}$ are resp. the pullbacks of

$$\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z},\ \frac{dx}{x}\wedge\frac{dy}{y},\ \frac{dx}{x},\ 1$$

under $X_5\to\mathbb{P}^3_5$. We note that $\omega(s_0)$ represents a class in $F^{-1}H_3(X_{s_0},\mathbb{C})$.

Now (III.A.34) tells us that

$$\left\langle \mathrm{AJ}_{X_{s_0}}(\alpha(W_{f,g})),\frac{[\omega(s_0)]}{(2\pi\sqrt{-1})^3}\right\rangle=\langle \mathrm{AJ}_{\mathbb{C}}(3)(W_{f,g}),\omega_{2345}\rangle$$

and the right-hand term is visibly $R_{f,g}$. Since $[\alpha(W_{f,g})]_{\mathcal{M}}=[Z_{f,g}]_{\mathcal{M}}=[Z_{s_0}]_{\mathcal{M}}$ and (III.A.34) is well-defined,

$$\left\langle \mathrm{AJ}_{X_{s_0}}(Z_{s_0}),\frac{[\omega(s_0)]}{(2\pi\sqrt{-1})^3}\right\rangle\equiv\left\langle \mathrm{AJ}_{X_{s_0}}(\alpha(W_{f,g})),\frac{[\omega(s_0)]}{(2\pi\sqrt{-1})^3}\right\rangle$$

modulo $(2\pi\sqrt{-1})^2\times\{\text{periods of }\frac{\omega(s_0)}{(2\pi\sqrt{-1})^3}\}$. The difficulty is in showing that these periods are just the integers.

Let $\mathbb{X}$ be the *entire* semi-stable reduced pencil (containing $\mathcal{X}$ as an open set). In forthcoming work of the 3rd author with C. Doran [DK], a class $\Xi\in\mathrm{CH}^4(\mathbb{X}^*,4)$ is constructed with "cohomology class" $\frac{[\Omega]}{(2\pi\sqrt{-1})^4}\in H_4(\mathbb{X},X_{s_0})$. The homology class of $\mathrm{Res}\,\Xi\in\mathrm{CH}^3(X_{s_0},3)$, which is clearly $\frac{[\omega(s_0)]}{(2\pi\sqrt{-1})^3}$, must belong to $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0),H_3(X_{s_0},\mathbb{Z}))$. This says that $\langle -,\omega(s_0)\rangle$ induces a "splitting" morphism of MHS

$$H^3(X_{s_0},\mathbb{Z}(2))\twoheadrightarrow\mathbb{Z}(2)\ ,$$

and a corresponding map of Jacobians

$$J^2(X_{s_0})\twoheadrightarrow\mathbb{C}/\mathbb{Z}(2)\ .$$

This completes the proof. □

It is worth commenting that this map of MHS is something very special, and does not exist for the singular fiber over the maximal unipotent monodromy point of an arbitrary pencil of quintics. Another point we should make is that the content of this proof is basically contained in the upper right-hand square of (IV.C.6). To prove this commutes it suffices to verify that

$$\mathbb{C}/\mathbb{Z}(2)\hookrightarrow J(X_{s_0})\xrightarrow{\left\langle -,\frac{\omega(s_0)}{(2\pi\sqrt{-1})^3}\right\rangle}\mathbb{C}/\mathbb{Z}(2)$$

is the identity, which boils down to the splitting above plus a scaling check. For the latter, one need only observe that $\omega(s_0)/(2\pi\sqrt{-1})^3$ evaluates to unity against the vanishing cycle $\delta_{s_0}:=\{|x|=|y|=|z|=\epsilon\}$ (on X_5). □

IV.D Limit Abel-Jacobi computation for the Collino construction of the Ceresa cycle

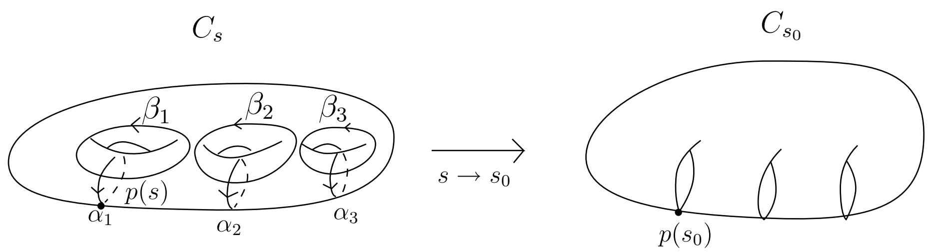
Let C be a smooth algebraic curve of genus $g = 3$ with Jacobian $J(C)$. Upon choice of a base point there is the standard embedding $i : C \hookrightarrow J(C)$. By definition, the *Ceresa cycle* is

$$W = i(C) - j(i(C)) \in Z^2(J(C)),$$

where $j : J(C) \to J(C)$ is the involution $u \mapsto -u$. It is homologous to zero and together with its variants has provided a very fertile example of many — perhaps even most — of the non-classical phenomena that occur for Abel-Jacobi mappings in higher codimension [Ba], [Ce], [Co]. In this section we will consider a degeneration

$$C_s \to C_{s_0} \qquad\qquad (s \in S = \{|s| < 1\})$$

of the type



where $p(s) \in C_s$ is a base point tending to a node $p(s_0) \in C_{s_0}$. We will construct a family $\mathcal{X} \to S$ related to $\{J(C_s)\}_{s \in S}$ together with cycles $Z_s \in Z^2(X_s)$ related to the Ceresa cycle. We shall then analyze the limit $\lim_{s \to s_0} \langle \mathrm{AJ}_{X_s}(Z_s), \omega(s) \rangle \in \mathbb{C}/\mathbb{Q}(2)$ where $\omega(s) \in H^0(\omega_{X_s})$ will be an explicit generator for all s. The result will be

$$\lim_{s \to s_0} \langle \mathrm{AJ}_{X_s}(Z_s), \omega(s) \rangle = -(64\sqrt{-1})C - (16\pi\sqrt{-1}) \log 2 \qquad \text{(IV.D.1)}$$

where C is Catalan's constant. Since the RHS is non-zero this shows in particular that

$$\mathrm{AJ}_{X_s}(Z_s) \neq 0$$

for s close to s_0, provided we can show $\langle -, \omega(s_0) \rangle$ induces a splitting of the MHS $H^3(X_{s_0})$ as in the last example. The point is again to illustrate how Abel-Jacobi maps may be explicitly evaluated in the limit.

Step one: Normalizations: We choose a standard basis $\omega_1(s), \omega_2(s), \omega_3(s)$ for $H^0(\Omega^1_{C_s})$, $s \neq s_0$, so as to normalize the period matrix $\int_{\alpha_i} \omega_j(s) = \delta_{ij}$. The monodromy is given by

$$\begin{cases} T\alpha_i = \beta_i + \alpha_i \\ T\alpha_i = \alpha_i \ . \end{cases}$$

Denoting by π_{α_i}, π_{β_i} the period vectors relative to the above basis for $H^0(\Omega^1_{C_s})$ and setting $J_s = J(C_s)$

$$J_s = \mathbb{C}^3 / \mathbb{Z} \left\langle \{\pi_{\alpha_i}\}, \{\pi_{\beta_i}\} \right\rangle \ .$$

For the limit curve we will take the description

$$C_{s_0} = \mathbb{P}^1 \Big/ \left\{ \begin{array}{c} 0 = \infty \\ 1 = -1 \\ \sqrt{-1} = -\sqrt{-1} \end{array} \right\}$$

where the expression in brackets means to identify the designated points where $\mathbb{P}^1$ has coordinate z.

Then

$$\begin{cases} \omega_1(s_0) &= \frac{1}{2\pi\sqrt{-1}} d\log z \\ \omega_2(s_0) &= \frac{1}{2\pi\sqrt{-1}} d\log\left(\frac{z-1}{z+1}\right) \\ \omega_3(s_0) &= \frac{1}{2\pi\sqrt{-1}} d\log\left(\frac{z-\sqrt{-1}}{z+\sqrt{-1}}\right) . \end{cases}$$

Step two: Construction of the family $\mathcal{X} \to S$ and cycles $Z_s \in Z^2(X_s)$: The family of Jacobians $J(C_s)$ on which the Ceresa cycle W_s lives degenerates to a compactification of $(\mathbb{C}^*)^3$. Using (u_1, u_2, u_3) as coordinates on $\mathbb{C}^3$, in the limit the lattice is generated by

$$(1,0,0),\ (0,1,0),\ (0,0,1) \qquad\qquad (\alpha\text{-periods})$$

and

$$\left(\text{“}\sqrt{-1}\infty\text{”}, \frac{1}{2}, \frac{1}{2}\right),\ \left(\frac{1}{2}, \text{“}\sqrt{-1}\infty\text{”}, \frac{1}{2}\right),$$
$$\left(\frac{1}{2}, \frac{1}{2}, \text{“}\sqrt{-1}\infty\text{”}\right) \qquad\qquad (\beta\text{-periods});$$

that is, the compactification of the generalized Jacobian $(\mathbb{C}^*)^3$ of C_{s_0} is not $(\mathbb{P}^1/\{0=\infty\})^3$, but rather one identifies $(0, z_2, z_3)$ with $(\infty, -z_2, -z_3)$, etc. To remedy this we shall use a lattice Λ whose limiting β-periods are

$$(\sqrt{-1}\infty, 0, 0),\ \ (0, \sqrt{-1}\infty, 0),\ \ (0, 0, \sqrt{-1}\infty)\ .$$

To this end, we define a lattice Λ in $\mathbb{C}^3$ by

$$\Lambda = \mathbb{Z}\left\langle \{\pi_{\alpha_i}\}, \pi_{\beta_1} - \frac{1}{2}(\pi_{\alpha_2}+\pi_{\alpha_3}), \pi_{\beta_2} - \frac{1}{2}(\pi_{\alpha_1}+\pi_{\alpha_3}), \pi_{\beta_3} - \frac{1}{2}(\pi_{\alpha_1}+\pi_{\alpha_2}) \right\rangle$$

and set $X_s = \mathbb{C}^3/\Lambda$, where the dependence of $\Lambda = \Lambda_s$ on s, $s \neq s_0$, will be understood. Denoting by u_1, u_2, u_3 coordinates in $\mathbb{C}^3$, we have an isogeny

$$\begin{array}{ccc} J_s & \xrightarrow{\mu} & X_s \\ \Cup & & \Cup \\ (u_1, u_2, u_3) & \longrightarrow & (2u_1, 2u_2, 2u_3) \end{array}$$

and involution

$$\begin{array}{ccc} X_s & \xrightarrow{j} & X_s \\ \Cup & & \Cup \\ (u_1, u_2, u_3) & \longrightarrow & (-u_1, -u_2, u_3)\ . \end{array}$$

We define the cycle

$$Z_s = \mu \circ i(C_s) - j(\mu \circ i(X_s)) \in Z^2(X_s)\ .$$

This cycle is obtained by first sending

$$z \to \left(\log z, \log\left(\frac{z-1}{z+1}\right), \log\left(\frac{z-\sqrt{-1}}{z+\sqrt{-1}}\right)\right)\ ,$$

then exponentiating, next squaring the entries corresponding to applying μ, and finally taking the curve given by this operation together with subtracting the curve obtained by applying the involution $(z_1, z_2, z_3) \to (z_1^{-1}, z_2^{-1}, z_3^{-1})$. The explicit form of the cycle is given in the lemma below.

We set

$$\omega(s) = du_1 \wedge du_2 \wedge du_3 \in H^0(\Omega^3_{X_s})\ ,$$

and note that

$$\begin{aligned}\Delta_s &= \alpha_1 \times \alpha_2 \times \alpha_3 \\ &= \left\{(u_1, u_2, u_3) \in \mathbb{R}^3 \mod \Lambda\right\}\end{aligned}$$

is the *unique* invariant 3-cycle in $H_3(X_s, \mathbb{Z})$. We have

$$\lim_{s \to s_0} \int_{\Delta_s} \omega(s) = 1 \in \mathbb{Q} \, .$$

We next have a general

Splitting Principle. *For a general family $\mathcal{X} \xrightarrow{\pi} S$, let $\Delta_{1,s}, \ldots, \Delta_{k,s}$ be a basis over $\mathbb{Q}$ for the invariant cycles*

$$\Gamma(S, R_\pi^{2n-1}\mathbb{Q}) \cong \operatorname{Ker} N_{\mathbb{Q}} \, ,$$

and let $\omega(s) \in \mathcal{F}_{s_0}^n$ satisfy

$$\lim_{s \to s_0} \int_{\Delta_{i,s}} \omega(s) \in \mathbb{Q}$$

for all i. Referring to (III.B.7), set

$$\omega(s_0) := \sigma(\lim_{s \to s_0} \omega(s)) \in F^{-n+1} H_{2n-1}(X_{s_0})$$

where we recall the Clemens-Schmid sequence ($\mathbb{Q}$-coefficients)

$$H^{2n-1}(X_{s_0}) \xrightarrow{\rho} H^{2n-1}_{\lim}(X_s) \xrightarrow{N} H^{2n-1}_{\lim}(X_s) \xrightarrow{\sigma} H_{2n-1}(X_{s_0}) \, .$$

Then

$$\begin{cases} \omega(s_0) \in \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H_{2n-1}(X_{s_0}, \mathbb{Q})) \\ \lim_{s \to s_0} \omega(s) \in \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), \operatorname{Coker} N \otimes \mathbb{Q}(n-1)) \end{cases} \tag{IV.D.2}$$

and these two induce dual splittings (morphisms of MHS)

$$\begin{tikzcd} H^{2n-1}(X_{s_0}) \arrow[rr, two heads] \arrow[dr] & & \mathbb{Q}(0) \\ & \operatorname{Ker} N. \arrow[ur] & \end{tikzcd} \tag{IV.D.3}$$

Moreover, tensoring (IV.D.2) with $\mathbb{Q}(n)$ and taking $\operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \bullet)$ we have

$$\begin{tikzcd} J(X_{s_0}) \arrow[rr, two heads, "\delta"] \arrow[dr, "J(\rho)"'] & & \mathbb{C}/\mathbb{Q}(n) \\ & J(\operatorname{Ker} N) \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

where δ, ϵ are induced by

$$(2\pi\sqrt{-1})^n \left\langle \bullet, \omega(s_0) \right\rangle$$

and

$$(2\pi\sqrt{-1})^n \left\langle \bullet, \lim_{s \to s_0} \omega(s) \right\rangle = \lim_{s \to s_0} (2\pi\sqrt{-1})^n \left\langle \bullet, \omega(s) \right\rangle$$

respectively.

Remark. A more complete discussion of "splitting principles" with proofs and further applications will appear in a future work of the 3rd author.

We next recall that

$$\begin{cases} J(X_{s_0}) \cong \left(F^{-n+1}H_{2n-1}(X_{s_0})\right)^\vee / H^{2n-1}(X_{s_0},\mathbb{Z}) \\ J(\operatorname{Ker} N) = J_{e,s_0} \cong \check{F}^n_{e,s_0} / \operatorname{Ker} N_{\mathbb{Z}} \ . \end{cases}$$

Prior to semistable reduction the fibre over s_0 in the family $\mathcal{X} \to S$ is the compactified semi-abelian variety

$$(\mathbb{P}^1/\{\infty = 0\})^3 =: Y \ .$$

Denoting by (z_1, z_2, z_3) the coordinates on Y, away from the codimension two strata $\bigcup_{i\neq j}(z_i = z_j = 0) =: \mathrm{Sing}^2(Y)$ the map

$$X_{s_0} \to Y$$

is an isomorphism. Also set $\mathrm{Sing}(Y) := \underset{i}{\cup}(z_i = 0)$. We now have the

Lemma. *The specialization Z_{s_0} of the family of cycles Z_s is given parametrically by*

$$\begin{aligned} Z_{s_0} = &\left\{\left(z^2, \left(\frac{z-1}{z+1}\right)^2, \left(\frac{z-\sqrt{-1}}{z+\sqrt{-1}}\right)^2\right) : z \in \mathbb{P}^1\right\} \\ &-\left\{\left(z^{-2}, \left(\frac{z+1}{z-1}\right)^2, \left(\frac{z+\sqrt{-1}}{z-\sqrt{-1}}\right)^2\right) : z \in \mathbb{P}^1\right\} \ . \end{aligned}$$

In particular, since $Z_{s_0} \cdot \mathrm{Sing}(Y) = 0$ we have $Z_{s_0} \in Z^2_{\#}(Y)$; moreover, the support $|Z_{s_0}|$ does not meet $\mathrm{Sing}^2(Y)$.

We shall label the two terms in Z_{s_0} as Z', Z'' so that $Z_{s_0} = Z' - Z''$. As long as we stay away from $\mathrm{Sing}^2(Y)$ we need not be concerned with the specifics of the semistable reduction.

We note that the projection of Z_{s_0} on each of the factors in Y cancels out; this shows that $[Z_{s_0}]_{\mathcal{M}} \in H^4_{\mathcal{M}}(Y, \mathbb{Q}(2))_{\mathrm{hom}}$. It follows [KLM, §5.8] that we may choose a chain $\Gamma_{s_0} \in C_3^{\#}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ such that

$$\begin{cases} \partial\Gamma_{s_0} = Z_{s_0} \\ \Gamma_{s_0} \cdot \left(\bigcup_{i=1}^3 \{z_i = 0 \ \text{ or } \ \infty\}\right) = \emptyset \end{cases}$$

and

$$\int_{\Gamma_{s_0}} \omega(s_0) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^3 \int_{Z_{s_0}} R_3(z_3, z_1, z_2) \tag{IV.D.4}$$

where

$$\begin{aligned} R(z_3, z_1, z_2) = &\log z_3 \, d\log z_2 \wedge d\log z_2 \\ &+(2\pi\sqrt{-1}) \log z_1 \, d\log z_2 \cdot \delta_{T_{z_3}} - (4\pi^2) \log z_2 \delta_{T_{z_3} \cap T_{z_1}} \ . \end{aligned}$$

Here, we have used the notation $T_f = f^{-1}(\mathbb{R}^- \cdot e^{-i\epsilon})$ for the current assosicated to a map $Y \xrightarrow{f} \mathbb{P}^1$ obtained by pulling back a small peturbation of $\mathbb{R}^-$, the reason for the peturbation being to avoid $-\sqrt{-1}$. The integral of the first term in R is zero by type and the sum of the contributions from the third term cancel in the limit as $\epsilon \to 0$. There is some tedium involved here in keeping track of branches of log, so we omit the details. It then follows that $(2\pi\sqrt{-1})^2$ times the RHS of (IV.D.4) is

$$\int_{Z_{s_0}} \log z_1 \, d\log z_2 \wedge \delta_{T_{z_3}}$$

which is equal to

$$2\int_{T_{\left(\frac{z-\sqrt{-1}}{z+\sqrt{-1}}\right)^2}} \log z^2 \, d\log\left(\frac{z-1}{z+1}\right) - 2\int_{T_{\left(\frac{z+\sqrt{-1}}{z-\sqrt{-1}}\right)^2}} \log(z^{-2}) \, d\log\left(\frac{z+1}{z-1}\right)$$

$$= -32(L(\sqrt{-1}) - L(-\sqrt{-1}))$$
$$= 64\sqrt{-1}C - 16\pi\sqrt{-1}\log 2$$

where

$$L(x) = Li_2(x) + \frac{1}{2}\log x \log(1-x)$$

is the Rogers dilogarithm and

$$C = \beta(2) = \sum_{m \geqq 0} \frac{(-1)^m}{(2m+1)^2}$$

is Catalan's constant. It follows that

$$(2\pi\sqrt{-1})^2 \int_{\Gamma_{s_0}} \omega(s_0) \in \mathbb{C}/\mathbb{Q}(2)$$

is purely imaginary and non-zero, hence nontrivial in $\mathbb{C}/\mathbb{Q}(2)$. □

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