NÉRON MODELS AND BOUNDARY COMPONENTS FOR DEGENERATIONS OF HODGE STRUCTURE OF MIRROR QUINTIC TYPE

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To Roy Smith, on his 60^{th} *birthday*

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Introduction

By a *family of Hodge structures of mirror quintic type* we shall mean a variation of polarized Hodge structures (VHS) of weight $w = 3$ and Hodge numbers $h^{3,0} = h^{2,1} = 1$. We shall assume throughout that *the polarization is principal.* It is well known that the number of essential parameters in the variation is one. Therefore we shall assume that the parameter space for the VHS is either

the punctured disc $S^* = \{0 < \|s\| < 1\}$	(local case)
a smooth algebraic curve S^*	(global case) .

In each case S^* has a canonical smooth completion S obtained by adding the puctures. We shall assume throughout that the local monodromy around a puncture is *unipotent*.

Families of Hodge structure of mirror quintic type are the simplest non-classical VHS's of odd weight. In some ways they may be thought of as the first non-classical analogues of families of elliptic curves; as such they may be expected to indicate some of the deeper aspects of VHS's of odd weight and the associated families of intermediate Jacobians and normal functions.

Our primary objective in this paper is to work out in detail the general theory of Néron models in [GGK] and extended period maps in [KU], and to relate these two, for the case of families of Hodge structures of mirror quintic type. For this, we shall build on and complement the extensive literature growing out of the original work by Candelas et al [CdOGP]; cf. the book [CoK] for a valuable account of the theory up until 1999. These families are also discussed briefly from different perspectives in [GGK] and [KU].

This paper is partly expository and partly devoted to some new material. Regarding the latter we shall emphasize

(i) Insofar as it is well-defined, we shall keep precise track of the integral structure, as this is essential for the Néron model;
(ii) In the case at hand, relate the general theory from [GGK] and [KU] in response to the question: How much of the fine structure of a degeneration of a VHS of mirror quintic type is captured by the limit of the period mapping?
(iii) Illustrate all possible degenerations of a VHS of mirror quintic type by geometric examples, in part building on and refining the work of the physicists and that in [Mo] and [CoK].

As we shall see below, there are three types of local degenerations of families of Hodge structures of mirror quintic type, classified according to their monodromy logarithms N by

Case I: $N^3 \neq 0$
Case II_1: $N^2 = 0$ and rank $N = 1$
Case II_2: $N^2 = 0$ and rank $N = 2$.

Our labelling here is motivated by the classical work [Ko] on degenerations of elliptic curves.

Starting from [GP] and [CdOGP] and continuing to the present [KSV], case I has been the object of the most study. To this we shall add the anaysis of the possible Néron models, their relation to the boundary components, and the detailed structure in geometric examples.

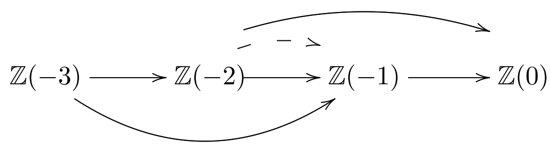
Case II_1 also occurs in a geometric example in the literature (loc. cit and [Mo]) but appears to have been less studied. We shall again determine the possible Néron models and their relation to the boundary components, and illustrate these in a geometric example.

Case II_2 appears in a geometric example in [Bo] and [Vo]. Here, the integral, symplectic linear algebra — i.e., the monodromy in an adapted, integral symplectic basis, is formally the same as for Hodge-Tate degenerations of genus two curves. However, the limiting mixed Hodge structure and boundary components are quite different and, we feel, illustrate interesting contrasts between classical and non-classical degenerations of Hodge structures.

Our main technique is, quite naturally, the close analysis of the *limiting mixed Hodge structure* (LMHS). Here, the one variable theory of [Sc] and its geometric analogue [St] are sufficient, although to draw the aforementioned contrast in the discussion of case II_2 does require some relatively elementary several variable theory. Usually, the LMHS is only defined over $\mathbb{Q}$. Here for geometric reasons we will keep track of the integral information.

In [KU] boundary components are defined group theoretically in terms of certain rational nilpotent cones and their admissible nilpotent orbits. Here, we shall identify these geometrically as "the part of the LMHS that is invariantly defined".

For degenerations of type I the LMHS is of the much studied Hodge-Tate type and may be pictured as



Using the canonical choice of a degeneration parameter suggested by the physicists, we find that the extensions depicted by the dotted arrow and by the two solid arrows — the latter of which are dual — are well-defined and are given by ratios of entries in the rational matrix N, which implies that they are torsion. There is one remaining variable complex parameter in the LMHS, which serves as a local coordinate in the KU boundary component.

More specifically, we may uniquely normalize the matrix N and the period matrix Ω of the LMHS relative to an integral, symplectic basis adapted to the weight filtration to be given by

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix}$$

where $a, b \in \mathbb{Z}$ are non-zero with $b > 0$ and $e \in \mathbb{Z}[\frac{1}{2}]$ and $f \in \mathbb{Z}$ may be uniquely specified, and then

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f/2a & e/a & 1 & 0 \\ \pi & f/2a & 0 & 1 \end{pmatrix} .$$

Then

$$\begin{cases} e/a \text{ corresponds to } \dashrightarrow \\ f/2a \text{ corresponds to } \longrightarrow \\ \pi \in \mathbb{C} \text{ is a local coordinate for the boundary component.} \end{cases}$$

Moreover, the group G of components of the Néron model has a composition series with successive quotients

$$\mathbb{Z}/a\mathbb{Z},\ \ \mathbb{Z}/b\mathbb{Z},\ \ \mathbb{Z}/a\mathbb{Z}$$

(cf. [GGK]). We also determine the condition that two boundary components are conjugate under $G_{\mathbb{Q}}$; this relates to the question of how much of the Néron model is captured by the limiting period mapping.

For degeneration of type II_1, the LMHS may be pictured as

$$\begin{cases} \mathbb{Z}(-2) \to \mathbb{Z}(-1) \\ \qquad H^3 \end{cases}$$

where H^3 is a Hodge structure of weight three with $h^{3,0} = 1$, $h^{2,1} = 0$. After again determining a canonical degeneration parameter, the then normalized N and the first two[1] columns of Ω are given by

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

where $0 \neq a \in \mathbb{Z}$, and

$$\omega_3 = \begin{pmatrix} 0 \\ 1 \\ \tau \\ \delta - \tau\gamma \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ 0 \end{pmatrix}, \quad \operatorname{Im}\tau \neq 0 \text{ and } \gamma, \delta \in \mathbb{R} .$$

The boundary component has dimension two with coordinates τ and $(\gamma, \delta) \in \mathbb{R}^2$, and

$$G \cong \mathbb{Z}/a\mathbb{Z} .$$

The extension group for the LMHS turns out to be isomorphic to $\mathbb{R}^2/\mathbb{Z}^2$ and $\binom{\gamma}{\delta}$ gives the extension class.

For degenerations of type II_2, the LMHS may be pictured as

$$H_1^2(-1) \to H_2^2 ,$$

where H_1^2 and H_2^2 are two Hodge structures of weight two with $h^{2,0} = 1$ and $h^{1,1} = 0$. Then N is given by

$$N = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

where $B = {}^tB > 0$ is an integral matrix which induces an isogeny between H_1^2 and H_2^2. The Néron model has a group G of components with $G \cong \mathbb{Z}^2/B(\mathbb{Z}^2)$; in particular, $|G| = \det B$. The first two columns of the normalized period matrix[2] are

$$\omega_3 = \begin{pmatrix} 1 \\ i\alpha \\ 0 \\ \gamma \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha^{-1} \end{pmatrix}$$

where B has been normalized to $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, $a \geqq c > 0$ and we have set $\alpha = \sqrt{a/c}$, $i = +\sqrt{-1}$. The complex parameter γ serves both as a local coordinate on the boundary component and to parametrize the extension class of the LMHS.

In all cases we shall determine the fibre over the origin $\tilde{J}_0$ of the Néron model, as well as its identity component J_0 (cf. [GGK]). In cases I and II_1, $\dim J_0 = 1$ while $\dim J_0 = 2$ in case II_2.

In cases II_1 and II_2 we shall also determine the conditions that two boundary components be conjugate under $G_{\mathbb{Q}}$. This turns out to always be so in case II_1 but, as expected, there are arithmetic conditions in case II_2.

In part III.A we shall give geometric examples of each of the three types of degenerations and from this compute the group of components of the Néron model. Case I is of course much studied and we have adapted and elaborated on, in our

[1] The first two columns of Ω are sufficient to specify the Hodge filtration in the LMHS.

[2] In contrast to cases I and II_1 we are *not* saying that there is a canonical parameter, only that there are natural choices to have ω_3 and ω_2 as above.

setting, the treatment in [CdOGP], [Mo], and [CoK]. Case II_1 also appears in the physics literature, and case II_2 is discussed in [Bo] and [Vo].

Finally, in part IV we discuss a somewhat different topic, which is to determine the complete set of differential invariants in case I. The situation is analogous to the classical curvature and torsion for ordinary space curves, and the method (due to Darboux and E. Cartan) is to construct a canonical framing locally associated to a non-degenerate complex VHS of mirror quintic type. It turns out that there is one invariant, related to the Yukawa coupling, so that such complex VHS's may be said to locally depend on one arbitrary function of one variable.

Three topics not discussed here include (i) the global properties of a VHS of mirror quintic type, (ii) the normal functions that arise in geometric examples (cf. [BHHW]), and (iii) the aritmetic aspects of LMHS (cf. [KSV]). Under (i) one may suspect that

> The global monodromy group is of finite index in the full arithmetic group.

Additionally there is some heuristic evidence to suggest an affirmative answer to the question

> Is a VHS of mirror quintic type defined over a *complete* curve S locally trivial; i.e. does it trivialize on a finite covering of S?

This would be an analogue of the well-known phenomenon for elliptic curves.[3] We note that in general a non-locally trivial VHS over a complete curve S implies that the genus $g(S) \geqq 2$ (cf. [GS]), and this is the best possible in general. A positive answer to the above question will therefore have to rely on special features of the $w = 3$, $h^{3,0} = h^{2,1} = 1$ case.

Under (ii) there is the question

> Are all components of Néron models reached by graphs of admissable normal functions (cf. [GGK]), especially in the geometric examples?

Under (iii), since a geometric VHS of mirror quintic type is defined over a number field, one may suspect that the methods of [KSV] may shed some light on the question

> Which LMHS's are motivic?

In section III.B we shall formulate this question precisely and shall suggest and illustrate a conjectural answer.

Finally, at the suggestion of the referee, in order to set a context for this work, a section reviewing the material from [GGK] has been added.

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[3] This question has now been ansewred in the affirmative.

Part I: Hodge-theoretic analysis

I.A. Notations and general background

We will be studying local degenerating Hodge structures of mirror quintic type by their approximating nilpotent orbits [Sc]. These are canonical up to a scaling of the parameter, and it is in terms of these that the LMHS is defined. In this section we will establish notations and determine the possible LMHS's that might occur in degenerations of Hodge structures of mirror quintic type (cf. [KU]).

We denote by $H_{\mathbb{Z}}$ a lattice of rank four, and by

$$Q = H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$$

a non-degenerate unimodular, alternating form on $H_{\mathbb{Z}}$. In terms of a choice of a suitable integral basis for $H_{\mathbb{Z}}$ giving an isomorphism

$$H_{\mathbb{Z}} \cong \mathbb{Z}^4 ,$$

and representing elements of $H_{\mathbb{Z}}$ as column vectors, in sections II and III of this work Q will be, up to ± 1, one of the two matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{I.A.1}$$

$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{I.A.2}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In section IV it will be a minor sign variation of (I.A.1). We set

$$\Gamma_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q) ,$$

and for $\mathbb{A} = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$ we use the notation

$$H_{\mathbb{A}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{A}$$

and set

$$G_{\mathbb{A}} = \mathrm{Aut}(H_{\mathbb{A}}, Q) .$$

We denote by D the *period domain* ([CMP]) for polarized Hodge structures built on $(H_{\mathbb{Z}}, Q)$ of weight $w = 3$ and with Hodge numbers

$$h^{3,0} = h^{2,1} = 1 .$$

We shall sometimes refer to these as *Hodge structures of mirror quintic type.* A point $F \in D$ is given by a filtration

$$\{0\} \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H_{\mathbb{C}}$$

where the quotients are all of dimension one; i.e., F is a flag, and where the Hodge-Riemiann bilinear relations are satisfied. The first of these bilinear relations is

$$\begin{cases} \text{(i)} & Q(F^1, F^3) = 0 \quad \Leftrightarrow \quad F^3 = (F^1)^{\perp} \\ \text{(ii)} & Q(F^2, F^2) = 0 \quad \Leftrightarrow \quad F^2 = (F^2)^{\perp} . \end{cases} \tag{I.A.3}$$

We shall also denote by $\check{D}$ the *dual period domain* of all flags satisfying only the bilinear relations (I.A.3). Then

$$D \subset \check{D}$$

is an open subset.

Remark: In this paper we shall only make implicit use, via the existence of the approximating nilpotent orbit and the polarization of the LMHS, of the 2^{nd} bilinear relation.

It is known ([CMP]) that, upon choice of a reference Hodge structure $F_0 \in D$, both D and $\check{D}$ are homogeneous spaces

$$\begin{array}{ccc} D & \cong & G_{\mathbb{R}}/H \\ \cap & & \cap \\ \check{D} & \cong & G_{\mathbb{C}}/P \end{array}$$

where H is a compact subgroup of $G_{\mathbb{R}}$ and P is a parabolic subgroup of $G_{\mathbb{C}}$. The group $\Gamma_{\mathbb{Z}}$ acts properly discontinuously on D and the quotient

$$\mathcal{M}_{MQ} =: \Gamma_{\mathbb{Z}} \backslash D$$

may be thought of as the *moduli space for equivalence classes of Hodge structures of mirror quintic type.*

We set $\mathcal{G}_{\mathbb{Z}} = \mathrm{End}_{\mathbb{Z}}(H_{\mathbb{Z}}, Q)$ and denote by

$$\mathcal{G}_{\mathbb{A}} = \mathrm{End}_{\mathbb{A}}(H_k, Q)$$

the Lie algebra of $G_{\mathbb{A}}$. Of particular importance are nilpotent elements

$$N \in \mathcal{G}_{\mathbb{Q}}$$

satisfying the conditions

$$\text{(I.A.4)} \qquad \left\{ \begin{array}{rl} \text{(i)} & N^4 = 0;^{\,4} \\ \text{(ii)} & T = \exp N \in \Gamma_{\mathbb{Z}} \text{ is an integral matrix;} \\ \text{(iii)} & \text{there exists } F \in D \text{ such that} \\ & \text{(a)} \quad N(F^i) \subseteq F^{i-1} \qquad i = 3, 2; \text{ and} \\ & \text{(b)} \quad (\exp zN)F \in D \text{ for } \mathrm{Im}\, z \gg 0\ . \end{array} \right.$$

Definition: We set $Z = \{\exp zN : z \in \mathbb{C}\}$ and call the pair (Z, N) a *nilpotent orbit.*

By rescaling, which has the effect of changing F, we may assume that (iii,b) holds for $\mathrm{Im}\, z > 0$. We set

$$\left\{ \begin{array}{l} s = e^{2\pi\sqrt{-1}z} \\ l(s) = \log s / 2\pi\sqrt{-1}\ . \end{array} \right.$$

Then for $s \in S^* =: \{0 < |s| < 1\}$ and $\Gamma_T =: \{T^n : n \in \mathbb{Z}\}$

$$F_s = (\exp l(s)N)F \in \Gamma_T \backslash D$$

gives a local variation of Hodge structure of mirror quintic type. By the results of [Sc], any degenerating VHS of mirror quintic type is asymptotic to one defined by a nilpotent orbit as above. See also [GS], [G], and especially [Ha] for a recent clear exposition of Schmid's results.

[3] This is automatic, since $\dim H_{\mathbb{Q}} = 4$.

Associated to a nilpotent orbit and a parameter, which we take to be s as above, there is a limiting *mixed Hodge structure* (LMHS) $(W_\bullet(N), F)$. The weight filtration is only defined over $\mathbb{Q}$ and is uniquely characterized by the properties

$$\text{(I.A.5)} \qquad \begin{cases} \text{(i)} & W_{-1} = 0 \text{ and } W_6 = H_{\mathbb{Q}} \\ \text{(ii)} & N : W_m \longrightarrow W_{m-2} \\ \text{(iii)} & N^k : \mathrm{Gr}_{3+k} \xrightarrow{\sim} \mathrm{Gr}_{3-k} \quad \text{is an isomorphism} \\ & \text{for } k = 1, 2, 3\ . \end{cases}$$

As is evident from the construction, the limit Hodge filtration depends on the choice of parameter. A new choice of parameter $t = t(s)$ changes F by $\exp(-t'(0)N)F$, so that only rescalings

$$s \to e^{2\pi\sqrt{-1}\alpha} s\ , \qquad \alpha \in \mathbb{C}\ ,$$

are relevant.[5] Since

$$N(F^i) \subset F^{i-1}$$

the Hodge structures on Gr_m are well-defined, as are the "adjacent" extensions

$$0 \to \mathrm{Gr}_{m-1} \to W_m/W_{m-2} \to \mathrm{Gr}_m \to 0\ .$$

In the situation where the odd weight graded pieces are zero, as happens in case I, the adjacent extensions are defined to be

$$0 \to \mathrm{Gr}_{m-2} \to W_{m/W_{m-4}} \to \mathrm{Gr}_m \to 0$$

and they are also well defined independently of the scaling.

In the general case, all of this is over $\mathbb{Q}$. Below we shall refine it to be over $\mathbb{Z}$ in the case of families of Hodge structures of mirror quintic type.

For later use we want to recall the relation between the LMHS and canonical extension associated to a VHS over $S^* = \{0 < |s| < 1\}$. Recall that a VHS is given by the data $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^p)$ where

(i) $\mathcal{H}_{\mathbb{Z}}$ is a local system over S^*, and
(ii) setting

$$\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{S^*}$$

the $\mathcal{F}^p$ give a filtered system of holomorphic sub-bundles of $\mathcal{H}$ such that $\nabla\mathcal{F}^i \subseteq \mathcal{F}^{i-1} \otimes \Omega_{S^*}$, and for each $s \in S^*$ the fibres

$$F^i_s =: \mathcal{F}^i_s/\mathbf{m}_s\mathcal{F}^i_s$$

define a Hodge structure on $H_s = \mathcal{H}_s/\mathbf{m}_s\mathcal{H}_s$. Here, we are identifying locally free sheaves and vector bundles, and $\mathbf{m}_s \subset \mathcal{O}_{S^*,s}$ denotes the maximal ideal.

Fix a base point $\mathbf{s}$ and a basis γ_i for $\mathcal{H}_{\mathbb{Z},\mathbf{s}}$. Then over S^* the γ_i analytically continue to a *multi-valued* frame $\gamma_i(s)$ for $\mathcal{H}_{\mathbb{Z}}$, and

$$e_i(s) = e^{-l(s)N}\gamma_i(s)$$

gives a *single-valued* frame for $\mathcal{H}$ over S^*. By definition, the $e_i(s)$ give a frame for the *canonical extension* $\mathcal{H}_e$ of $\mathcal{H}$. If

$$j : S^* \hookrightarrow S$$

denotes the inclusion, we set

$$\mathcal{F}_e = j_*(\mathcal{F}) \cap \mathcal{H}_e\ .$$

[5] More formally, the LMHS is well defined on $T_0(S)$.

Then letting $F_0^i = \mathcal{F}_{e,0}^i/\mathbf{m}_0\mathcal{F}_{e,0}^i$ and similarly for H_0, we *define*

$$\begin{aligned} H_{0,\mathbb{Z}} &= \operatorname{span}_{\mathbb{Z}}\{e_i(0)\} \\ W_{0,\bullet} &= \left\{ \begin{array}{c} \text{filtration on } H_{0,\mathbb{Q}} \text{ corresponding to the} \\ \text{monodromy weight filtration on } \mathcal{H}_{\mathbb{Q},\mathbf{s}} \\ \text{under the correspondence } e_i(0) \leftrightarrow \gamma_i \end{array} \right\} . \end{aligned}$$

We note that $W_{0,\bullet}$ is just $\mathcal{W}_{e,0}/\mathbf{m}_0\mathcal{W}_{e,0}$ where $\mathcal{W}_e$ is the natural extension of the filtration on $\mathcal{H}_e$ induced from the monodromy weight filtration on H_s. Then we have

> $(H_{0,\mathbb{Z}}, W_{0,\bullet}, F_0^i)$ *defines a MHS which corresponds to the LMHS described above.*

Here, "corresponds to" means that for $U = \{z : \operatorname{Im} z > 0\}$ and $\mathbf{z} \in U$ lying over $\mathbf{s}$, we identify

$$\left\{ \begin{array}{c} \text{VHS on } S^* \text{ with} \\ \text{unipotent monodromy} \\ T \text{ on } \mathcal{H}_{\mathbb{Z},\mathbf{s}} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{diagrams of holomorphic maps} \\ \begin{array}{ccc} U & \xrightarrow{\tilde{\varphi}} & D \\ \downarrow & & \downarrow \\ S^* & \xrightarrow{\varphi} & \Gamma_T\backslash D \end{array} \end{array} \right\}$$

where D is the period matrix domain constructed from $H_{\mathbb{Z}} =: \mathcal{H}_{\mathbb{Z},\mathbf{s}}$ and with reference point $\mathbf{F}^i =: F_s^i$. Then we have

$$\mathrm{MHS}\{H_{0,\mathbb{Z}}, W_{0,\bullet}, F_0^i\} = \mathrm{MHS}\{H_{\mathbb{Z}}, W_\bullet(N), \mathbf{F}^i\} .$$

This identification depends on a choice of parameter s. Rescaling by $e^{2\pi\sqrt{-1}\alpha}$ as above induces the changes

$$\begin{aligned} &\mathcal{H}_{0,\mathbb{Z}} \to \exp(-\alpha N)\mathcal{H}_{0,\mathbb{Z}} && \text{on the LHS} \\ &\mathbf{F}^i \to \exp(\alpha N)\mathbf{F}^i && \text{on the RHS} . \end{aligned}$$

In the above we are implicitly assuming that all VHS's are polarized and have omitted reference to the transversality condition, which is to be understood.

Finally, using Q we shall identify $\mathcal{H}_{\mathbb{Z}}$ and $\mathcal{H}_{\mathbb{Z}}^*$, the latter of which we shall think of as "homology". We shall also use the notations $\gamma_i(s)$ and $e_i(s)$ to denote the corresponding elements of $\mathcal{H}_{\mathbb{Z},s}^*$. Then for $\omega(s) \in \mathcal{H}_s$ we shall symbolically write the pairings $Q(\omega(s), \gamma_i(s))$ and $Q(\omega(s), e_i(s))$ as

$$\text{(I.A.6)} \qquad \int_{\gamma_i(s)} \omega(s), \quad \int_{e_i(s)} \omega(s)$$

respectively. This is for an arbitrary VHS of mirror quintic type. In the geometric case the quantities (I.A.6) are the usual integrals over cycles, integer valued in the first case and generally complex valued in the second.

A crucial point is that the LMHS is *polarized.* This means the following: first one defines the *primitive spaces*

$$P_{3+k} = \operatorname{Ker} N^{k+1} : \mathrm{Gr}_{3+k} \to \mathrm{Gr}_{3-k-2} ,$$

and denoting by C the Weil operator given on a Hodge structure of weight r

$$\left\{ \begin{array}{l} H^r = \underset{p+q=r}{\oplus} H^{p,q} \\ H^{p,q} = \overline{H}^{q,p} \end{array} \right.$$

by

$$C(u) = (\sqrt{-1})^{p-q}u, \qquad u \in H^{p,q}\ ,$$

then the real quadratic form defined by

$$Q(u,v) = Q(N^k u, Cv)$$

is symmetric and positive definite. Thus:

> *the Hodge structures on P_{3+k} and P_{3-k} are canonically dual, which using the primitive decomposition $\mathrm{Gr}_{3+k} \cong \oplus N^l P_{3+k+2l}$ implies the same duality between the Hodge structure on Gr_{3+k} and Gr_{3-k}.*

We shall now determine the possibilities for the monodromy weight filtration $W_\bullet = W_\bullet(N)$ for the LMHS. We shall also obtain some information on the LMHS itself; the complete picture will be given below in section I.C.

Case I: $N^3 \neq 0$. In this case, since $\dim H_{\mathbb{Q}} = 4$ the only possibility is pictured by

$$\mathrm{Gr}_6 \xrightarrow[N]{} \mathrm{Gr}_4 \xrightarrow[N]{} \mathrm{Gr}_2 \xrightarrow[N]{} \mathrm{Gr}_0, \qquad N^3 : \mathrm{Gr}_6 \to \mathrm{Gr}_0$$

where each of the Gr_{2k} has dimension one. Since a one dimensional Hodge structure is of Hodge-Tate type, the LMHS must be, using the notations as in [GGK],

(I.A.7) $$\mathbb{Q}(-3) \xrightarrow{N} \mathbb{Q}(-2) \xrightarrow{N} \mathbb{Q}(-1) \xrightarrow{N} \mathbb{Q}(0)\ .$$

In the next section we shall determine how the integral structure coming from $H_{\mathbb{Z}}$ relates to this picture, and in section I.C we shall determine that part of the extension data that is intrinsic after rescaling.

We next claim that the possibility

(I.A.8) $$N^3 = 0, \qquad N^2 \neq 0$$

cannot occur. The reason is that *a non-trivial Hodge structure of odd weight must be even dimensional.* If (I.A.8) holds then we have the picture

$$\mathrm{Gr}_5 \xrightarrow[N]{} \mathrm{Im} N_{\mathbb{Q}} \xrightarrow[N]{} \mathrm{Gr}_1, \qquad N^2 : \mathrm{Gr}_5 \to \mathrm{Gr}_1, \qquad \mathrm{Im} N_{\mathbb{Q}} \subset \mathrm{Gr}_3$$

and since the horizontal arrows are isomorphisms

$$4 = \dim H_{\mathbb{Q}} \geqq 3 \dim \mathrm{Gr}_5 \geqq 6\ . \qquad \square$$

If

(I.A.9) $$N^2 = 0, \qquad N \neq 0$$

then we have

$$\begin{cases} \mathrm{Im}\, N_{\mathbb{Q}} = \mathrm{Ker}\, N_{\mathbb{Q}} \\ N \in \mathcal{G}_{\mathbb{Z}} \text{ is integral}\ . \end{cases}$$

From an integral symplectic point of view — i.e., without reference to the Hodge filtration — this situation is the same as the much studied degeneration of genus two curves (cf. [O], [Na]). There are two cases according to the rank of N.

Case II$_1$: Rank $N = 1$. In this case the LMHS has the picture

$$\mathbb{Q}(-2) \xrightarrow{N} \mathbb{Q}(-1)$$

$$H^3$$

where $H^3 = H^{3,0} \oplus H^{0,3}$ is a Hodge structure of weight three with $h^{3,0} = 1$, $h^{2,1} = 0$. We shall see that this possibility occurs both Hodge-theoretically and geometrically. The integral structure will be discussed in section I.D.

Case II$_2$: Rank $N = 2$. Then the picture of the LMHS must be

$$H^2(-2) \xrightarrow{N} H^2(-1)$$

where $H^2 = H^{2,0} \oplus H^{0,2}$ is a Hodge structure of weight two with $h^{2,0} = 1$, $h^{1,1} = 0$. We shall see that this case also occurs Hodge-theoretically and geometrically. The integral structure will be discussed in section I.E.

Although the integral symplectic story for (I.A.9) looks the same as the $g = 2$ curve case, as we shall see below the Hodge-theoretic picture is completely different.

Review of material from [GGK].

At the suggestion of the referee, in order to set a context for this work, we shall briefly review some of the material from [GGK]. For this we consider a polarized VHS $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^p, S^*)$ of odd weight $2n-1$ over a smooth, quasi-projective curve S^*. Denoting by S the smooth completion of S^*, we assume that the monodromy around each puncture on $S \backslash S^*$ is unipotent. We then define the sheaf associated to the family of intermediate Jacobians to be

$$\mathcal{J} =: \mathcal{H}_{\mathbb{Z}} \backslash \mathcal{H} / \mathcal{F}^n \cong \mathcal{H}_{\mathbb{Z}} \backslash \check{\mathcal{F}}^n_s \ ,$$

the isomorphism resulting from the polarization. Given $\nu_s \in \mathcal{J}_s$ where $s \in S^*$, we may lift ν_s to $\tilde{\nu}_s \in \check{\mathcal{F}}^n_s$ and we define the sheaf of normal functions

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by the condition

$$\nabla \tilde{\nu}_s \in \check{\mathcal{F}}^{n-1}_s \otimes \Omega^1_{S^*,s} \ . \tag{I.A.10}$$

By definition, a *normal function* defined over S^* is given by $\nu \in \Gamma(S^*, \mathcal{J}_\nabla)$.

Denote by H_s, F^p_s the fibres of the Hodge bundles and for $s \in S^*$ set

$$J_s = \mathcal{H}_{\mathbb{Z},s} \backslash H_s / F^n_s \cong \mathcal{H}_{\mathbb{Z},s} \backslash \check{F}^n_s \ .$$

Then the family of intermediate Jacobians $J = \bigcup_{s \in S^*} J_s$ forms naturally an analytic fibre space

$$J \to S^* \tag{I.A.11}$$

of compact, complex tori such that

$$\mathcal{J} \cong \mathcal{O}_{S^*}(J) \ .$$

With the notation

$$\mathcal{O}_{S^*}(J)_\nabla =: \mathcal{J}_\nabla$$

we have

$$\Gamma(S^*, \mathcal{J}_\nabla) = \Gamma\left(\mathcal{O}_{S^*}(J)_\nabla\right) \ ;$$

we shall express this by saying that

$$\textit{(I.A.11) graphs normal functions}$$

the condition (I.A.10) being understood.

The main objective of [GGK] is to extend this picture across the singularities to S. We denote by $\mathcal{H}_{\mathbb{Z},e}, \mathcal{H}_e, \mathcal{F}_e^p$ the canonical extensions of $\mathcal{H}_{\mathbb{Z}}, \mathcal{H}, \mathcal{F}^p$ to sheaves on S (Chapter IV in [G]), and then

$$\mathcal{J}_e =: \mathcal{H}_{\mathbb{Z},e} \backslash \mathcal{H}_e / \mathcal{F}_e^n \cong \mathcal{H}_{\mathbb{Z},e} \backslash \check{\mathcal{F}}_e^n$$

gives a canonical extension of $\mathcal{J}$ to a sheaf on S. Thus, for the inclusion $j : S^* \hookrightarrow S$ we have canonically

$$\mathcal{J}_e \subset j_*(\mathcal{J}) \ .$$

We may also canonically extend $\mathcal{J}_\nabla$ to $\mathcal{J}_{e,\nabla}$, the condition (I.A.10) being replaced by

$$\nabla \tilde{\nu}_{s_0} \in \check{\mathcal{F}}^n_{e,s_0} \otimes \Omega^1_{S,s_0}(\log s_0)$$

for a puncture $s_0 \in S \backslash S^*$.

Definition: *A* **normal function** *is given by* $\nu \in \Gamma(S, \mathcal{J}_{e,\nabla})$.

This is the standard definition — cf. [Z1] and [El-Z]. It turns out that it is unsatisfactory in two ways:

(i) Denote by H_{e,s_0} and F^p_{e,s_0} the fibres of the canonically extended Hodge bundles. Then there is an injection

$$\mathcal{H}_{\mathbb{Z},e;s_0} \hookrightarrow H_{e,s_0}/F^n_{e,s_0} \cong \check{F}^n_{e,s_0} \ , \tag{I.A.12}$$

and perhaps by inserting $\mathcal{H}_{\mathbb{Z},e;s_0} \backslash \check{\mathcal{F}}^n_{e,s_0}$ over the punctures one might hope to complete (I.A.11) to a fibre space of not-necessarily-compact complex Lie groups that graphs normal functions defined over S. One problem is that for $n \geqq 2$ the image of (I.A.12) is not discrete, so that we do not obtain a Lie group. A related issue is that $\mathcal{H}_{\mathbb{Z},e;s_0} \backslash \check{\mathcal{F}}^n_{e,s_0}$ is too big — there are constraints on the "value" of a normal function at a puncture.

(ii) Perhaps more seriously, there are over S^* geometrically defined normal functions that do not extend to normal functions, as defined above, over S. One must enlarge $\mathcal{J}_{e,\nabla}$ to a sheaf of *admissable normal functions* (ANF's) $\tilde{\mathcal{J}}_{e,\nabla}$. Even once this is done, there is the issue of constructing a fibration

$$\tilde{J}_e \to S \tag{I.A.13}$$

that graphs ANF's. In the classical $n = 1$ this is accomplished by the Néron model.

Both issues (i) and (ii) are treated in [GGK].

For (i), we let N denote the logarithm of the monodromy T around a singular point s_0. In a neighborhood of s_0 the sheaves

$$\operatorname{Ker} N \cap \mathcal{H}_e, \ \ \operatorname{Ker} N \cap \mathcal{F}_e^p$$

are well-defined and arise from sub-bundles of $\mathcal{H}_e$, $\mathcal{F}_e^p$ respectively. We denote the fibres over s_0 by H^N_{e,s_0}, $F^{p,N}_{e,s_0}$. Analysis of the LMHS shows that the map

$$\mathcal{H}_{\mathbb{Z},e;s_0} \hookrightarrow H^N_{e,s_0}/F^{n,N}_{e,s_0}$$

is injective with image a *discrete* subgroup.[6] We set

$$J_{e,s_0} = \mathcal{H}_{\mathbb{Z},e;s_0} \backslash H^N_{e,s_0} / F^{n,N}_{e,s_0} \ ,$$

which we may think of as $J(\operatorname{Ker} N)$ — i.e., the generalized intermediate Jacobian constructed from $\operatorname{Ker} N$. We note that, except in the classical case where $N^2 = 0$, plus one other technical condition when $n \geqq 2$

$$\dim J_{e,s_0} < \dim J_{e,s}, \qquad s \neq s_0 \ . \tag{I.A.14}$$

Nonetheless, we may insert J_{e,s_0} over the puncture to obtain a *slit analytic fibre space*[7]

$$J_e \to S \tag{I.A.15}$$

of connected, commutative complex Lie groups. The important points are

$$\text{(I.A.16)} \qquad \begin{cases} \bullet & J_e \text{ is separated, and one may "do geometry" on it;} \\ \bullet & \text{(I.A.15) graphs normal functions.} \end{cases}$$

In particular, the value

$$\nu(s_0) \in J_{e,s_0}$$

is well-defined. Because of (I.A.14) we may say that when $n \geqq 2$ the value of an ANF has an a priori constraint not present classically.

Turning to (ii), for U a small neighborhood of a puncture s_0 and setting $U^* = U \cap S^*$, we assume given a normal function ν defined over U^* and choose a multi-valued lifting

$$\tilde{\nu}_s \in \check{\mathcal{F}}^n_{e,s}, \qquad s \neq s_0 \ .$$

Denoting by $T - I$ analytic construction around s_0, we have

$$(T - I)\nu_s \in \mathcal{H}_{\mathbb{Z},s} \qquad s \neq s_0 \ . \tag{I.A.17}$$

Definition: *ν defines an* **admissable normal function** *(ANF) if*

(i) *ν_s has moderate — i.e. logarithmic — growth as $s \to s_0$, and*
(ii) *we have*

$$(T - I)\nu_s \in \operatorname{Ker}(T - I)^\perp \ .$$

Since $\tilde{\nu}_s$ is defined up to $\mathcal{H}_{\mathbb{Z},s}$ and

$$(T - I)\mathcal{H}_{\mathbb{Z},s} \subseteq \operatorname{Ker}(T - I)^\perp \ ,$$

the second condition is well-defined. Moreover, the value

$$[\nu(s_0)] \in \operatorname{Ker}(T - I)^\perp / \operatorname{Im}(T - I) \tag{I.A.18}$$

is well-defined. We denote by G_{s_0} the RHS of (I.A.18) and note that, again by the local invariant cycle theorem, *G_{s_0} is a finite group.*

We shall denote by $\tilde{\mathcal{J}}_{e,\nabla}$ the sheaf of ANF's, and note that we have an exact sheaf sequence

$$0 \to \mathcal{J}_{e,\nabla} \to \tilde{\mathcal{J}}_{e,\nabla} \to \mathcal{G} \to 0$$

---

[6]By the local invariant cycle theorem, for s close to s_0

$$\mathcal{H}_{\mathbb{Z},e;s_0} \cong \operatorname{Ker} N \cap \mathcal{H}_{\mathbb{Z},e;s} \ .$$

[7]Slit analytic spaces have appeared in [KU], where they are defined as log-analytic spaces, i.e., it is discussed in a way one may "do geometry as usual."

where $\mathcal{G}$ is a skyscraper sheaf supported on $S\backslash S^*$ and with stalk

$$\mathcal{G}_{s_0} \cong G_{s_0}$$

at s_0. In [GGK] there is constructed a corresponding slit analytic fibre space

$$\tilde{J}_e \to S$$

of not-necessarily-connected, commutative complex Lie groups whose sub-fibre-space of connected components is J_e and which has the two corresponding properties to (I.A.16). Moreover, there is a fibrewise exact sequence of groups

$$0 \to J_e \to \tilde{J}_e \to G \to 0$$

where $G = \Gamma(S, \mathcal{G})$.

Definition: *$\tilde{J}_e$ is the* **Néron model** *associated to the VHS.*

To explain the geometric motivation we assume given

$$f : X \to S \tag{I.A.19}$$

where X is a smooth variety, S is a smooth curve, and the finitely many singular fibres of (I.A.19) are assumed to be NCD's. Over S^* there is an associated VHS where

$$\begin{cases} \mathcal{H}_{\mathbb{Z}} = R_f^{2n-1}\mathbb{Z}/\text{torsion} \\ \mathcal{F}^p = R_f^{2n-1-p}\Omega^p_{X/S} \ . \end{cases}$$

We consider the relative Chow group $\mathrm{CH}^n(X/S)$ constructed using codimension-n cycles on X and rational equivalences among them that meet the fibres of (I.A.19) properly. For such a cycle Z, $U \subset S$ an open set and

$$\begin{cases} X_U = f^{-1}(U) \\ Z_U = Z \cap X_U \end{cases}$$

we define

$$\mathrm{CH}^n(X/S)_{\mathrm{Hom}} = \left\{ \begin{array}{c} \text{classes represented by cycles } Z \\ \text{such that } [Z_U] = 0 \text{ in } H^{2n}(X_U, \mathbb{Z}) \\ \text{for } U \text{ a neighborhood of any } s \in S \end{array} \right\}$$

$$\mathrm{CH}^n(X/S)_{\mathrm{Hom}} = \left\{ \begin{array}{c} \text{classes represented by cycles } Z \\ \text{such that } [Z_U] = 0 \text{ in } H^{2n}(X_U, \mathbb{Z}) \\ \text{for } U \text{ a neighborhood of any } s \in S^* \end{array} \right\} .$$

The first is equivalent to the *co*homology class

$$[Z_s] = 0 \ \text{ in } \ H^{2n}(X_s, \mathbb{Z}) \qquad \text{for all } s \in S \ ,$$

and the second to

$$[Z_s] = 0 \ \text{ in } \ H^{2n}(X_s, \mathbb{Z}) \qquad \text{for all } s \in S^* \ .$$

In the first case care must be taken in defining the *co*homology class of Z_{s_0} when X_{s_0} is singular.

For X_s smooth we have the Abel-Jacobi image

$$\mathrm{AJ}_{X_s}(Z_s) \in J(X_s)$$

in the intermediate Jacobian

$$J(X_s) =: H^{2n-1}(X_s, \mathbb{Z})\backslash F^n \check{H}^{2n-1}(X_s, \mathbb{C}) \ ,$$

and these define a normal function ν_Z over S^*. Then ν_Z extends to give homomorphisms

$$\begin{cases} \mathrm{CH}^n(X/S)_{\mathrm{Hom}} \to \Gamma(S, \mathcal{J}_{e,\nabla}) \\ \mathrm{CH}^n(X/S)_{\mathrm{Hom}} \to \Gamma(S, \tilde{\mathcal{J}}_{e,\nabla})\ . \end{cases}$$

In particular, *classes in* $\mathrm{CH}^n(X/S)_{\mathrm{Hom}}$ *define admissable normal functions.*

Recalling the assumption that X_{s_0} is a NCD, in [GGK] there is defined an Abel-Jacobi map

$$\mathrm{AJ}_{X_{s_0}} : Z^n_{\#}(X_{s_0})_{\mathrm{Hom}} \to J_{e,s_0}$$

where $Z^n_{\#}(X_{s_0})_{\mathrm{Hom}}$ are the codimension n algebraic cycles on X_{s_0} that are in general position relative to the strata of X_{s_0} and whose *co*homology classes are zero, the latter being definable because of the general position assumption. For a class in $\mathrm{CH}^n(X/S)_{\mathrm{Hom}}$ represented by a cycle Z with $Z_s = Z \cdot X_s$ and where $Z_{s_0} \in Z^n_{\#}(X_{s_0})_{\mathrm{Hom}}$, in [GGK] it is shown that in a precise sense

$$\lim_{s \to s_0} \mathrm{AJ}_{X_s}(Z_s) = \mathrm{AJ}_{X_{s_0}}(Z_{s_0})\ .$$

One may say that, *for normal functions* ν_Z *arising from geometry, the value* $\nu_Z(s_0)$ *at a singular point is given by the image of an Abel-Jacobi on the singular variety* X_{s_0}.

As stated above, the purposes of this paper are

- to work out explicitly all possible Néron models for VHS's of mirror quintic type;
- to illustrate all the possibilities by geometric examples; and
- to begin to relate the Néron models to the Kato-Usui boundary components, as defined in [KU].

We say "begin to" in the last statement, because the fine structure corresponding to the congruence subgroups in the elliptic curve has yet to be investigated.

I.B. Integral symplectic linear algebra background

In this section, building on and refining results in the literature (cf. [KU], [Mo], [CoK]) we shall begin by establishing the following proposition.

(I.B.1) **Proposition:** *(i) There exists an integral basis for* $H_{\mathbb{Z}}$ *adapted to the monodromy weight filtration. (ii) This integral basis may be chosen so that the symplectic form is given by*

$$\begin{cases} Q \text{ as in (I.A.1) in case I and II}_1 \\ Q \text{ as in (I.A.2) in case II}_2\ . \end{cases}$$

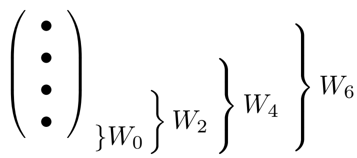
Explanation: We shall denote the integral basis in (i) by e_0, e_1, e_2, e_3 where

$$\begin{aligned} e_i &\in W_{2i}, \quad i = 0,1,2,3 && \text{in case I} \\ e_0 &\in W_{2,\mathbb{Z}}, \quad e_1 \text{ and } e_2 \in W_{3,\mathbb{Z}} && \text{in case II}_1 \\ e_0, e_1 &\in W_{2,\mathbb{Z}} && \text{in case II}_2\ . \end{aligned}$$

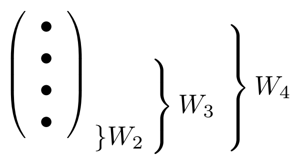
We shall write elements in $H_{\mathbb{Z}}$ as column vectors where

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

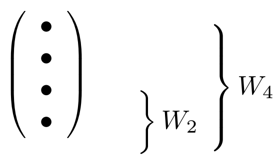
Thus the weight filtration may be pictured by



in case I.



in case II_1.



in case II_2.

As an application of the proposition we shall

(i) analyze the possibilities for the matrix for N in the three cases;
(ii) determine the group G of components of the Néron model in each case.

Both of these rely on the interplay between the weight filtration and the integral structure.

Proof of Proposition (I.B.1): We shall proceed by establishing a series of lemmas.

(I.B.2) **Lemma:** *Given $H_{\mathbb{Z}} \cong \mathbb{Z}^4$ and a flag*

$$W_0 \subset W_2 \subset W_4 \subset W_6$$

defined over $\mathbb{Q}$, there exists a $\mathbb{Z}$-basis e_0, e_1, e_2, e_3 adapted to the flag.

Proof: Using the isomorphism $H_{\mathbb{Z}} \cong \mathbb{Z}^4$ to write elements of $H_{\mathbb{Z}}$ as column vectors, we recall that $e = {}^t(a_3, a_2, a_1, a_0)$ is *primitive* if g.c.d. $\{a_i\} = \pm 1$. Equivalently, $e \neq 0$ in $H_{\mathbb{Z}}/a \cdot H_{\mathbb{Z}}$ for all $a \in \mathbb{Z}$, $a \geqq 2$. This is in turn equivalent to the existence of $f = (b_3, b_2, b_1, b_0) \in H^*_{\mathbb{Z}}$ with

(i) $$\langle f, e \rangle = \sum b_i a_i = 1 .$$

Finally, given a unimodular $Q \in \Lambda^2 H^*_{\mathbb{Z}}$, e is primitive if, and only if, there exists $f \in H_{\mathbb{Z}}$ with

(ii) $$Q(e, f) = 1;$$

this is because the unimodularity of Q is equivalent to the map

$$H_{\mathbb{Z}} \to H_{\mathbb{Z}}^*$$

given by Q being an isomorphism.

Next, e is primitive if, and only if, it is part of a $\mathbb{Z}$ basis for $H_{\mathbb{Z}}$. This is because the sequence

$$0 \longrightarrow \mathbb{Z}e \longrightarrow H_{\mathbb{Z}} \underset{\curvearrowleft}{\longrightarrow} H_{\mathbb{Z}}/\mathbb{Z}e \longrightarrow 0$$

splits by using that $H_{\mathbb{Z}}/\mathbb{Z}e$ is torsion free to have over $\mathbb{Z}$

$$H_{\mathbb{Z}} \cong \mathbb{Z}e \oplus (f)^{\perp} \ .$$

Here, either $f \in H_{\mathbb{Z}}^*$ with (i) holding or, in case we are given a Q as above, $f \in H_{\mathbb{Z}}$ with (ii) holding. Thus

$$H_{\mathbb{Z}}/\mathbb{Z}e \cong (f)^{\perp} \cong \mathbb{Z}^3 \ .$$

To prove the lemma, we choose a primitive vector $e_0 \in W_0 \cap H_{\mathbb{Z}}$ and proceed inductively, noting that in

$$0 \to \mathbb{Z}e_0 \to H_{\mathbb{Z}} \to H_{\mathbb{Z}}/\mathbb{Z}e_0 \to 0$$

a primitive vector e_1' in $H_{\mathbb{Z}}/\mathbb{Z}e_0 \cong \mathbb{Z}^3$ lifts to a primitive vector e_1 in $H_{\mathbb{Z}}$ such that

$$H_{\mathbb{Z}}/\mathbb{Z}e_0 + \mathbb{Z}e_1$$

is torsion free. □

We shall refer to e_0, e_1, e_2, e_3 as an *adapted integral basis* for $H_{\mathbb{Z}}$.

(I.B.3) **Lemma:** *Given a unimodular Q such that, over $\mathbb{Q}$,*

(I.B.4)
$$\begin{cases} W_0^{\perp} = W_4 \\ W_2^{\perp} = W_2 \end{cases},$$

we may choose an adapted integral basis so that

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

as in (I.A.1).

Proof: Using (I.B.4), for any adapted integral basis e_0, e_1, e_2, e_3 we have

$$Q = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & -b & 0 & d \\ -a & -c & -d & 0 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}$ and where the entries in the matrix are the $Q(e_i, e_j)$ for $i, j = 0, 1, 2, 3$; e.g. $a = Q(e_0, e_3)$. From $\det Q = 1$ we have

$$a = \pm 1, \qquad b = \pm 1 \ .$$

Changing signs on the e_i we may assume that $a = b = 1$. Replacing e_1 by $e_1 + ce_0$ we will have $c = 0$. Once $c = 0$ we may replace e_2 by $e_2 + de_0$ to have $d = 0$. □

This completes the proof of Proposition (1.B.1) in case I. In case II_1 the monodromy weight filtration on $H_{\mathbb{Q}}$ is

$$W_2 \subset W_3 \subset W_4$$

where $\dim W_2 = 1$, $\dim \mathrm{Gr}_3 = 2$, and $\dim \mathrm{Gr}_4 = 1$. We note that by definition

$$\begin{cases} W_2 = \mathrm{Im}\, N_{\mathbb{Q}} \\ W_3 = \mathrm{Ker}\, N_{\mathbb{Q}} \end{cases}$$

and

$$N_{\mathbb{Q}} : \mathrm{Gr}_4 \xrightarrow{\sim} \mathrm{Gr}_2 \ .$$

We note that $N = T - I$ is an integral matrix but

$$\mathrm{Im}\, N_{\mathbb{Z}} / (\mathrm{Im}\, N_{\mathbb{Q}}) \cap H_{\mathbb{Z}}$$

may be torsion.

(I.B.5) **Lemma:** *We may choose an adapted integral basis*

$$\underbrace{\underbrace{e_0;}_{W_2}\ e_1, e_2}_{W_3}, e_3$$

such that

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

Proof: This is similar to the preceeding pair of lemmas, using that over $\mathbb{Q}$

$$W_2^{\perp} = W_3 \ .$$

Indeed, we may insert a $\mathbb{Q}$-subspace W to have a flag

$$W_2 \subset W \subset W_3 \subset W_4$$

and the previous argument may be repeated. □

In case II_2 the monodromy weight filtration over $\mathbb{Q}$ is

$$W_2 \subset W_4$$

where $\dim W_2 = \dim \mathrm{Gr}_4 = 2$ and

$$W_2^{\perp} = W_4 \ .$$

In this again

$$N_{\mathbb{Q}} : \mathrm{Gr}_4 \xrightarrow{\sim} W_2$$

is an isomorphism over $\mathbb{Q}$, but

$$\mathrm{Im}\, N_{\mathbb{Z}} / (\mathrm{Ker}\, N_{\mathbb{Q}}) \cap H_{\mathbb{Z}}$$

is in general a finite group.

(I.B.6) **Lemma:** *We may choose an adapted integral basis*

$$\underbrace{e_0, e_1}_{W_2}, e_3, e_4$$

for $H_{\mathbb{Z}}$ *such that*

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} .$$

Proof: As in the previous lemma we may choose an adapted integral basis e_0, e_1, e_2, e_3 by completing $W_2 \subset W_4$ to a flag for $H_{\mathbb{Q}}$. In terms of this basis

$$Q = \begin{pmatrix} 0 & 0 & a & b \\ 0 & a & c & d \\ -a & -c & * & * \\ -b & -d & * & * \end{pmatrix} .$$

From $1 = \det Q = (ad - bc)^2$, we may change e_2, e_3 in $\mathrm{SL}_2(\mathbb{Z})$ and choose signs to have $a = d = 1$, $c = d = 0$. Then in this basis

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & f \\ 0 & -1 & f & 0 \end{pmatrix} , \qquad f \in \mathbb{Z} .$$

Replacing e_3 by $e_3 + fe_2$ sends f to zero. □

In all three cases the adapted, integral symplectic bases are unique up to transformations $A \in \Gamma_{\mathbb{Z}}$ which, when viewed as elements of $G_{\mathbb{Q}}$, preserve the monodromy weight filtration. We denote this subgroup by

$$G_{\mathbb{Z}}(W_\bullet) = \{A \in \Gamma_{\mathbb{Z}} : A(W_m) \subseteq W_m\} .$$

We shall now

(i) determine the possible matrices for N in an adapted, integral symplectic basis in each of the three cases;
(ii) determine the group $G_{\mathbb{Z}}(W_\bullet)$ in each of the three cases;
(iii) determine the action of $G_{\mathbb{Z}}(W_\bullet)$ on N and use this to uniquely and canonically normalize N.

Case I: This is the most intricate and most interesting case. To begin with we have for the matrix for N that

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & g & c & 0 \end{pmatrix}$$

where

(i) $a, b, c, e, f, g \in \mathbb{Q}$;
(ii) $a, b, c \neq 0$ since the monodromy weight filtration of N is the standard flag in terms of our basis. Then

$$NQ + Q^t N = 0$$

gives

$$c = -a, \quad g = e$$

so that

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix} . \tag{I.B.7}$$

The polarization condition

$$Q(N^3 e_3, e_3) > 0$$

and condition that

$$e^N \in \Gamma_{\mathbb{Z}}$$

give respectively

$$b > 0 \tag{I.B.8}$$

$$\begin{cases} a, b, f \in \mathbb{Z} \\ e + ab/2 \in \mathbb{Z} \\ a^2 b/6 \in \mathbb{Z} . \end{cases} \tag{I.B.9}$$

We set

$$m = \text{g.c.d. } (a, b)$$

and shall prove the

(I.B.10) **Proposition:** *Under the group $G_{\mathbb{Z}}(W_\bullet)$*

(i) *b is invariant and a is invariant up to ± 1*

(iia) *if $ab \equiv (2)$ then $[e] \in \mathbb{Z}/m\mathbb{Z}$ is invariant*

(iib) *if $ab \equiv 1(2)$ then $[2e] \in \mathbb{Z}/2m\mathbb{Z}$ is invariant*

(iii) *using (iia) and (iib) we may normalize to have*

$$\begin{cases} 0 \leqq e < m & \text{(case a)} \\ 0 \leqq 2e < m & \text{(case b)} \end{cases}$$

and then $[f] \in \mathbb{Z}/2a\mathbb{Z}$ is invariant, and we may choose A so that

$$0 \leqq f < 2a.$$

What (iii) means is that under the subgroup of $G_{\mathbb{Z}}(W_\bullet)$ that fixes e, $[f] \in \mathbb{Z}/2a\mathbb{Z}$ is invariant.

Proof: Any $A \in G_{\mathbb{Z}}(W_\bullet)$ is represented by a matrix which is lower triangular with ± 1's on the diagonal. By fiddling with signs we may reduce to the case of all $+1$'s. Any such A is then of the form

$$A = e^M$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ r & q & 0 & 0 \\ s & r & -p & 0 \end{pmatrix}$$

where p, q, r, s satisfy the analogous conditions to a, b, e, f in (I.B.9). Then

$$A^{-1} = e^{-M}$$

and under the transformation

$$N \to ANA^{-1}$$

we may compute that a, b are invariant and

$$\text{(I.B.11)} \qquad \begin{cases} e \to e + pb - qa \\ f \to f - (2r + pq)a + p^2 b + 2pe \ . \end{cases}$$

From the first relation in the case (iia) we see that e is well-defined modulo the ideal $m\mathbb{Z}$ in $\mathbb{Z}$ generated by a and b, and since $p, q \in \mathbb{Z}$ are arbitrary we may choose them to have

$$0 \leqq e + pb - qa < m \ .$$

Relabelling, we may assume that $0 \leqq e < m$. Any transformation in $G_{\mathbb{Z}}(W_\bullet)$ that preserves this condition must then have $p = q = 0$. Recalling that in general

$$r + pq/2 \in \mathbb{Z}$$

if $p, q = 0$ we have

$$f \to f - 2ra$$

where $r \in \mathbb{Z}$ is arbitrary, which implies the proposition in the case (iia).

The case (iib) is similar: Writing $e = e'/2$ where $e' \in \mathbb{Z}$ is odd we obtain

$$e' \to e' + 2(pb - qa)$$

so that $[e'] \in \mathbb{Z}/2m\mathbb{Z}$ is well-defined. Choosing p, q to have $0 \leqq e' < 2m$ and restricting as before to the subgroup that preserves e, $[f] \in \mathbb{Z}/2a\mathbb{Z}$ is well-defined. □

Definition: *We shall call N given by (I.B.7)* **normalized** *if $a < 0$ and e, f satisfy the conditions in proposition (I.B.10).*

The normalized N is clearly unique. The reason for the sign choice for a is that this occurs in the geometric example; cf. section III.A.

Case II$_1$: In this case, in terms of any adapted, integral symplectic basis

$$\text{(I.B.12)} \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

where $a \in \mathbb{Z}$ and $a > 0$ by the polarization condition. The transformations in $G_{\mathbb{Z}}(W_\bullet)$ leave N invariant. For the record, $G_{\mathbb{Z}}(W_\bullet)$ is generated by the subgroup

$$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & b^{-1} \end{pmatrix}$$

where $b > 0$ and $\|c_{ij}\| \in \mathrm{SL}_2(\mathbb{Z})$, together with the abelian subgroup

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ e & b_2 & -b_1 & 1 \end{pmatrix} .$$

Case II$_2$: In this case, in terms of an adapted, integral symplectic basis and using 2×2 block matrices

$$N = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \tag{I.B.13}$$

where B is an integral 2×2 matrix satisfying

$$B = {}^tB > 0\ . \tag{I.B.14}$$

The group $G_{\mathbb{Z}}(W_\bullet)$ is given by integral matrices

$$A = \begin{pmatrix} C^{-1} & 0 \\ D & {}^tC \end{pmatrix}$$

where $CD = {}^t(CD)$. Under the transformation $N \to A^{-1}NA$ we have

$$B \to CB\,{}^tC\ , \tag{I.B.15}$$

and although it is not the most general case we shall assume that we may bring B to the form

$$B = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad a \geqq c > 0\ .$$

The general case may be worked out with somewhat more complicated notations which tend to obscure the basic geometry.

Having determined $N \in \mathcal{G}_{\mathbb{Q}}$ and with $\exp N = T \in \Gamma_{\mathbb{Z}}$, we may determine the group

$$G =: \frac{(\mathrm{Ker}(T-I)^{\perp})_{\mathbb{Z}}}{\mathrm{Im}(T-I)_{\mathbb{Z}}}$$

of components of the Néron model (cf. [GGK]).

Case I: In this case G has a composition series[8] with factors

$$\mathbb{Z}/a\mathbb{Z},\ \ \mathbb{Z}/b\mathbb{Z},\ \ \mathbb{Z}/a\mathbb{Z}\ .$$

Thus, non-canonically

$$G \cong (\mathbb{Z}/a\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/b\mathbb{Z}\ .$$

Case II$_1$: In this case

$$G \cong \mathbb{Z}/a\mathbb{Z}\ .$$

Case II$_2$: In this case

$$G \cong \mathbb{Z}^2/B(\mathbb{Z}^2)\ .$$

In particular

$$|G| = \det B\ .$$

Finally, we want to describe the fibre $\tilde{J}_0$ of the Néron model in each of the three cases. We recall from [GGK], and the review of that material in the section following the introduction, that there is an exact sequence

$$0 \to J_0 \to \tilde{J}_0 \to G \to 0 \tag{I.B.16}$$

where J_0 is the connected, complex Lie group that is inserted over the origin to fill in the family of intermediate Jacobians in a way that allows us to graph normal

[8]It is a general feature of limits of Abel-Jacobi mappings in higher codimension that one lands in a filtered object — cf. [GGK] and the aforementioned summary of that work for the explanation.

functions.[9] The extension $\tilde{J}_0$ is inserted over the origin in the family of intermediate Jacobians in order to be able to graph admissable normal functions ν. In the geometric case when ν arises from a family of algebraic cycles

$$Z_s \in Z^2(X_s)_{\text{hom}} ,$$

it is proved in [GGK] that

$$\nu(0) \in \tilde{J}_0$$

is defined. In case the cohomology class of $Z_0 \in Z^2(X_0)$ is zero,

$$\nu(0) \in J_0$$

and may be evaluated as

$$\nu(0)\text{``}=\text{''}\mathrm{AJ}_{X_0}(Z_0) .$$

The quotation marks mean that the RHS should be understood as the image of $\mathrm{AJ}_{X_0}(Z_0)$ under the natural map induced from the retraction map in the Clemens-Schmid exact sequence.

With this understood we have for J_0 that

Case I: $J_0 \cong \mathbb{C}^*$.

Case II$_1$: In this case we have an extension

$$0 \to \mathbb{C}^* \to J_0 \to J(H^3) \to 0$$

where $J(H^3)$ is the intermediate Jacobian constructed from the somewhat strange weight three Hodge structure H^3 with

$$\begin{cases} H^3_{\mathbb{C}} = H^{3,0} \oplus \overline{H^{3,0}} \\ \dim H^{3,0} = 1 . \end{cases}$$

We note that in the geometric case any geometric variation of $\nu(0)$ projects to a constant value in $J(H^3)$ — i.e., it is "in the $\mathbb{C}^*$-direction."

Case II$_2$: In this case, setting $H^2_2 = H^2$ where

$$\begin{cases} H^2_{\mathbb{C}} = H^{2,0} \oplus \overline{H^{2,0}} \\ \dim H^{2,0} = 1 \end{cases}$$

we have

$$J_0 = J(H^2)$$

where

$$J(H^2) = F^2 \backslash H^2_{\mathbb{C}} / H^2_{\mathbb{Z}} .$$

Again, in the geometric case the value $\nu(0)$ at the origin of a normal function arising from a family of cycles is rigid.

We observe that *complex analytically* $J(H^3)$ and $J(H^2)$ are both isomorphic to complex tori $\mathbb{C}/\Lambda$, but this isomorphism is not algebro-geometric.

---

[9]We may think of J_0 as $J(\operatorname{Ker} N)$. In general we will have $\dim J_0 < \dim J_s$, $s \neq 0$. To then say that $\nu(0) \in J_0$ provides a constraint on the value of the normal function at the origin — cf. [GGK]. We have here simplified the notation from the review section — thus

$$\begin{cases} J_0 \text{ is the } J_{e,s_0} \\ \tilde{J}_0 \text{ is the } \tilde{J}_{e,s_0} \end{cases}$$

from there. The notation in the review section **TEXT CUT OFF FROM SCANNED MATERIAL** to that in [GGK].

We also note that

$$\dim J_0 < \dim J_s, \qquad s \neq 0$$

in cases I and II_2.

I.C. Analysis of the LMHS in the three cases

In each of the cases I, II_1, II_2 we have determined a canonical set of integral, symplectic bases $\{e_i\}$ of $H_{\mathbb{Z}}$ that are adapted to the weight filtration and in terms of which N is unique. In each case, in terms of $\{e_i\}$ we shall

(i) determine the complex bases $\{\omega_i\}$ for $H_{\mathbb{C}}$ that are adapted to the Hodge filtration;[10]

(ii) show that, by suitably rescaling the parameter s, we may canonically determine the basis $\{\omega_i\}$ in case I, or partial bases in cases II_1 and II_2, for the equivalent LMHS;[11]

(iii) writing $\{\omega_i\}$ in terms of $\{e_i\}$ gives a *period matrix* Ω which will be seen to contain the information of the Hodge structures on the Gr_k, as well as the intrinsic part of the extension data in the LMHS; and

(iv) in the next section we will see that Ω contains a natural parameter, or in case II_1 two parameters, that give local coordinate(s) in the corresponding boundary component.

Analysis of case I: We shall denote the to-be-determined basis by

$$\underbrace{\underbrace{\underbrace{\omega_3}_{F^3}, \omega_2}_{F^2}, \omega_1}_{F^1}, \omega_0 \; .$$

The reason for the notation is that in the geometric case when we have a family $\{X_s\}_{s\in S}$ as in [GGK] we will have

$$Q(\omega_i, e_j) = \pm \lim_{s\to 0} \int_{e_j(s)} \omega_i(s) \tag{I.C.1}$$

where $e_j(s) = e^{-l(s)N}\gamma_j$ and where here we interpret

$$\gamma_j \in H^*_{\mathbb{Z}}$$

as a homology class using the isomorphism $H_{\mathbb{Z}} \cong H^*_{\mathbb{Z}}$ given by the basis e_i. In particular

$$\begin{aligned} Q(\omega_3, e_0) &= \lim_{s\to 0} \int_{e_0(s)} \omega_3(s) \\ &= \lim_{s\to 0} \int_{\gamma_0} \omega_3(s) \end{aligned}$$

where $\gamma_0 = e_0(s)$, γ_0 is the invariant cycle, and where, for all s, $\omega_3(s) \in H^0(\omega_{X_s})$ is a generator. In the following we denote by $[u]$ the projection of $u = W_{2i,\mathbb{C}}$ to

[10] Actually, we shall only determine the full basis in case I; in the other cases we shall determine adapted bases for $F^3 \subset F^2$, as that is sufficient to determine the full Hodge filtration. This basis for F^2 may be naturally completed to an adapted basis for $H_{\mathbb{C}}$ but that will not be necessary for our purposes.

[11] The partial basis in case II_2 will be natural but only becomes canonical upon a further choice.

$\mathrm{Gr}_{2i,\mathbb{C}}$. When necessary to avoid confusion, we shall denote by $[M]_g$ the matrix of a linear transformation M in a basis $\{g_i\}$.

(I.C.2) **Proposition:** *There exists a unique basis $\omega_i \in F^i$ such that*

$$[\omega_i] = [e_i] \ \textit{in} \ \mathrm{Gr}_{2i,\mathbb{C}} \ .$$

In terms of this basis

$$[N]_\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix}$$

and

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \pi_{32} & 1 & 0 & 0 \\ \pi_{31} & \pi_{21} & 1 & 0 \\ \pi_{30} & \pi_{20} & \pi_{10} & 1 \end{pmatrix} .$$

Proof: The unique choice of ω_i follows from

$$\begin{cases} \dim(F^i \cap \mathrm{Gr}_{2i,\mathbb{C}}) = 1 \\ [e_i] \neq 0 \ \text{in} \ F^i \cap \mathrm{Gr}_{i,\mathbb{C}} \ , \end{cases}$$

since e_i projects to a generator of

$$G_{2i,\mathbb{Q}} \cong \mathbb{Q}(-i) \ .$$

The form of $[N]_\omega$ is a consequence of

$$\begin{cases} N(F^i) \subseteq F^{i-1} \\ N(W_{2i}) \subseteq W_{2i-2} \end{cases}$$

which together imply that

$$N\omega_i = \mu_i \omega_{i-1} \ .$$

The constants μ_i are determined by

$$[N\omega_i] = [Ne_i]$$

and the form (I.B.7) of $[N]_e$.

The form of Ω follows from

$$\mathrm{span}_{\mathbb{C}}\{\omega_0, \ldots, \omega_i\} = \mathrm{span}_{\mathbb{C}}\{e_0, \ldots, e_i\}$$

for $i = 2, 1, 0$. □

Remark: In the geometric case the *Yukawa coupling* is, by definition ([CoK]), the function

(I.C.3) $$Y(s) = -Q(\omega_3(s), \delta_s^3 \omega_3(s))$$

where

$$\delta_s = \left(-\frac{1}{2\pi\sqrt{-1}} \right) s \nabla_{d/ds} \ .$$

Since

$$\mathrm{Res}_0(\nabla) = N ds$$

we may take the limit as $s \to 0$ in (I.C.3) to obtain

$$Y(0) = Q(\omega_3, N^3\omega_3) = -a^2 b\ .$$

Conclusion: *The value of the Yukawa coupling at $s = 0$ is equal to minus the order of the group of components of the Néron model.*

We now set

$$\pi_2 = \pi_{32},\ \ \pi_1 = \pi_{31},\ \ \pi_0 = \pi_{30}$$

and shall prove the

(I.C.4) **Proposition:** *The period matrix is given by*

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \pi_2 & 1 & & 0 \\ \pi_1 & (b/a)\pi_2 + e/a & 1 & 0 \\ \pi_0 & (e/a)\pi_2 + f/a - \pi_1 & -\pi_2 & 1 \end{pmatrix}.$$

Proof: This follows from the expression (I.B.7) for $[N]_e$ and the expression in proposition (I.C.2) for $[N]_\omega$ by using

$$[N]_\omega = \Omega [N]_e \Omega^{-1}.$$ □

(I.C.5) **Proposition** *(following the physicists): There is a canonical choice of parameter so that $\pi_2 = 0$.*

Proof: We recall that on S^* we have denoted by $\gamma_3(s), \gamma_2(s), \gamma_1(s), \gamma_0(s)$ the multi-valued frame for $\mathcal{H}^*_{\mathbb{Z},s}$ obtained from the basis $e_3, e_2, e_1, e_0 \in H_{\mathbb{Z}}$ by dualizing to have homology classes, and then by analytic continuation. Then

$$\begin{cases} T\gamma_0 &= \gamma_0 \\ T\gamma_1 &= \gamma_1 + a\gamma_0\ . \end{cases}$$

It follows that

$$\int_{\gamma_1(s)} \omega_3(s) = al(s) \int_{\gamma_0} \omega_3(s) + h(s)$$

where $h(s)$ is holomorphic. Rescaling by

(I.C.6) $$s \to e^{-2\pi\sqrt{-1}h(0)} s$$

we may assume that $h(0) = 0$. Now

(I.C.7) $$\pi_2 = Q(\omega_3, e_1) = \lim_{s\to 0} \int_{e_1(s)} \omega_3(s)\ ,$$

while

$$e_1(s) = \gamma_1(s) - al(s)\gamma_0(s)\ .$$

Thus the RHS of (I.C.7) is

$$\lim_{s\to 0}\left(\int_{\gamma_1(s)}\omega_3(s) - al(s)\int_{\gamma_0(s)}\omega_3(s)\right) = 0\ . \tag{I.C.8}$$

To make the choice of parameter canonical, we shall modify the choice (I.C.6) by using the parameter t defined by the implicit equation

$$t = \exp\left(\left(2\pi\sqrt{-1}\int_{\gamma_1(s(t))}\omega_3(s(t))\right)\Big/ a\int_{\gamma_0(s(t))}\omega_3(s(t))\right)\ . \tag{I.C.9}$$

Since the framings $\{e_i\}, \{\omega_i\}$ are canonical, so is t. Moreover, from (I.C.8) it follows that

$$\begin{cases} s(0) &= 0 \\ s'(0) &= 1\ , \end{cases}$$

so that canonical parameter t given by (I.C.9) and the arbitrary parameter s, rescaled by (I.C.6), agree to 2^{nd} order at the origin. □

Remark: Rescaling by (I.C.6) eliminates the term π_2 in Ω. This is canonical to 1^{st} order and is sufficient for canonically normalizing the period matrix Ω. The physicists construction gives much more, in that the entire parameter is canonical and may therefore be intrinsically used for the series expansions used in mirror symmetry calculations.

At this stage the period matrix is

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi_1 & e/a & 1 & 0 \\ \pi_0 & f/a - \pi_1 & 0 & 1 \end{pmatrix}\ .$$

From the 2^{nd} bilinear relation $\Omega Q\,{}^t\Omega = 0$ we have

$$\begin{aligned} 0 = Q(e_3, e_2) = -(f/a - \pi_1) + \pi_1 &= 2\pi_1 - f/a \\ \Rightarrow \pi_1 &= f/2a \end{aligned}$$

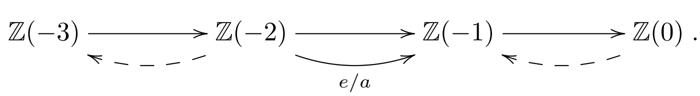
so that, setting $\pi = \pi_0$, we obtain finally

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f/2a & e/a & 1 & 0 \\ \pi & f/2a & 0 & 1 \end{pmatrix}\ . \tag{I.C.10}$$

Definition: *We shall refer to Ω given by (I.C.10) as the (canonically) normalized period matrix.*

We now relate the entries in (I.C.10) to the extension data for the LMHS, which we picture as

(I.C.11)



The zeroes below the diagonal in (I.C.10) imply that the "adjacent" extensions at the two ends, which are dual, are in fact split as indicated by the dotted arrows. The middle adjacent extension has

$$e/a \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), \mathbb{Z}(-1)) \cong \mathbb{C}/\mathbb{Z}$$

as its extension class. By our normalization (I.B.10), it is non-zero if $e \neq 0$.

Going further, we would like to "extract" from H Tate extensions with classes

$$f/2a \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), \mathbb{Z}(0)), \text{ or dually } \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), \mathbb{Z}(-1))$$

and

$$\pi \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), \mathbb{Z}(0))$$

where we are identifying all three Ext-groups with $\mathbb{C}/\mathbb{Z}$.

We will carry this out explicitly for π, by means of a "push-pull" operation on H and the corresponding period matrix. Since the period matrices of $W_2 H$ and H/W_2 are both the 2×2 identity, the maps

$$\begin{array}{ll} W_2 H \overset{\theta_1}{\twoheadrightarrow} \mathbb{Z}(0), \qquad & \mathbb{Z}(-3) \overset{\theta_2}{\hookrightarrow} H/W_2 \\ e_1 \mapsto 0 & (2\pi i)^3 \mapsto e_3 \\ e_0 \mapsto 1 & \end{array}$$

give "splitting" morphisms of MHS's. Hence we may identify H with an extension of H/W_2 by $W_2 H$ and project

$$\begin{array}{rcl} \mathrm{Ext}^1_{\mathrm{MHS}}(H/W_2, W_2 H) & \overset{\theta_2^*}{\twoheadrightarrow} & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), W_2 H) \\ & \overset{(\theta_1)_*}{\twoheadrightarrow} & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), \mathbb{Z}(0)) \cong \mathbb{C}/\mathbb{Z}(3) \ . \end{array}$$

In a diagram the picture is

$$\begin{array}{ccccc} W_2 & \longrightarrow & H & \longrightarrow & H/W_2 \\ \| & & \uparrow \text{pullback} & & \uparrow \theta_2 \\ W_2 & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{Z}(-3) \\ \theta_1 \downarrow & \text{pushforward} & \downarrow & & \| \\ \mathbb{Z}(0) & \longrightarrow & \overline{\mathbb{E}} & \longrightarrow & \mathbb{Z}(-3) \end{array}$$

and the period matrix for $\overline{\mathbb{E}}$ is clearly

$$\begin{pmatrix} 1 & 0 \\ (2\pi\sqrt{-1})^3 \pi & 1 \end{pmatrix} ,$$

which is equivalent to the result claimed.

The parameter π will be discussed further in section III.B below.

Remark: An interesting issue arises concerning the canonical parameter (I.C.10). Namely, it is canonically defined *once the adapted, integral symplectic basis* $\{e_i\}$ *has been specified.* Similarly for the extension classes represented by e/a and $f/2a$.

However, as noted in the proof of proposition (I.B.10), such a basis is not unique unless we specify the conditions on e and f in the proposition. These conditions are not met in the basis used in the literature for the geometric mirror quintic family ([Mo]), with the result that s is defined only up to a a^{th} root of unity.

For a transformation $A = e^M$ as in the proof of proposition (I.B.10) where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ r & q & 0 & 0 \\ s & r & -p & 0 \end{pmatrix} \in \mathcal{G}_{\mathbb{Q}}$$

and $A \in G_{\mathbb{Z}}$, which in particular implies that

$$p, q \in \mathbb{Z} \ ,$$

we will have a new adapted, integral symplectic basis $\{\hat{e}_i\}$ with new canonical parameter $\hat{s}$. After computation one finds that

$$\hat{s} = \zeta_a^{-p} s$$

where

$$\zeta_a = \exp(2\pi\sqrt{-1}/a) \ .$$

This follows from

$$\hat{s} = \exp\left((2\pi\sqrt{-1}) \int_{\hat{\gamma}_1(s)} \omega_3(s) \Big/ a \int_{\hat{\gamma}_0(s)} \omega_3(s) \right)$$

together with

$$\begin{cases} \hat{\gamma}_0 &= \gamma_0 \\ \hat{\gamma}_1 &= \gamma_1 - p\gamma_0 \ . \end{cases}$$

To verify that extensions in the normalized LMHS do not change, we compute that the period matrix for the basis $\{\omega_i\}$ relative to the transformed basis $\{\hat{e}_i\}$ is

$$\text{(I.C.12)} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & 0 \\ \frac{f}{2a} + \frac{pq}{2} + r & e/a + q & 1 & 0 \\ \pi - \frac{p^2 q}{\gamma} + s - \frac{pf}{2a} & f/2a - \frac{pq}{2} + r - pe/a & -p & 1 \end{pmatrix} .$$

To normalize this one does a *row* operation on (I.C.12) corresponding to (I.C.10). Specifically,

$$p \times (\text{third row}) + (\text{fourth row})$$
$$= \left(\pi + \underbrace{\left(\frac{p^2 q}{2} - \frac{p^2 q}{6} \right) + pr + s}, \frac{f}{2a} + \underbrace{\frac{pq}{2} + r}, 0, 1 \right)$$

and the terms above the brackets are integral so that the corresponding extensions do not change.

Analysis of case II_1: We recall that in this case the monodromy weight filtration is

$$\begin{cases} W_2 \subset W_3 \subset W_4 \\ \dim \mathrm{Gr}_2 = \dim \mathrm{Gr}_4 = 1, \ \ \dim \mathrm{Gr}_3 = 2 \,. \end{cases}$$

In terms of an adapted, integral symplectic basis

$$\underbrace{\underbrace{e_0,}_{W_2}\ e_1}_{W_3}, e_2, e_3$$

we write vectors in $H_{\mathbb{Z}} \cong \mathbb{Z}^4$ as column vectors and have

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix} .$$

(I.C.13) **Proposition:** *We may uniquely choose the scaling parameter and independent vectors $\omega_3 = F^3$ and $\omega_2 \in F^2$ such that*

$$\omega_3 = \begin{pmatrix} 0 \\ 1 \\ \tau \\ \delta - \tau\gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ 0 \end{pmatrix}$$

where $\mathrm{Im}\,\tau \neq 0$ *and* $\gamma, \delta \in \mathbb{R}$.

Proof: For any independent $\omega_3 \in F^3$ and $\omega_2 \in F^2$ we write

$$\omega_3 = \begin{pmatrix} \pi_{33} \\ \pi_{32} \\ \pi_{31} \\ \pi_{30} \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} \pi_{23} \\ \pi_{22} \\ \pi_{21} \\ \pi_{20} \end{pmatrix} .$$

Then

$$\pi_{33} = \pi_{23} = 0 \Rightarrow F^2 \subseteq W_{3,\mathbb{C}}$$

which is not the case, so we have:

not both π_{33} and π_{23} are zero.

On the other hand, since $\mathrm{Gr}_4 \cong \mathbb{Q}(-2)$ we have an inclusion

$$F^3 \cap W_{3,\mathbb{C}} \hookrightarrow \mathrm{Gr}_{3,\mathbb{C}}$$

which, since $\dim F^3 = \dim F^3 \cap W_{3,\mathbb{C}}$, implies that

$$F^3 \subset W_{3,\mathbb{C}} \,.$$

Then from

$$N(F^3) \subset F^2 \cap W_{1,\mathbb{C}} = (0)$$

we obtain

$$0 = N\omega_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a\pi_{33} \end{pmatrix}$$

$$\Rightarrow \pi_{33} = 0 \, .$$

Thus $\pi_{23} \neq 0$, and by scaling ω_2 we will have

$$\omega_3 = \begin{pmatrix} 0 \\ \pi_{32} \\ \pi_{31} \\ \pi_{30} \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 1 \\ \pi_{22} \\ \pi_{21} \\ \pi_{20} \end{pmatrix} .$$

Now $\mathrm{Gr}_{3,\mathbb{C}} \cong H^{3,0} \oplus \overline{H}^{3,0}$, from which it follows that

$$\begin{pmatrix} \pi_{32} \\ \pi_{31} \end{pmatrix}, \overline{\begin{pmatrix} \pi_{32} \\ \pi_{31} \end{pmatrix}} \in \mathbb{C} \ \text{ are independent } \, .$$

Scaling ω_3 we may assume that

$$\pi_{32} = 1, \ \ \pi_{31} = \tau \ \text{ where } \ \mathrm{Im}\,\tau \neq 0 \, .$$

Next we may uniquely determine ω_2 by the condition that π_{21} and π_{22} are real. Relabelling we have

$$\omega_3 = \begin{pmatrix} 0 \\ 1 \\ \tau \\ \beta \end{pmatrix} , \qquad \omega_2 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ \epsilon \end{pmatrix} .$$

Rescaling s by $s \to e^{-\lambda s}$ gives

$$\begin{cases} \omega_3 \to \omega_3 \\ \omega_2 \to \omega_2 - \lambda N \omega_3 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ \epsilon - \lambda a \end{pmatrix} \end{cases} .$$

Thus we may make $\epsilon = 0$. In fact, defining a new parameter t implicitly by

$$t = \exp\left(2\pi\sqrt{-1} \int_{\gamma_3(s)} \omega_2(s) \Big/ a \int_{\gamma_0(s)} \omega_2(s) \right) \, ,$$

as in the discussion of case I we see that t agrees with the previous scaled parameter s to 2^{nd} order at the origin, and in addition it is canonical.

Next we use $\Omega Q^t \Omega = 0$ in the form

$$0 = Q(\omega_3, \omega_2) = -\delta + \tau\gamma + \beta$$

to have $\beta = \delta - \gamma\tau$, and then

$$\omega_3 = \begin{pmatrix} 0 \\ 1 \\ \tau \\ \delta - \tau\gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ 0 \end{pmatrix}$$

where $\operatorname{Im}\tau \neq 0$ and $\delta, \gamma \in \mathbb{R}$. $\square$

For the LMHS we have

$$0 \to \mathrm{Gr}_3 \to W_4/W_2 \to \mathrm{Gr}_4 \to 0 \tag{I.C.14}$$

where $\mathrm{Gr}_{4,\mathbb{Z}} \cong \mathbb{Z}(-2)$ and $\mathrm{Gr}_3 \cong H^3$.

(I.C.15) **Proposition:** *$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), H^3) \cong \mathbb{R}^2/\mathbb{Z}^2$ and the extension class of (I.C.14) is given by*

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \in \mathbb{R}^2/\mathbb{Z}^2 \,.$$

Proof: We have

$$\mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(-2), H^3\right) \cong \frac{H^3_{\mathbb{C}}}{F^2 + H^3_{\mathbb{Z}}} \,.$$

Identifying $H^3_{\mathbb{C}} = W_{3,\mathbb{C}}/W_{2,\mathbb{C}} \cong \mathbb{C}^2$ to be the middle two entries in the column vectors the denominator is

$$\mathbb{C}\begin{pmatrix} 1 \\ \tau \end{pmatrix} + \mathbb{Z}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \,.$$

Since γ, δ are real, $\binom{\gamma}{\delta}$ can lie in this span if, and only if, γ and δ are in $\mathbb{Z}$. $\square$

Remark: In contrast to case I, the LMHS does not depend on the particular monodromy logarithm with $N^2 = 0$, rank $N = 1$. The *normalized* LMHS, in the form given by Proposition (I.C.12), depends on the complex parameter τ and the two real parameters γ, δ. This will be further discussed in section II below.

Analysis of case $\mathbf{II}_2$: In this case we have an adapted, symplectic integral basis

$$\underbrace{e_1, e_2}_{W_2}, e_3, e_4$$

for the weight filtration $\{0\} \subset W_2 \subset W_4$ in terms of which

$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

where B is integral and

$$B = {}^tB > 0 \,.$$

As in section I.B we will normalize B to have

$$B = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad a \geqq c > 0 \,.$$

(I.C.16) **Proposition:** *We may choose adapted bases ω_3, ω_2 for $F^3 \subset F^2$ of the form*

$$\omega_3 = \begin{pmatrix} 1 \\ i\alpha \\ 0 \\ \gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha^{-1} \end{pmatrix}$$

where $\alpha = \sqrt{a/c}$ and $i = +\sqrt{-1}$.

Proof: Writing

$$\omega_3 = \begin{pmatrix} \pi_{34} \\ \pi_{33} \\ \pi_{32} \\ \pi_{31} \end{pmatrix} \in F^3 W_{1,\mathbb{C}} \cong F^3$$

we see that since

$$\operatorname{span}\left\{ \begin{pmatrix} \pi_{34} \\ \pi_{33} \end{pmatrix}, \begin{pmatrix} \overline{\pi}_{34} \\ \overline{\pi}_{33} \end{pmatrix} \right\} = \mathbb{C}^2$$

we have $\pi_{34} \neq 0$, $\pi_{33} \neq 0$. Scaling ω_3 we may assume that

$$\omega_3 = \begin{pmatrix} 1 \\ i\alpha \\ \beta \\ \gamma \end{pmatrix} .$$

Next,

$$0 \neq \omega_2 \in F^2 W_{2,\mathbb{C}}$$

and as before we may arrange that

$$\omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \delta \end{pmatrix} .$$

Now

$$0 \neq N\omega_3 \in F^2 W_{2,\mathbb{C}}$$

which gives $N\omega_3 = \mu\omega_2$ or

$$\begin{pmatrix} 0 \\ 0 \\ a \\ \alpha c \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ 1 \\ \delta \end{pmatrix}$$

$$\Rightarrow \mu = a, \quad \delta = \frac{\alpha c}{a} .$$

For the next step, scaling $s \to e^{-\lambda} s$ induces

$$\text{(I.C.17)} \qquad \begin{cases} \beta \to \beta - \lambda a \\ \gamma \to \gamma - \lambda \alpha c . \end{cases}$$

Choosing $\lambda = \beta/a$ and relabelling as

$$\omega_3 = \begin{pmatrix} 1 \\ i\alpha \\ 0 \\ \gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha c/a \end{pmatrix}$$

then $\Omega Q^t \Omega = 0$ in the form $Q(\omega_3, \omega_2) = 0$ gives

$$1 - \frac{\alpha^2 c}{a} = 0$$

$$\Rightarrow \alpha = \pm\sqrt{\frac{a}{c}}\ .$$

Choosing the + sign gives the proposition. □

Remark: In contrast to the previous two cases, the choice of ω_3, ω_2 seems to be "natural" but not canonical. Indeed, from (I.C.17) by scaling we may make any linear combination

$$\rho\beta + \sigma\gamma = 0$$

provided

$$\rho\sqrt{a} \pm \sigma\sqrt{-c} \neq 0\ .$$

For the LMHS we have

(I.C.18)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}_2 & \longrightarrow & W_4 & \longrightarrow & \mathrm{Gr}_4 & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & H_1^2 & & & & H_2^2(-1) & & \end{array}$$

where H_1^2 and H_2^2 are two isogeneous weight two Hodge structures of the special type

$$h^{2,0} = 1, \quad h^{1,1} = 0\ ,$$

the isogeny being induced by

$$B : \mathbb{Z}^2 \to \mathbb{Z}^2\ .$$

We shall now determine the extension class for the LMHS in this case, where we initially shall not fix the scaling parameter.

(I.C.19) **Proposition:** *(i) We may choose an adapted basis $\omega_3, \omega_2, \omega_1, \omega_0$ so that the period matrix is*

(I.C.20)
$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ \beta & 1 & 0 & 0 \\ \gamma & i\alpha^{-1} & \delta & 0 \end{pmatrix}.$$

(ii) We have

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H_2^2(-1), H_1^2) \cong \mathbb{C}^2/\Lambda$$

where $\Lambda \cong \mathbb{Z}^3$ is generated by $(1,0), (0,1)$ and $(i\alpha, -i\alpha^{-1})$. The extension class of the MHS with period matrix (I.C.20) is represented by (β, δ).

The result of normalizing to make $\beta = 0$ is given by the

(I.C.21) **Proposition:** *Define $a^{\#}$ by $c/a = c^{\#}/a^{\#}$ where $a^{\#}$ and $c^{\#}$ are relatively prime. Then when (I.C.20) is normalized to make $\beta = 0$,*

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H_2^2(-1), H_2^2) \cong \mathbb{C}/\Lambda^{\#}$$

where $\Lambda^{\#}$ is the lattice generated by $1/a^{\#}$ and $2i\alpha^{-1}$, and the extension class is represented by $\delta = -i\alpha^{-1}\gamma$.

Proof of (i) in (1.C.19): As in the proof of proposition (1.C.10), without normalizing we have

$$\omega_3 = \begin{pmatrix} 1 \\ i\alpha \\ \beta \\ \gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha^{-1} \end{pmatrix}$$

where $\alpha = \sqrt{a/c}$. We shall complete ω_3, ω_2 to an adapted basis $\omega_3, \omega_2, \omega_1, \omega_0$ for $F^3 \subset F^2 \subset F^1 \subset H_{\mathbb{C}}$. Choosing an

$$\omega_1 = \begin{pmatrix} \zeta \\ \theta \\ \epsilon \\ \delta \end{pmatrix} \in F^1$$

with $\omega_1 \neq 0$ in F^1/F^2, we have $\omega_1 \neq 0$ in W_1/W_2 which gives

$$\begin{pmatrix} \zeta \\ \theta \end{pmatrix} \neq 0 \, .$$

Subtracting $\zeta\omega_3$ and scaling, after relabeling we may assume that

$$\omega_1 = \begin{pmatrix} 0 \\ 1 \\ \epsilon \\ \delta \end{pmatrix} \, .$$

Subtracting $\epsilon\omega_2$ and relabeling again, we will have

$$\omega_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \delta \end{pmatrix} \, .$$

A similar argument shows that we may take

$$\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \, .$$

Thus the period matrix is then

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ \beta & 1 & 0 & 0 \\ \gamma & i\alpha^{-1} & \delta & 1 \end{pmatrix} \, . \qquad \square$$

Remark: The polarization condition $Q(\omega_3, \omega_1) = 0$ gives

$$\delta = -i\alpha^{-1}\gamma\ . \tag{I.C.22}$$

Proof of (ii) in proposition (1.C.19): We write the extension as

$$0 \to H_1^2 \to V \to H_2^2(-1) \to 0$$

where $V \cong \mathbb{C}^4$. Recalling our notations from the proof of the proposition (1.B.1), if $\bar{e}_3, \bar{e}_2$ is an integral basis for $H_2^2(-1)$ relative to which its period matrix is

$$\begin{pmatrix} 1 & 0 \\ i\alpha & 1 \end{pmatrix}\ , \tag{I.C.23}$$

then $\bar{e}_3, \bar{e}_2$ have integral lifts to e_3, e_2. Hodge-theoretic lifts of the adapted basis of the Hodge filtration for $H_2^2(-1)$ given by (I.C.23) are

$$\begin{pmatrix} 1 \\ i\alpha \end{pmatrix} \to \begin{pmatrix} 1 \\ i\alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \delta \end{pmatrix}\ .$$

Thus, since $\gamma = i\alpha\delta$ by (I.C.22)

$$\bar{e}_3 \to \begin{pmatrix} 1 \\ i\alpha \\ \beta \\ \gamma \end{pmatrix} - \begin{pmatrix} 0 \\ i\alpha \\ 0 \\ i\alpha\delta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

and

$$\bar{e}_2 \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \delta \end{pmatrix}\ .$$

The extension class map

$$H_2^2(-1) \to H_1^2$$

is thus

$$\bar{e}_3 \to \begin{pmatrix} \beta \\ 0 \end{pmatrix}$$

$$\bar{e}_2 \to \begin{pmatrix} 0 \\ \delta \end{pmatrix}\ .$$

We now consider the effect of a change of integral basis on V that preserves the integral bases for H_1^2 and $H_2^2(-1)$ as well as the poloarization form. Such a change is given by

$$\begin{cases} e_3 \to e_3 + Ae_1 + Be_0 \\ e_2 \to e_2 + Ce_1 + De_0 \\ e_1 \to e_1 \\ e_0 \to e_0 \end{cases}$$

where $A, B, C, D \in \mathbb{Z}$. Preservation of the polarization gives

$$C = B\ ,$$

and there are no further conditions on A, B, C, D.

Now

$$\begin{array}{rcl} e_3 + i\alpha e_2 + \beta e_1 + \gamma e_0 & \to & e_3 + Ae_1 + Be_0 + i\alpha(e_2 + Be_1 + De_0) + \beta e_1 + \gamma e_0 \\ & = & e_3 + i\alpha e_2 + (\beta + A + i\alpha B)e_1 + (\gamma + B + i\alpha D)e_0 \end{array}$$

and

$$\begin{array}{rcl} e_2 + \delta e_0 & \to & e_2 + Be_1 + De_0 + \delta e_0 \\ & = & e_2 + Be_1 + (\delta + D)e_0 \\ e_1 + i\alpha^{-1}e_0 & \to & e_1 + i\alpha^{-1}e_0 \\ e_0 & \to & e_0\ , \end{array}$$

from which it follows that the new period matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ \beta + A + i\alpha B & 1 & B & 0 \\ \gamma + B + i\alpha D & i\alpha^{-1} & \delta + D & 1 \end{pmatrix}.$$

Subtracting a multiple of the second column from the third gives the equivalent period matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ i\alpha & 0 & 1 & 0 \\ \beta + A + i\alpha B & 1 & 0 & 0 \\ \gamma + B + i\alpha D & i\alpha^{-1} & \delta + D - i\alpha^{-1}B & 1 \end{pmatrix}.$$

Thus

$$\text{(I.C.24)} \qquad \begin{cases} \beta \to \beta + A + i\alpha B \\ \gamma \to B + i\alpha D \\ \delta \to \delta + D - i\alpha^{-1}B\ . \end{cases}$$

It follows that the extension class

$$\begin{pmatrix} \beta \\ \delta \end{pmatrix} \in \mathbb{Z}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i\alpha \\ -i\alpha^{-1} \end{pmatrix} \right\}$$

as claimed. □

Proof of proposition (1.C.21): If we reparameterize the VHS we get, as noted above,

$$\begin{array}{rcl} \beta & \to & \beta + \lambda a \\ \delta & \to & \delta + \lambda c\ . \end{array}$$

Normalizing to make $\beta = 0$, when we make an admissable integral change of adapted basis to have (I.C.24) we should take

$$\lambda = -\left(\frac{1}{a}\right)(A + i\alpha B)\ .$$

Then

$$\begin{aligned}\delta \quad &\to \quad \delta + D - i\alpha^{-1}B - \left(\frac{c}{a}\right)(A + i\alpha B) \\ &= \quad \delta + D - \left(\frac{c}{a}\right)A - \left(i\alpha^{-1}B + \left(\frac{c}{a}\right)i\alpha\right)B\ .\end{aligned}$$

Since $N\binom{1}{i\alpha}$ is a multiple of $\binom{1}{i\alpha^{-1}}$ we have

$$\left(\frac{c}{a}\right)i\alpha = i\alpha^{-1}$$

which gives

$$\delta \to \delta + D - \left(\frac{c}{a}\right)A - 2i\alpha^{-1}B\ .$$

It follows that

$$\left\{D - \left(\frac{c}{a}\right)A : A, D \in \mathbb{Z}\right\} = \mathbb{Z}\left(\frac{1}{a^{\#}}\right)\ ,$$

and

$$\delta \in \mathbb{C}/\mathbb{Z}\left\{\left(\frac{1}{a^{\#}}\right), 2i\alpha^{-1}\right\}$$

is well-defined. □

Part II: Boundary component structure

The purpose of this part is twofold:

(i) We want to analyze the boundary components, in the sense of [KU], for each of the three types of degenerations, with emphasis on their geometric interpretation in terms of LMHS's; and

(ii) We want to formulate and analyze the question of how much of the Néron model is captured by the extended period map

$$S \xrightarrow{\varphi} \Gamma\backslash D_{\Sigma} \tag{II.1}$$

of [KU].

Item (i) is discussed in [KU], and our main contribution will be to give an exposition from a geometric perspective keeping track of the role of the integral structure.

Under item (ii), in case II_1 and II_2 the integral symplectic linear algebra is the same as in the genus 2 curve case, but the LMHS and boundary component structures are completely different and we shall discuss the interesting contrasts.

II.A. Review of material from [KU]

The compactifications of the quotients of bounded symmetric domains by arithmetic groups is a very rich and much studied subject — cf. [AMRT] and the references cited therein. The particular case of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, or equivalently the moduli space of equivalence classes of principally polarized Hodge structures of weight one, is perhaps the most important special case, one in which there has been significant recent progress from an algebro-geometric perspective — cf. [Al]. A particularly important result, originating with Mumford, is that the *Torelli map*

$$\mathcal{M}_g \to \mathcal{A}_g$$

extends to a regular map

$$\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$$

from the Deligne-Mumford compactification of the moduli space of genus g stable curves to Alexeev's canonical compactification of $\mathcal{A}_g$, which is the one corresponding to the second Voronoi fan.

Continuing a program initiated by Cattani and Kaplan [CaK1], in a major work Kato and Usui [KU] have defined partial compactifications

$$\Gamma\backslash D_\Sigma$$

of the moduli spaces of $\Gamma\backslash D$ of polarized Hodge structures of arbitrary weight w. In particular, given a VHS

$$S^* \xrightarrow{\varphi} \Gamma\backslash D \tag{II.A.1}$$

and a smooth compactification S of S^* such that the divisor at infinity $S\backslash S^*$ has local normal crossings with unipotent monodromies, for a suitable choice of fan Σ (which conjecurally always exists), together with the technical condition of "neatness", the map (II.A.1) extends to

$$S \xrightarrow{\varphi} \Gamma\backslash D_\Sigma\ . \tag{II.A.2}$$

Typically, (II.A.1) will arise as the Torelli map for a family $\{X_s\}_{s\in S^*}$ of smooth projective varieties, given as a smooth fibration

$$\mathcal{X}^* \to S^*\ ,$$

and then semi-stable reduction [AK] will give a completion of the above type

$$\mathcal{X} \to S\ ,$$

and (II.A.2) will be the extended Torelli map (cf. [U]). Except in the classical case when D is a bounded symmetric domain, the Kato-Usui spaces $\Gamma\backslash D_\Sigma$ are a new type of object, which may be informally described as *slit log-analytic varieties.* They are Hausdorff but not compact in and of themselves; they are however *relatively compact* in the sense that period maps (II.A.1) extend to (II.A.2) under the above conditions. The important point is that

> *It appears that one may "do geometry" on* $\Gamma\backslash D_\Sigma$ *as in the classical cases.*

For example, in [U] it is proved that the image is a proper, separated algebraic space. Building on, and "geometrizing", section 12.3 in [KU] we want to examine closely the maps (II.A.2) in the case of Hodge theoretic mirror quintics; i.e., when

$$w = 1, \qquad h^{3,0} = h^{2,1} = 1$$

and the polarization is principal. For this we need to recall, and to some extent recast, some of the definitions from [KU]. Remark that here we will only be concerned with $\Gamma\backslash D_\Sigma$ as a set — i.e., we will not discuss the log-analytic structure.

Retaining the notations from section I.A, we have the general

Definitions: (i) A *rational nilpotent cone* is

$$\sigma = \mathbb{R}_{\geqq 0} N_1 + \cdots + \mathbb{R}_{\geqq 0} N_k$$

where $N_i \in \mathcal{G}_{\mathbb{Q}}$ is nilpotent, in fact

$$(N_i)^{w+1} = 0 \ ,$$

and all

$$[N_i, N_j] = 0$$

and finally

$$\sigma \cap -\sigma = (0) \ .$$

(ii) A subset $Z \subset \check{D}$ is a *σ-nilpotent orbit* if

$$Z = (\exp \sigma_{\mathbb{C}}) F$$

where $\sigma_{\mathbb{C}} = \mathbb{C}N_1 + \cdots + \mathbb{C}N_k$ and $F = \{F^p\} \in \check{D}$, and where the conditions

$$\begin{cases} N_i(F^p) \subseteq F^{p-1} & \\ \exp(\sum_i z_i N_i) F \in D & \text{for } \operatorname{Im} z_i \gg 0 \end{cases}$$

are satisfied.

(iii) A *fan* Σ is a non-empty set of rational nilpotent cones such that

- $\sigma \in \Sigma \Rightarrow \tau \in \Sigma$ for any face τ of σ.
- $\sigma, \sigma' \in \Sigma \Rightarrow \sigma \cap \sigma'$ is a face of σ and σ'.

(iv) For a subgroup $\Gamma \subset \Gamma_{\mathbb{Z}}$, the pair (Γ, Σ) is *admissible* if

$$\sigma \in \Sigma \ \text{ and } \ \gamma \in \Gamma \Rightarrow Ad\gamma(\sigma) \in \Sigma \ .$$

(v) For a fan Σ, the space D_Σ of *nilpotent orbits in the directions in* Σ is defined, as a set, to be

$$D_\Sigma = \left\{ (\sigma, Z) : \sigma \in \Sigma, \quad Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit} \right\} \ .$$

Since $\{0\}$ is a face of any $\sigma \in \Sigma$, tautologically we have

$$D \subset D_\Sigma \ .$$

If (Γ, Σ) is admissible, then the quotient

$$\Gamma \backslash D_\Sigma$$

exists as a set. Here the action of Γ is given by

$$\gamma \cdot (\sigma, Z) = \{ Ad\gamma(\sigma), \gamma \cdot Z \} \ .$$

Definition: The *boundary component* $B(\sigma)$ corresponding to a non-trivial rational nilpotent cone σ is given by

$$\bigcup_Z \{ (\sigma, Z) : Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit} \} \ .$$

An alternate description (cf. [Ca] and [CCK]) is

$$B(\sigma) = (\exp \sigma_{\mathbb{C}}) \cdot D \big/ \exp \sigma_{\mathbb{C}} \ , \tag{II.A.3}$$

where the RHS denotes all $\sigma_{\mathbb{C}}$ orbits of all $F \in D$, modulo rescalings.

We observe that

$$D_\Sigma = \bigcup_{\sigma \in \Sigma} B(\sigma)$$

where $D = B(\{0\})$. With our notations we have

$$\sigma_1 < \sigma_2 \Rightarrow B(\sigma_1) \subset B(\sigma_2)$$

where the notation $\sigma_1 < \sigma_2$ means that σ_1 is a face of σ_2. If we set

$$B(\hat{\sigma}) = \left\{ \begin{array}{c} \text{nilpotent } Z\text{-orbits, modulo rescaling} \\ \text{of } N = \Sigma_i \lambda_i N_i \text{ where all } \lambda_i \neq 0 \end{array} \right\}$$

then we may think of $B(\hat{\sigma})$ as the "proper" boundary component corresponding to σ.

A third description, the one that we shall use here, is the following: First, denoting by σ° the interior of σ, for every $N \in \sigma^\circ$ there is defined over $\mathbb{Q}$ a monodromy weight filtration $W_i(N)$, $0 \leqq i \leqq 2w$, uniquely characterized by the conditions

$$N : W_i(N) \to W_{i-2}(N)$$

and that

$$N^k : \mathrm{Gr}_{w+k} \xrightarrow{\sim} \mathrm{Gr}_{w-k} \tag{II.A.4}$$

is an isomorphism. For those N that arise in a LMHS, which are the only ones we shall consider, a fundamental result of Cattani-Kaplan [CaK2] is that *the monodromy weight filtration is independent of* $N \in \sigma^\circ$. We shall denote it by $W_\bullet(\sigma)$.

Alternate definitions of $B(\sigma)$: (i) A *LMHS associated to* $N \in \sigma^\circ$ is given by $F \in \check{D}$ such that $W_\bullet(F)$ gives a MHS that is polarized using the primitive decomposition arising from (II.A.4). (ii) Two LMHS's $(W_\bullet, F_1)$ and $(W_\bullet, F_2)$ associated to $N \in \hat{\sigma} =: \sigma^\circ \otimes \mathbb{C}$ are identified if

$$F_2 = (\exp \lambda N) F_1, \qquad \lambda \in \mathbb{C} \ .$$

(iii) $B(\hat{\sigma}) = \{$equivalence classes of LMHS's associated to all $N \in \hat{\sigma}\}$.

The following example — the "classical case" — illustrates the construction and how in practice one may compute $\dim B(\sigma)$. Computations in non-classical cases are given in [GG].

Example: For weight $w = 1$ we have

$$\{0\} \subset W_0(\sigma) \subset W_1(\sigma) \subset W_2(\sigma) = H_{\mathbb{Q}}$$

and for $\sigma = \mathbb{R}_{\geqq 0} N_1 + \cdots + \mathbb{R}_{\geqq 0} N_k$ each N_i induces

$$B_i : \mathrm{Gr}_2(\sigma) \to \mathrm{Gr}_0(\sigma)$$

where

$$B_\lambda =: \sum_i \lambda_i B_i = {}^t B_\lambda > 0$$

for $\lambda_i > 0$. If $\dim H_{\mathbb{Q}} = 2g$ and $\dim \mathrm{Gr}_1(\sigma) = 2g_1$, we have

$$\mathrm{Gr}_2(\sigma) \cong \mathbb{Q}(-1)^{\oplus h} \ , \qquad\qquad h = g - g_1 \ .$$

For the polarized HS on $\mathrm{Gr}_1(\sigma)$

$$\dim \mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Gr}_2(\sigma), \mathrm{Gr}_1(\sigma)) = h g_1 \ .$$

We set

$$r = \dim \left(\mathrm{span}\{B_1, \dots, B_k\}^\perp \right) \ .$$

Then

$$\dim B(\hat{\sigma}) = r + hg_1 + \frac{g_1(g_1+1)}{2}\ . \tag{II.A.5}$$

Intuitively, for $s = (s_1, \dots s_k) \in (\Delta^*)^k$ we write the asymptotic period matrix as

$$Z(s) = \begin{pmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{pmatrix}$$

where $Z_{22}(s)$ is holomorphic and $\operatorname{Im} Z_{22}(0) > 0$, $Z_{12}(s) = {}^tZ_{21}(s)$ is holomorphic, and

$$Z_{11}(s) = \sum_{i=1}^{k} l(s_i)B_i + H(s)$$

where $l(s_i) = \log s_i/2\pi\sqrt{-1}$ and $H(s)$ is holomorphic. The corresponding nilpotent orbit is given by

$$\begin{pmatrix} \sum_{i=1}^{k} l(s_i)B_i + H(0) & Z_{12}(0) \\ Z_{21}(0) & Z_{22}(0) \end{pmatrix}\ . \tag{II.A.6}$$

Rescaling $s_i \to e^{\lambda_i}s_i$ induces

$$H(0) \to H(0) + \sum_{i=1}^{k} \lambda_i B_i\ .$$

Thus the number of independent parameters in (II.A.6) modulo rescalings of the nilpotent orbit is

$$\underbrace{\frac{h(h+1)}{2} - \dim\left(\operatorname{span}\{B_1, \dots B_k\}\right)}_{r} + hg_1 + \frac{g_1(g_1+1)}{2}$$

in agreement with (II.A.5).

When $g = 2$ and the LMHS is of Hodge-Tate type, so that $g_1 = 0$ and $h = g$, the possibilities are

(i)	$k = 1 \Rightarrow r = 2$	(e.g., $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$)
(ii)	$k = 2 \Rightarrow r = 1$	(e.g., $B_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), B_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$)
(iii)	$k = 3 \Rightarrow r = 0$	(e.g., B_1 and B_2 as above, $B_3 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$).

Note that, with the hopefully obvious notation

$$\sigma(\mathrm{i}) \subset \sigma(\mathrm{ii}) \subset \sigma(\mathrm{iii})\ .$$

Example: For a local, one parameter variation of Hodge structure with unipotent monodromy T over the punctured disc S^*, we may take as fan

$$\Sigma_N =: \{0, \mathbb{R}_{\geqq 0}N\}$$

where $N = \log T$. Then for $\sigma_N = \mathbb{R}_{\geqq 0} N$

$$\begin{aligned} \sigma_N^\circ &= \mathbb{R}_{>0} N \\ D_{\Sigma_N} &= D \cup B(\hat{\sigma}_N) \\ B(\hat{\sigma}_N) &= \exp(\mathbb{C}^* \cdot N) D / \mathbb{C}^* \ , \end{aligned}$$

where $\mathbb{C}^*$ acts by rescaling. For $\Gamma_T =: \{T^n\}_{n \in \mathbb{Z}}$, as a consequence of the work of [Sc] the extended period mapping

$$\varphi \cdot S \to \Gamma_T \backslash D_{\Sigma_N}$$

is defined, where

$$\varphi(0) = \{\text{LMHS modulo rescalings}\} \ .$$

Example: A "tautological" example from [KU] is given by

$$\Sigma = \bigcup_{\begin{cases} N \in \mathcal{G}_\mathbb{Q} \\ N \text{ nilpotent} \end{cases}} \sigma_N \tag{II.A.7}$$

where $\sigma_N = \mathbb{R}_{\geqq 0} N$ and $N = 0$ is included. This fan is especially relevant when nontrivial VHS's can only be one-dimensional. It is conceivably true in some generality that, in this case, *any* fan is contained in (II.A.7). This is trivially true when the weight $w = 1$, and it is a result of [KU] that we shall derive from our perspective in section II.B that this is also true in the $w = 3$, $h^{3,0} = h^{2,1} = 1$ case (cf. (i) in proposition (II.B.1)).

Remark: A fan does not keep track of *level structures*, but rather these are encoded in the group Γ that one uses to form the quotient

$$\Gamma \backslash D_\Sigma \ .$$

For example, in the classical $w = g = 1$ case and for

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the fan (II.A.7) is

$$\bigcup_{g \in \mathrm{SL}_2(\mathbb{Z})} \mathrm{Ad} g(\sigma_N)$$

and thinking of $D_\Sigma = U \cup \mathbb{P}^1(\mathbb{Q})$ where $U = \{\mathrm{Im}\, z > 0\}$, if we take for Γ the full $\Gamma_\mathbb{Z} = \mathrm{SL}_2(\mathbb{Z})$ then in $\Gamma \backslash D_\Sigma$ there is only one boundary component.

For example, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ correspond to the cusps at ∞ and 0, which are identified in the quotient. The extended period map does not detect the different degenerations I_m in Kodaira's list. For this one must use congruence subgroups in order that $\varphi(0)$ capture information on the Néron model.

In the $w = 3$, $h^{3,0} = h^{2,1} = 1$ case we will see that even if we take the full $\Gamma_\mathbb{Z}$ some, but not all, of the information in the Néron model is captured by $\varphi(0)$. The issue of level structures beyond the division points in intermediate Jacobians — we might say *non-classical level structures* — seems to us a potentially interesting one.

II.B. Boundary component structure for degenerations of Hodge structures of mirror quintic type

(II.B.1) **Proposition:** *(i) Let σ be a non-trivial rational nilpotent cone in $\mathcal{G}_{\mathbb{R}}$ and (σ, Z) a σ nilpotent orbit. Then for any $N \in \sigma^{\circ}$*

$$\sigma = \mathbb{R}_{\geqq 0} N \ .$$

In particular

$$\dim \sigma = 1 \ .$$

(ii) We have

$$\begin{cases} \dim B(\sigma^{\circ}) = 1 & \text{in cases I and II}_2 \\ \dim B(\sigma^{\circ}) = 2 & \text{in case II}_1 \ . \end{cases}$$

In all cases the LMHS is rigid.

To say that the LMHS is *rigid* means there is no non-constant admissable VMHS in the boundary component. Henceforth we shall omit the adjective admissable, as this will be understood.

In addition we shall consider the equation

$$c\mathrm{Adg}(N) = N' \tag{II.B.2}$$

where $N, N' \in \mathcal{G}_{\mathbb{Q}}$ are nilpotent in the same class and where

$$\begin{cases} c \in \mathbb{Q}_{>0} \\ g \in G_{\mathbb{Q}} \ . \end{cases} \tag{II.B.3}$$

(II.B.4) **Proposition:** *(i) In case I, denoting by G and G' the group of components of the corresponding Néron models, (II.B.2) has a solution if and only if*

$$|G| = m^{\pm 2} |G'|, \qquad m \in \mathbb{Z} \ . \tag{II.B.5}$$

(ii) *In case* II$_1$*, (II.B.2) always has a solution.*

(iii) *In case* II$_2$*, (II.B.2) has a solution if, and only if, the condition (II.B.5) holds.*

We shall prove the two propositions simultaneously and shall proceed case by case.

Case I: Suppose that $N, N' \in \sigma^{\circ}$ define the same weight filtration and satisfy

$$\begin{cases} [N, N'] = 0 \\ N(F^i) \subseteq F^{i-1}, \quad N'(F^i) \subseteq F^{i-1} \ . \end{cases} \tag{II.B.6}$$

We want to show that N' is a multiple of N.

We may choose an adapted, integral symplectic basis so that

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix}, \quad N' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a' & 0 & 0 & 0 \\ e' & b' & 0 & 0 \\ f' & e' & -a' & 0 \end{pmatrix} .$$

The first equation in (II.B.6) computes to give

$$a/b = a'/b', \qquad e/a = e'/a' \ .$$

From the first relation we may scale N' to have $a' = a$, $b' = b$. The second relation gives $e = e'$.

Using the normalization (I.C.9) we then have

$$(N - N')\omega_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f - f' \end{pmatrix} .$$

Then from (I.C.9), $(N - N')F^3 \subseteq F^2$ implies that $f = f'$. This establishes (i) in proposition (II.B.1) in this case.

To establish (ii) we see from (I.C.9) that the entry π in ω_3 in the normalized period matrix is the only variable quantity. Thus $\dim B(\sigma) = 1$. Moreover, letting $' = d/d\pi$ we have

$$\omega_3' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which does not lie in F^2; thus varying π does not give a VMHS.

We now turn to equation (II.B.2). We shall establish (i) in proposition (II.B.4) in the special case where

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix} , \qquad N' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

The general case is similar with a more complicated computation. We recall that

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and set

$$g = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \lambda & 0 & 0 \\ \gamma & \mu & \zeta & 0 \\ \delta & \epsilon & \theta & \omega \end{pmatrix} \in G_{\mathbb{Q}} .$$

We shall show that the equations

(II.B.7)
$$\begin{cases} \text{(i)} & gQ\,{}^t g = Q \\ \text{(ii)} & gN = N'g \end{cases}$$

can be solved if, and only if,

$$b = m^2, \qquad m \in \mathbb{Z} .$$

From this we will infer the proposition, as follows: First, since N and N' define the same monodromy weight filtration, any g satisfying $gNg^{-1} = N'$ will have the above form. If $b = m^2$ then setting $g = mg'$ we have $g' \in G_{\mathbb{Q}}$ and

$$\mathrm{Adg}'(N) = N' .$$

The constant c in (II.B.2) plays no role in this equation; i.e., if we can solve (II.B.2) we can solve it with $c = 1$.

The equations (i) in (II.B.7) are

(ia) $\gamma\lambda = \beta\mu + \alpha\epsilon$
(ib) $\beta\zeta + \alpha\theta = 0$
(ic) $\alpha\omega = 1$
(id) $\lambda\zeta = 1$

and the equations (ii) in (II.B.7) are

(iia) $\lambda a = \alpha$
(iib) $b\zeta = \lambda$
(iic) $a\omega = \zeta$
(iid) $\beta = a\mu + e\zeta$
(iie) $-\gamma = a\epsilon + e\theta + f\omega$
(iif) $-\mu = b\theta + e\omega$.

Now (id) and (iib) give

$$\lambda^2 = b\ .$$

Since $\lambda \in \mathbb{Q}$, this equation has a solution if, and only if, it has an integral solution $b = m^2$, $m \in \mathbb{Z}$. Then

$$\lambda = m, \quad \zeta = m^{-1}\ ,$$

and (iia), (iic) give

$$\alpha = ma, \qquad \omega = m^{-1}a^{-1}\ .$$

This determines the diagonal entries in g.

Next, (ib) and (iid), (iif) give

$$\begin{aligned} \beta &= -m^2 a\theta \\ \beta &= -a(b\theta + em^{-1}a^{-1}) + em^{-1} = -m^2 a\ . \end{aligned}$$

A similar calculation gives that equations (ia), (iie) are compatible. This gives the

Conclusion: Equations (II.B.7) can be solved if, and only if, $b = m^2$ for $m \in \mathbb{Z}$. The solutions depend on an arbitrary rational parameter θ.

This completes the proof of (i) in proposition (II.B.4). □

Case II$_1$: Over $\mathbb{Z}$ we have, in terms of an adapted, integral symplectic basis

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$N_a =: \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

and for the first two columns in Ω

$$\omega_3 = \begin{pmatrix} 0 \\ 1 \\ \tau \\ \delta - \tau\gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 1 \\ \gamma \\ \delta \\ 0 \end{pmatrix}$$

where $\operatorname{Im}\tau \neq 0$ and $\delta, \gamma \in \mathbb{R}$. Thus $\dim_{\mathbb{R}} B(\sigma) = 4$ with local coordinates, $\tau \in \mathbb{C}$ and $(\gamma, \delta) \in \mathbb{R}^2$. This establishes (ii) in proposition (II.B.1), while (i) follows simply from the observation that any $N' \in \mathcal{G}_{\mathbb{Z}}$ with $N^{'2} = 0$, rank $N' = 1$ and

$$W_\bullet(N') = W_\bullet(N)$$

must have as matrix in the adapted, integral symplectic basis

$$N' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a' & 0 & 0 & 0 \end{pmatrix}.$$

We will now observe that given any $N' \in \mathcal{G}_{\mathbb{Z}}$ with $N^{'2} = 0$ and rank $N' = 1$, there exists $g \in \Gamma_{\mathbb{Z}}$

(II.B.8) $$\operatorname{Adg}(N') = N_1 .$$

The reason is that chosing adapted, integral symplectic basis for the weight filtrations $W_\bullet(N_1)$ and $W_\bullet(N')$ and conjugating by an element in $\Gamma_{\mathbb{Z}}$ to take one basis to the other, we may assume that the weight filtrations coincide and $N' = N_a$. Then we may use $g = (1/a)\mathrm{Id}$ to have (II.B.8).

(II.B.9) **Corollary:** *In case II_1, any two non-trivial rational nilpotent cones are conjugate under $\Gamma_{\mathbb{Z}}$.*

In particular the information in $G \cong \mathbb{Z}/a\mathbb{Z}$ is lost when we map to $\Gamma_{\mathbb{Z}}\backslash D_\Sigma$. To retain it, one must introduce some sort of "level structure".

Case II$_2$: We first want to show that if we have $N, N' \in \sigma^\circ$ with

(II.B.10) $$\begin{cases} W_\bullet(N) = W_\bullet(N') \\ N(F^3) \subseteq F^2, \quad N'(F^3) \subseteq F^2 \end{cases}$$

then

(II.B.11) $$N' = \lambda N .$$

We note that, in this case,

$$[N, N'] = 0$$

is automatic from the first condition in (II.B.10). In terms of adapted, integral symplectic bases we will have

$$N = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \qquad N' = \begin{pmatrix} 0 & 0 \\ B' & 0 \end{pmatrix}$$

where, after normalization as in section I.B

$$B = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad B' = \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}$$

where $a \geq c > 0$ and $B' > 0$.

Next, setting $\alpha = i\sqrt{a/c}$ $(i = +\sqrt{-1})$ we have normalized adapted bases ω_3, ω_2 for $F^3 \subset F^2$ where

$$\omega = \begin{pmatrix} 1 \\ i\alpha \\ 0 \\ \gamma \end{pmatrix}, \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha^{-1} \end{pmatrix} .$$

To check signs, which will be important here, we have

$$N\omega_3 = \begin{pmatrix} 0 \\ 0 \\ a \\ i\alpha c \end{pmatrix} = a\omega_2$$

since $i\alpha c = ai\alpha^{-1}$. Now we have from the second relation in (II.B.10)

$$N'\omega_3 = \begin{pmatrix} 0 \\ 0 \\ a' + i\alpha b' \\ b + i\alpha c' \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\alpha^{-1} \end{pmatrix}$$

which gives

$$a' + i\alpha b' = \lambda = -i\alpha b' + \alpha^2 c'$$
$$\Rightarrow b' = 0 \text{ and } a' = \alpha^2 c' .$$

The last relation is

$$a'/c' = a/c$$

which gives (II.B.11).

We clearly have $\dim B(\sigma) = 1$ with γ as local coordinate, and setting $' = d/d\lambda$ from

$$\omega_3' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we see that LMHS's in $B(\sigma)$ are rigid; i.e., there is no non-trivial VMHS in $B(\sigma)$.

Finally, turning to (II.B.2) and working over $\mathbb{Q}$,[12] if that equation holds then g induces

$$W_\bullet(N) \to W_\bullet(N')$$

and a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_2 & \longrightarrow & \mathrm{Gr}_2' \\ {\scriptstyle cN}\big\downarrow & & \big\downarrow{\scriptstyle N'} \\ \mathrm{Gr}_2 & \xrightarrow{\;g\;} & \mathrm{Gr}_2' . \end{array}$$

It follows that

$$c^2 \det B = \det B'$$

or equivalently

$$c^2|G| = |G'| ,$$

[12] The "c" in (II.B.2) is not to be confused with the same letter in the normalized B.

from which (iii) in proposition (II.B.4) follows.

Discussion: It is interesting to contrast this case of the boundary component of structure of degenerations of Hodge structures of mirror quintic type with Hodge-Tate degenerations of genus two curves. In both cases the integral symplectic linear algebra is the same, but the Hodge theory is very different.

Following [Na] and retaining the notations from the end of section II.A, there are two possibilities for the above type of boundary component in

$$\Gamma_{\mathbb{Z}}\backslash D_{\Sigma} =: \overline{\mathcal{A}_2}$$

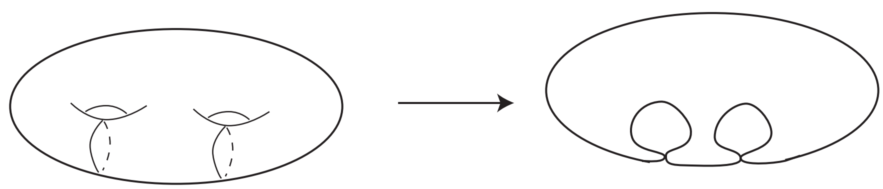
where Σ is the 2^{nd} Voronoi form. These are

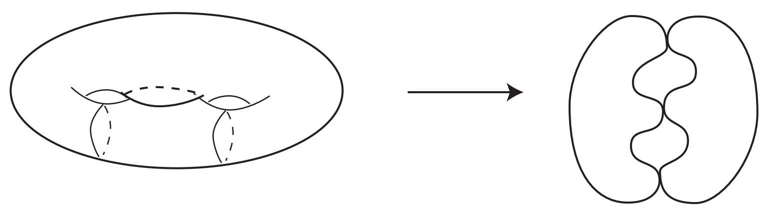
$$\begin{aligned}\sigma_1 &= \operatorname{span}_{\mathbb{R}_{\geqq 0}}\{N_1, N_2\}\\ \sigma_2 &= \operatorname{span}_{\mathbb{R}_{\geqq 0}}\{N_1, N_2, N_3\}\end{aligned}$$

where

$$N_i = \begin{pmatrix} 0 & 0 \\ B_i & 0 \end{pmatrix} .$$

Geometrically, general degenerations to $B(\sigma_1)$ and $B(\sigma_2)$ are given by the Jacobian varieties of stable curve degenerations





We have

> $\dim B(\sigma_1) = 1$ with a local coordinate being given by the cross ratio of the four points on $\mathbb{P}^1$ lying over the nodes on the limit curve,

while $\dim B(\sigma_2) = 0$. We note that any positive definite integral matrix lies in σ_2^0. When we factor by $\Gamma_{\mathbb{Z}}$, there is consequently only one boundary component of each type in $\overline{\mathcal{A}_2}$.

Part III: Geometric setting

In this section we will discuss geometric examples for each of the three degeneration classes for Hodge structures of mirror quintic type. In each case there are naturally — and in two of the cases canonically — defined parameters in the normalized period matrix Ω that gives the LMHS.

To be more precise, in cases there are entries in $\mathbb{Q}$ that are ratios of entries in N and that give the (torsion) extension data for adjacent or next-to-adjacent extensions in the LMHS. In all cases there are additional continuous parameters in Ω that give local coordinates in the corresponding boundary component. We have seen that these parameters are rigid; i.e., they do not vary in geometric families.

A natural question is:

What is the "value" of these parameters for motivic degenerations?

By "motivic degenerations" we shall intuitively mean "arising from geometric families where the field of definition is kept track of".[13] This question will be formulated precisely and a conjectural answer suggested and illustrated in section III.B.

III.A. Examples of the three cases arising from degenerations of Calabi-Yau threefolds

The first two cases, namely I and II_1, have been extensively studied, especially in the physics literature — cf. [Ba1], [Ba2], [CdOGP], [CoK], [GL], and [Mo]. The remaining case II_2 has been studied in [Bo] and [Vo]. Here we shall

(i) briefly review the standard physics construction of the quintic mirror; and also the approach of Batyrev including the fundamental period;
(ii) recall the monodromy of the quintic mirror about the singular points $0, 1, \infty$;
(iii) recast the homology basis used in [CdOGP] in terms of our adapted, integral symplectic basis, and from that determine the normalized LMHS in the maximally unipotent case; and
(iv) discuss briefly the physics example in case II_1 and the Borcea-Voisin example in case II_2.

Of particular interest in (iii) is the complete story of the extension data associated to the LMHS.

(i) We begin with the Fermat/pentahedron pencil of quintics in $\mathbb{P}^4$, given in the standard notation in the literature by

$$\tilde{X}_\psi = \left\{F_\psi =: \psi\left(x_0^5 + \cdots + x_4^5\right) - 5x_0\cdots x_4 = 0\right\} \subset \mathbb{P}^4 . \tag{III.A.1}$$

There is an obvious symmetry group

$$G \cong (\mathbb{Z}/5\mathbb{Z})^3 ,$$

defined to be the quotient

$$\left\{(a_0,\ldots,a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 : \sum_i a_i = 0(5)\right\} \Big/ \{(a,\ldots,a) : a \in \mathbb{Z}/5\mathbb{Z}\}$$

and acting on $\mathbb{P}^4$ by

$$g[x_0,\ldots,x_4] = [\zeta^{a_0}x_0,\ldots,\zeta^{a_4}x_4]$$

where $\zeta = e^{2\pi\sqrt{-1}/5}$. Then the quotients

$$\hat{X}_\psi =: \tilde{X}_\psi/G \subset \mathbb{P}^4/G$$

fit together to form a family

$$\hat{\mathfrak{X}} \subset \mathbb{P}^4/G \times \mathbb{P}^1 .$$

[13] Here, it is understood that the degenerations are local; i.e., they take place over a discrete valuation ring.

Further quotienting by the action

$$\mathbb{Z}/5\mathbb{Z} \times \hat{\mathfrak{X}} \to \hat{\mathfrak{X}}$$

given by

$$(a; [x_0, \dots, x_4]) \to ([\zeta^{-a} x_0, x_1, \dots, x_4], \zeta^a \psi)$$

results in a family parameterized by $z = \psi^5$. The toroidal resolution of singularities

$$X_\psi \to \hat{X}_\psi \ ,$$

very nicely and completely described in [Mo], is compatible with this action.

Recall that we are seeking a family of threefolds X_ψ which for general ψ have

$$\begin{cases} K_{X_\psi} \cong \mathcal{O}_{X_\psi} \\ h^{2,1}(X_\psi) = 1 \ . \end{cases}$$

To give a heuristic reason as to why the $\hat{X}_\psi$ are at least a first approximation to such a family we shall show that

$$\text{(III.A.2)} \qquad \begin{cases} \text{(i)} & H^{3,0}(\tilde{X}_\psi)^G = H^{3,0}(\tilde{X}_\psi) \\ \text{(ii)} & \dim(H^{2,1}(\tilde{X}_\psi)^G) = 1 \end{cases}$$

where $H^{p,q}(\tilde{X}_\psi)^G$ denotes the G-invariant part of $H^{p,q}(\tilde{X}_\psi)$. For simplicity of notation we shall do this for a fixed $\psi = \psi_0$ with X_{ψ_0} smooth. We set $F = F_{\psi_a}, \tilde{X} = \tilde{X}_{\psi_0}$ and denote by $\pi : \tilde{X} \to \hat{X}$ the projection to the quotient. The idea is to represent $H^{p,q}(\tilde{X})$ by the image under the *residue map*

$$\text{Res} : H^4(\mathbb{P}^4 \backslash \tilde{X}) \to H^3(\tilde{X})$$

of forms

$$\Phi_P =: \frac{P\Omega_0}{F^{q+1}}$$

in $\mathbb{P}^4$ with poles of order $q+1$ along $\tilde{X}$, where

$$\begin{cases} \Omega_0 = \sum_{i=0}^4 (-1)\gamma_i^i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4 \\ \deg P = 5q \end{cases}$$

(cf. [CoK, pp. 83–87] and the references cited therein). Denoting by J_F the Jacobian ideal generated by the $F_{x_i} = \partial F / \partial x_i$, so that

$$\text{(III.A.3)} \qquad J_F = \left\{ \text{ideal generated by the } F_{x_i} = 5x_i^4 - \prod_{j \neq i} x_j : i = 0, \dots, 4 \right\} ,$$

it is known (loc. cit.) that

$$\text{Res}(\Phi_p) = 0 \iff P \in J_F \ .$$

For $q = 0$, taking $P = 1$

$$\text{Res}\left(\frac{\Omega}{F}\right) = \psi^*(\omega)$$

is a G invariant holomorphic 3-form that induces a holomorphic form ω on *any* desingularization of $\hat{X}$ and is non-vanishing outside the image of the fixed points.[14]

[14]This is because

$$\int_X \omega \wedge \bar{\omega} = \frac{1}{|G|} \int \pi^*(\omega) \wedge \overline{\pi^*(\omega)}$$

is finite.

For $q = 1$, by inspection from (III.A.3) we see that

$$P = X_0 X_1 X_2 X_3 X_4 \notin J_F \ ,$$

while any other G-invariant monomial belongs to J_F. Since the action of G is completely reducible and the invariant subspace factors are spanned by images of monomials under the residue map, we may conclude (III.A.2).

Of course (III.A.2) only provides heuristic evidence that the cohomologies of the X_ψ give a VHS of mirror quintic type. An alternate way of proceeding is suggested by the second proof of theorem (IV.A.1) in [GGK]. Namely, in a G-equivariant manner one blows up $\tilde{X}$ successively along the strata of the fixed point set of G to obtain a diagram

$$\begin{array}{ccc} \tilde{X}^{\#} & \xrightarrow{\pi^{\#}} & X \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\pi} & \hat{X} \end{array}$$

where $X = \tilde{X}^{\#}/G$. Then

$$H^{2,1}(X) \cong H^{2,1}(\tilde{X}^{\#})^G \ ,$$

and one may check that the map $\tilde{X}^{\#} \to \tilde{X}$ does not create any new $H^{2,1}$, which then leads to showing that $h^{2,1}(X) = 1$.

In the Batyrev approach one considers the Laurent polynomial

$$\varphi(x_1, \ldots, x_4) = \sum_{i=1}^{4} \chi_i + \prod_{i=1}^{4} \chi_i^{-1}$$

with Newton polytope $\Delta \subset \mathbb{R}^4$ given by the convex hull of $(1,0,0,0), \ldots$ $(0,0,0,1)$, $(-1,-1,-1,-1)$. The equation

$$5 - \psi\varphi(x_1, \ldots, x_4) = 0 \qquad \text{(III.A.4)}$$

cuts out a hypersurface in $(\mathbb{C}^*)^4$. Denote its compactification in the toric Fano fourfold

$$\hat{\mathbb{P}}_\Delta =: \mathrm{Proj}\left(\mathbb{C}\left[\left\{\chi_0^l \boldsymbol{\chi}^{\mathbf{m}} : \mathbf{m} \in l\Delta \cap \mathbb{Z}^4, l \in \mathbb{Z}_{\geq 0}\right\}\right]\right) \supset (\mathbb{C}^*)^4$$

by $\hat{X}_{[\psi]}$. A maximal projective triangulation of the dual polytope $\Delta^\vee$ induces a partial desingularization (only terminal point singularities remain)

$$\mathbb{P}_\Delta \to \hat{\mathbb{P}}_\Delta \ .$$

Since this resolution is crepant, triviality of the dualizing sheaf of the anticanonical hypersurface (III.A.4) is preserved under the projection

$$X_{[\psi]} \to \hat{X}_{[\psi]} \ .$$

Thus $X_{[\psi]}$ is Calabi-Yau and $h^{2,1} = 1$, as well as $h^{1,1} = 101$, is proved in [Ba2]. We may identify $X_{[\psi]}$ with $X_{[\zeta\psi]}$ by sending

$$\chi_i \to \zeta^{-1}\chi_i \ ,$$

and we write X_z for the resulting fibre over the z-disc.

To see the equivalence with the first construction, one considers

$$M : \mathbb{Z}^4 \to \mathbb{Z}^4$$

given by the matrix

$$\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} .$$

The induced action on $\mathbb{R}^4$ identifies the fan of $\mathbb{P}^4$ with the fan of $\hat{\mathbb{P}}_\Delta$, i.e. the fan on the facets of $\Delta^\vee$, and the corresponding map

$$\mathbb{P}^4 \to \hat{\mathbb{P}}^4_\Delta$$

is a quotient by $\mathbb{Z}^4/M(\mathbb{Z}^4) \cong G$ which sends

$$[x_0, \ldots, x_4] \to \left(\frac{x_0^5}{\prod_{i=1}^4 x_i}, \ldots, \frac{x_3^5}{\prod_{i=0}^4 x_i} \right)$$

mapping

$$\tilde{X}_\psi \to \hat{X}_{[\psi]} .$$

From the Batyrev approach one obtains an explicit expression for the *fundamental period* in terms of the hypergeometric function ${}_4F_3$. Here the fundamental period refers to

$$\pi_0(z) =: \int_{\delta_0} (z) \omega_3(z)$$

where $\delta_0(z)$ is the vanishing cycle associated to an ordinary quadruple point on X_0 (cf. [Cl] for the original construction), and $\omega_3(z) \in F^3H^3(X_z, \mathbb{Z})$ is the generator given by

$$\left(\frac{1}{2\pi\sqrt{-1}} \right)^4 \mathrm{Res}_{X_z} \left(\frac{d\log\chi_1 \wedge \cdots \wedge d\log\chi_4}{1 - \left(\frac{\psi}{5}\right)\varphi} \right) .$$

Since the Clemens tube $\tau(\delta_0)$ associated to δ_0 is given by

$$\tau_\epsilon(\delta_0) = \{|\chi_1| = \cdots = |\chi_4| = \epsilon\} \subset (\mathbb{C}^*)^4$$

the above integral is

$$\begin{aligned} \pi_0(z) &= \left(\frac{1}{2\pi\sqrt{-1}} \right)^4 \int_{\tau_\epsilon(\delta_0)} \sum_{k=0}^\infty \frac{\psi^k}{5^k} \varphi^k d\log\chi_1 \wedge \cdots \wedge d\log\chi_4 \\ &= \sum_{l=0}^\infty \binom{5l}{l,l,l,l} \frac{\psi^{5l}}{5^{5l}} , \end{aligned}$$

where the term in parenthesis is the multifactorial symbol,

$$= {}_4F_3 \binom{1/5, 2/5, 3/5, 4/5}{1, 1, 1} (z)$$

and where one has used that $(5l)!/(l!)^5$ is the constant coefficient of φ^{5l}, while φ^k for $k \neq 0 (\mathrm{mod}\, 5)$ has no constant term.

(ii) For the monodromy around the singular fibres, over $z = 0, 1, \infty$, one may cut $\mathbb{P}^1$ along $\mathbb{R}_{\geqq 0} \cup \{\infty\}$ and talk about the monodromy operators T_0, T_1, T_∞ where

- $$(T_0 - I)^3 \neq 0 \leftrightarrow \left\{ \begin{array}{c} \text{maximal unipotent} \\ \text{monodromy of class I} \end{array} \right\}$$
- $$(T_1 - I)^2 = 0 \leftrightarrow \left\{ \begin{array}{c} \text{unipotent monodromy} \\ \text{of class } \mathrm{II}_1 \end{array} \right\}$$
- $$T_\infty^5 = 1 \leftrightarrow \{\text{finite monodromy of order five,}\}$$

and where

$$T_0 T_1 = T_\infty^{-1} \ .$$

An integral basis for the local system

$$\mathcal{H}_\mathbb{Z} = \left\{ H^3(X_z, \mathbb{Z}) \right\}_z \in \mathbb{P}^1 \backslash \mathbb{R}_{\geqq 0} \cup \{\infty\}$$

is given by

$$\delta_j = T_\infty^j \delta_0, \qquad j = 0, 1, 2, 4.$$

We set

$$\pi_j(z) = \int_{\delta_j(z)} \omega_3(z) \ .^{15}$$

Although $\{\delta_j\}$ is not a symplectic basis it is particularly convenient for direct computation in terms of the Meijer G-function formalism [GL].

The global monodromy group Γ is generated by T_∞ and T_1 and admits a faithful representation in $\mathrm{SL}_2(\mathbb{R})$ by

$$T_\infty \quad \to \quad \begin{pmatrix} \cos \pi/5 & \sin \pi/5 \\ -\sin \pi/5 & \cos \pi/5 \end{pmatrix}$$

$$T_1 \quad \to \quad \begin{pmatrix} 1 & -2 \tan 2\pi/5 \\ 0 & 1 \end{pmatrix}$$

([CdOGP]). This is *not* conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which has no elliptic elements of order five, such as is the case for T_∞. Hence, one should not expect the VHS to arise from a family of elliptic curves — e.g., as a sub-VHS of $H^3(\mathrm{Sym}\, E_z)$ where $\{E_z\}$ is a Kodaira family of type I_m in his notation [Ko]. Additionally, if it were the case that the LMHS was (over $\mathbb{Q}$) a sub-LMHS of $H^3(\mathrm{Sym}^3 E_z)$, then in terms of the canonical parameter it would be split. To see that this is *not* the case, we shall use data from [CdOGP] to compute the LMHS at $z = 0, 1$, and compare this with our analysis of the extension classes in section I.C above.

(iii) We now turn to the homology basis that will be needed to translate the computations of Candelas et al. into our framework. We will need integral, adapted symplectic bases for T_1 and T_0. For T_1, the basis $\{-A^2, -A^1, B_1, B_2\}$ in [CdOGP]

[15] This is $(5/2\pi\sqrt{-1})^3 \omega_j(z^{-1/5})$ in [CdOGP].

will be denoted by $\{\gamma'_0, \gamma'_1, \gamma'_2, \gamma'_3\}$. It is given in terms of the $\{\delta_j\}$ above by

(III.A.5)
$$\begin{cases} \gamma'_0 &= \delta_1 - \delta_0 \\ \gamma'_1 &= 8\delta_0 - \delta_2 + 3\delta_4 \\ \gamma'_2 &= \frac{1}{5}\{-2\delta_0 + \delta_1 + 3\delta_2 - 8\delta_4\} \\ \gamma'_3 &= \delta_0 \ . \end{cases}$$

For T_0 we need the different basis

(III.A.6)
$$\begin{cases} \gamma_0 &= \gamma'_3 = \delta_0 \\ \gamma_1 &= 2\gamma'_2 + \gamma'_1 = \frac{1}{5}\{2(\delta_1 - \delta_0) + \delta_2 - \delta_4\} \\ \gamma_2 &= \gamma'_2 \\ \gamma_3 &= -\gamma'_0 \ . \end{cases}$$

These are both symplectic bases with intersection form

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Case I: In this case we take as before in section I.C canonical parameter

$$s = \exp\left(2\pi\sqrt{-1}\int_{\gamma_1}\omega_3 \Big/ \int_{\gamma_0}\omega_3\right)$$

and recall our notation $l(s) = \log s/2\pi\sqrt{-1}$. From [CdOGP, p. 71] the period vector is

$$\left(1, l(s), \left(\frac{5}{2}\right) l(s)^2 + \left(\frac{11}{2}\right) l(s) - \frac{25}{12}, -\left(\frac{5}{6}\right) l(s)^3 - \left(\frac{25}{12}\right) l(s) + C\right) + O(s\,l(s)^3)$$

where

$$C = \frac{25i}{\pi^3}\zeta(3) \ .$$

This immediately gives as monodromy

(III.A.7)
$$\begin{aligned} (\gamma_0, \gamma_1, \gamma_2, \gamma_3)T_0 &= (\gamma_0, \gamma_1 + \gamma_0, \gamma_2 + 5\gamma_1 + 8\gamma_0, \gamma_3 - \gamma_2 + 3\gamma_1 - 5\gamma_0) \\ \Longrightarrow [T_0]_\gamma &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ -5 & 8 & 1 & 1 \end{pmatrix} \\ \Longrightarrow N &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 11/2 & 5 & 0 & 0 \\ -25/6 & 11/2 & 1 & 0 \end{pmatrix} . \end{aligned}$$

Using the notation (I.B.7) this gives

$$\text{(III.A.8)} \qquad \begin{cases} a = -1 \\ b = 5 \\ e = 11/2 \\ f = -25/6 \ , \end{cases}$$

and for the group G of components of the Néron model we have

$$G \cong \mathbb{Z}/5\mathbb{Z} \ ,$$

and the order of this group

$$|G| = -Y$$

where Y is the Yukawa coupling.

The form of the period vector for ω_3 also gives

$$[\omega_3]_\gamma = \begin{pmatrix} 1 \\ l(s) \\ (-5/2)l(s)^2 - \left(\frac{11}{2}\right) l(s) + \frac{25}{12} \\ (5/6)l(s)^3 + \left(\frac{25}{12}\right) l(s) - C \end{pmatrix} + O(s\, l(s)^3) \ .$$

The minus signs in the last two entries arise because we are using the pairing

$$Q : H^3(X_z, \mathbb{Z}) \otimes H^3(X_z, \mathbb{Z}) \to \mathbb{Z}$$

to identify $H^3(X_z, \mathbb{Z})$ and $H_3(X_z, \mathbb{Z})$.

We next recall our single-valued framing of $\mathcal{H}_e$ given by

$$e_j(s) = e^{-l(s)N} \gamma_j(s) \ .$$

Then

$$e^{l(s)[N]_\gamma} = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ -l(s) & 1 & 0 & 0 \\ \left(\frac{11}{2}\right) l(s) - \left(\frac{5}{2}\right) l(s)^2 & 5l(s) & 1 & 0 \\ -\left(\frac{25}{6}\right) l(s) - \left(\frac{5}{6}\right) l(s)^3 & \left(\frac{11}{2}\right) l(s) + \left(\frac{5}{2}\right) l(s)^2 & l(s) & 1 \end{array}\right)$$

so that

$$[\omega_3]_e = e^{l(s)N} [\omega_3]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 25/6 \\ -C \end{pmatrix} + O(s\, l(s)^3) \ .$$

Taking the limit as $s \to 0$ gives for the canonically normalized period matrix (I.C.10)

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 25/12 & -11/2 & 1 & 0 \\ -C & 25/12 & 0 & 1 \end{pmatrix} .$$

This encodes the following extensions, viewing all Ext's as lying in $\mathbb{C}/\mathbb{Z}$

$$25/12 \leftrightarrow \begin{cases} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), \mathbb{Z}(0)) \\ \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-3), \mathbb{Z}(-1)) \ , \end{cases}$$

these two extensions begin canonically dual

$$\begin{array}{rcl} -11/2 & \leftrightarrow & \mathrm{Ext}(\mathbb{Z}(-1),\mathbb{Z}(-2)) \\ -C & \leftrightarrow & \mathrm{Ext}(\mathbb{Z}(0),\mathbb{Z}(-3)) \ . \end{array}$$

The first two are torsion but the last is not, confirming our earlier claim that this VHS is not derived from a construction arising from an elliptic curve degeneration.

A better way of viewing the last, which will be discussed in section III.B, is

$$(-8\pi^3\sqrt{-1})(-C) = -200\zeta(3) \in \mathbb{C}/\mathbb{Z}(3) \ .$$

Case II_1: ([CdOGP], cf. also appendix C in [Mo]). Using the homology basis (III.A.5) one has

$$\text{(III.A.9)} \qquad [T_1]_{\gamma'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} ,$$

so that in the notation of section I.C

$$a = -1 \ .$$

Moreover, near $z = 1$ the vanishing cycle period has the form

$$\int_{\gamma_0'} \omega_3 = (z-1) f_0(z-1)$$

where f_0 is a holomorphic function with $f_0(0) \neq 0$; i.e., the period vanishes to exactly first order at $z = 1$. Setting $u = z - 1$ and

$$f_j(u) = \int_{\gamma_j'} \omega_3, \qquad j = 1, 2$$

from (II.A.10) we find that

$$\int_{\gamma_3'} \omega_3 = -ul(u) f_0(u) + f_3(u)$$

where

$$f_j(u) =: \sum_{k \geqq 0} b_{jk} u^k, \qquad j = 0, 1, 2, 3 \ .$$

By inspection

$$\text{(III.A.10)} \qquad [\omega_3]_{\gamma'} = \begin{pmatrix} b_{00} u \\ b_{10} + b_{11} u \\ -b_{20} - b_{21} 8 \\ -b_{30} - b_{31} u + ul(u) b_{00} \end{pmatrix} + O(u\, l(u)^2) \ ,$$

which, using the same method as in case I just above, leads to

$$[\omega_3]_{e'} = \begin{pmatrix} 0 \\ b_{10} \\ -b_{20} \\ -1 \end{pmatrix} \ .$$

Setting $\omega_2 = \nabla_{\partial/\partial u}\omega_3$, (III.A.10) gives

$$[\omega_2]_{\gamma'} = \begin{pmatrix} b_{00} \\ b_{11} \\ -b_{21} \\ -b_{31} + \left(\frac{1}{2\pi\sqrt{-1}}\right) b_{00} + l(u) b_{00} \end{pmatrix} + O(ul(u))$$

$$\implies [\omega_2]_{e'} = \begin{pmatrix} b_{00} \\ b_{11} \\ -b_{21} \\ \left(\frac{1}{2\pi\sqrt{-1}}\right) b_{00} - b_{31} \end{pmatrix} .$$

Next, as in section II.C we use the canonical parameter

$$s = \exp\left(-2\pi\sqrt{-1}\int_{\gamma_3'}\omega_2 \Big/ \int_{\gamma_0'}\omega_2\right) ,$$

which induces the change

$$e' \to (\exp([l(s) - l(u)])N)\, e'$$

where

$$(l(s) - l(u))\Big|_{z=1} = \frac{1}{2\pi\sqrt{-1}} - \frac{b_{31}}{b_{00}} .$$

From [CdOGP]

$$b_{00} = \frac{-\sqrt{5}}{2\pi\sqrt{-1}} ,$$

and setting

$$\tilde{\omega}_3 = \omega_3/b_{10}, \quad \tilde{\omega}_2 = \omega_2/b_{00}$$

we obtain

$$[\tilde{\omega}_3]_{e'} = \begin{pmatrix} 0 \\ 1 \\ -b_{20}/b_{10} \\ -b_{30}/b_{10} \end{pmatrix}$$

$$[\tilde{\omega}_2]_{e'} = \begin{pmatrix} 1 \\ -\left(\frac{2\pi\sqrt{-1}}{\sqrt{5}}\right) b_{11} \\ \left(\frac{2\pi\sqrt{-1}}{\sqrt{5}}\right) b_{21} \\ 0 \end{pmatrix} .$$

In [GL] there is an explicit calculation of the ω_3 periods in the form of Meijer G-functions, and these can be used to determine the remaining b_{j_0}. The resulting (complicated) expressions are linear combinations, with coefficients in $\mathbb{Q}\left(e^{\frac{\pi\sqrt{-1}}{10}}, \{\Gamma(k/5)\}_{k=1,\ldots,4}\right)$ of hypergeometric special values

$${}_4F_3\left(\begin{array}{c} k/5, k/5, k/5, k/5 \\ \frac{4+k}{5}, \ldots, \hat{1}, \ldots, \frac{1+k}{5} \end{array}\right), \qquad k = 1, 2, 3, 4 .$$

The first bilinear relation puts constraints, such as

$$b_{20}b_{11} + b_{30}b_{00} = b_{21}b_{00}$$

on the b_{ij}'s. In the notation of proposition (I.C.13) we have

$$\begin{cases} \tau = -b_{20}/b_{10} \\ \gamma = -\left(\frac{2\pi\sqrt{-1}}{\sqrt{5}}\right) b_{11}, \quad \delta = \left(\frac{2\pi\sqrt{-1}}{\sqrt{5}}\right) b_{21} \ . \end{cases}$$

Case II$_2$: This LMHS does not occur in the quintic mirror family, but it does occur naturally in another family of Calabi-Yau threefolds arising from mirror symmetry studied by Borcea and Voisin [Bo], [Vo]. Since our purpose is to illustrate that all Hodge-theoretic possibilities occur geometrically, we will only consider a special case of their construction.

For this we consider a family of elliptic curves E_s with semistable Kodaira-type I_m degeneration at $s = 0$. Let Y be a fixed exceptional K3 surface — i.e., one where the Picard number $\rho(Y) = 20$. Then for $\tilde{X}_s = E_s \times Y$

$$H^3_{\mathrm{tr}}(\tilde{X}_s) = H^2_{\mathrm{tr}}(Y) \otimes H^1(E_s)$$

gives a geometric VHS of class II$_2$. The entry $i\sqrt{a/c}$ in the normalized period vector for ω_3 as in proposition (I.C.16) reflects the fact that Y *admits a Shioda-Inose correspondence to a product* $E^{(1)} \times E^{(2)}$ *of isogeneous elliptic curves of CM type*, inducing a Hodge isometry between $H^2_{\mathrm{tr}}(Y)$ and $H^2_{\mathrm{tr}}(E^{(1)} \times E^{(2)})$. For example, if the complex multiplication is $e^{2\pi i/6}$, then $a/c = 3$, which recovers the quadratic irrationality in the sense that $\mathbb{Q}(e^{2\pi i/6}) = \mathbb{Q}(i\sqrt{3})$. In the notation of (I.C.10), $\gamma = 0$ since the LMHS of E_s is split.

To get an idea how the case when

$$E^{(1)} = E^{(2)} = \mathbb{C}/\mathbb{Z}\left\langle 1, e^{\pi i/3} \right\rangle$$

works, we let

$$\begin{cases} \alpha', \beta' & \text{be a canonical basis for } H_1(E_s, \mathbb{Z}) \\ \alpha, \beta & \text{be a canonical basis for } H_1(E^{(i)}, \mathbb{Z}), i = 2, 2 \ . \end{cases}$$

Then a $\mathbb{Z}[1/2]$ basis for $H^2_{\mathrm{tr}}(E^{(1)} \times E^{(0)})$ is

$$\begin{aligned} \gamma_2 &= \alpha \times \alpha - \beta \times \beta \\ \gamma_3 &= \alpha \times \alpha + \beta \times \beta - 2(\beta \times \alpha + \alpha \times \beta) \ . \end{aligned}$$

We also set

$$\begin{aligned} \gamma_0 &= \gamma_2/2 \\ \gamma_1 &= \gamma_3/6 \end{aligned}$$

and, in conformity with the notations in section I.C, we redefine the integral structure by taking

$$\begin{aligned} e_0 &= \alpha' \times \gamma_0 \\ e_1 &= \alpha' \times \gamma_1 \\ e_2 &= \beta' \times \gamma_2 \\ e_3 &= \beta' \times \gamma_3 \end{aligned}$$

as an integral symplectic (which it is) basis. The point is that with these choices

$$B = \begin{pmatrix} 6m & 0 \\ 0 & 2m \end{pmatrix}$$

so that

$$\alpha = i\sqrt{a/c} = i\sqrt{3}\ .$$

The family of elliptic curves E_s has the standard fiberwise involution $i_s =$ "-1" with fixed point set $\mathrm{FP}(i_s)$ the points of order two. Using the same i on one of the curves $E^{(i)}$ gives an involution $j : Y \to Y$ with $j = -1$ on $H^{2,0}(Y)$, so that

$$i_s \times j = \text{identity on } H^{3,0}(\tilde{X}_s)\ .$$

Now $\mathrm{FP}(j)$ is a disjoint union of smooth curves, so that (i_s, j) on $\tilde{X}_s$ also has as fixed point set a disjoint union of smooth curves. Blowing up $\tilde{X}_s/(i_s, j)$ along these gives a Calabi-Yau threefold ([CoK, 4.4.1]). The rational map

$$E_s \times Y \dashrightarrow X_s$$

induces a Hodge isometry from $H^2_{\mathrm{tr}}(\tilde{X}_s)[2]$ to its image in $H^3(Y_s)$, where the notation means that the intersection form gets multiplied by 2.

Conclusion: *For any a, c and positive integer m such that for $\alpha = i\sqrt{a/c}$, $\mathbb{C}/\mathbb{Z}\langle 1, \alpha\rangle$ gives an elliptic curve with complex multiplication, there is a geometric VHS of class II_2 with this α and* $B = \begin{pmatrix} ma & 0 \\ 0 & mc \end{pmatrix}$.

For these examples, $\gamma = 0$ so the LMHS is split. At present, we do not know any geometric examples where it is not split.

III.B. Towards motivation of the LMHS

First a few words about "motivating motivation". When an extension of MHS comes from a cycle on a smooth complete algebraic variety, there are two types of constraints on the extension class:

(III.B.1) constraints arising from the field of definition;

(III.B.2) more general algebro-geometric constraints (in this case, the failure of Abel-Jacobi to surject, since the extension class is the cycles Abel-Jacobi image).

This observation generalizes on the one hand to higher Chow cycles; i.e. algebraic K-theory classes, with regulator replacing AJ and (III.B.1) related to the Beilinson conjectures, and on the other to the extensions *present* in the cohomology, viewed as MHS, of more general varieties.

So it is natural to ask whether similar restrictions apply to the LMHS of a VHS arising from a degenerating family of algebraic varieties. In this section we will look at some evidence, including the type I example from III.A, for strong algebro-geometric constrants, focusing on the Tate extensions present in the LMHS, as well as a conjectural "motivic" explanation for their origin. Lest this appear far-fetched, consider that the analogue of Conjecture (III.B.5) below for limits of Abel-Jacobi classes is a *theorem* in [GGK, §III.B]. By way of notation, all fields k considered in this section are subfields of $\mathbb{C}$.

Roughly speaking, a motive $\mathbb{M}/k$ is a bounded complex of smooth quasi-projective varieties with arbitrary morphisms between them, all defined over k with arrows opposite to the actual morphisms:

(III.B.3) $$\varphi : X_1 \to X_2 \text{ becomes } \mathbb{Z}_{X_2} \xrightarrow{\varphi^*} \mathbb{Z}_{X_1} .$$

Through a "realization" process similar to hypercohomology, one can take the various (motivic, absolute Hodge, Deligne, étale, de Rham, Betti, etc.) cohomology groups of such a complex, which come endowed with a natural weight filtration. In particular, one gets weight and Hodge filtrations on $H^*_{\mathrm{dR}}(\mathbb{M}^{an}_{\mathbb{C}})$, and weight filtration on $M^*_{\mathrm{Betti}}(\mathbb{M})$, yielding a MHS, which we say to be *motivated* $/k$.

(III.B.4) **Definition:** A *semistable degeneration*$/k$ (of n-folds) is a diagram of smooth quasi-projective varieties and morphisms

$$\begin{array}{ccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_{p_0} := f^{-1}(p_0) \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow \\ C\backslash\{p_0\} =: C^* & \hookrightarrow & C & \hookleftarrow & \{p_0\} \\ & & {\scriptstyle g}\downarrow & & \\ & & \mathbb{P}^1 & & \end{array}$$

where

(i) $\dim_k(C) = 1$, $\dim(\mathcal{X}/C) = n$;
(ii) $g \in k(C)^*$ with $g(p_0) = 0$ and $\mathrm{ord}_{p_0}(g) = 1$;
(iii) $\bar{f}$ is proper and flat, and f is smooth;
(iv) $Y := X_{p_0}$ is a reduced NCD, i.e. $Y = \cup Y_i$ and the $Y_I := \cap_{i\in 1} Y_i$ are smooth and irreducible of dimension $n - |I|$ (or empty). Write $\iota_i^I : Y_I \hookrightarrow Y_{I\backslash\{i\}}$ for the inclusions; and
(v) everything (including the Y_I and C_i^I) is defined $/k$.

Remarks: (a) The choice of function g is taken as part of the definition of the SSD in order that "the LMHS of a SSD$/k$" be well-defined: it is computed with respect to the local algebraic parameter

$$t := g \circ \bar{f} .$$

For convenience later on, we will require that $g^{-1}(0) = p_0$ and $g^{-1}(\infty) = \emptyset$.

(b) We make an exception to (iv) in case Y is an irreducible curve with ordinary double point singularities. Then $Y^{[0]} = \tilde{Y}$ and $Y^{[1]} = \cup\{\text{double pts.}\}$.

What we mean by "motivation", then, is the following[16]

(III.B.5) **Conjecture:** *The LMHS (at s_0, in any degree) attached to a* SSD$/k$ *is motivated* $/k$.

For this to be plausible we have to enhance and refine our notion of motives in various ways. We will not do this rigorously but essentially have in mind the triangulated tensor category $D^b_{\mathrm{mot}}(Sm_k)$ of [Le], in which certain morphisms of motives are inverted to impose the axioms of a cohomology theory. Here are three key points:

[16][De, p. 695] essentially articulates this conjecture in reference to nilpotent orbits.

Refined morphisms: We formally defined arbitrary pullback morpshims in (III.B.3). In order for non-flat pullbacks φ^* (from X_2 to X_1) to be available on the level of cochains (for computing realizations), the cochains must be in good position with respect to the substrata of X_2 over which the relative dimension of φ changes. Instead of keeping track of substrata, [Le] adjoins to each X_i a map $f_i : X_i' \to X_i$ and demands that φ be compatible with a *flat* map $\tilde{\varphi} : X_1' \to X_2'$. He then formally inverts the morphisms forgetting such data, which reflects various moving lemmas in the theory of algebraic cycles, currents, etc. So while we will suppress these f_i, we want the reader to be aware of their necessity for computation.

Twisted motives: More generally one wants to have morphisms of "pure motives" $\mathbb{Z}_{X_1} \to \mathbb{Z}_{X_2}$ induced by arbitrary correspondences, including Gysin/pushforward maps. This requires a notion of cup-product with cycle-classes, since correspondences are algebraic cycles on $X_1 \times X_2$. If we enhance each "pure" motive by a twist by $p \in \mathbb{Z}$ and a shift by $q \in \mathbb{Z}$ (vis; $\mathbb{Z}_X(p)[q]$), then $Z \in Z^n(X/k)$ has a formal class[17]

$$\mathbb{Z}_{pt} \xrightarrow{[Z]} \mathbb{Z}_X(n)[2n] \ .$$

Strictly speaking, $[q]$ just shifts a motive q degrees to the left, so it is not really an "enhancement". The meaning of (p) will become clear in the context of realizations, e.g. as the Hodge twist. So for example, if $\iota : V \hookrightarrow X$ is a closed embedding of codim c, then the resulting Gysin morphism is written

$$\mathbb{Z}_V(-c)[-2c] \xrightarrow{i_*} \mathbb{Z}_{\mathfrak{X}}$$

and defined using deformation to the normal cone.

Arbitrary varieties: Using cubical hyperresolutions one may "construct" the motive of any scheme $/k$ of finite type (e.g. singular varieties). One also obtains motives for classical constructions like relative cohomology, cohomology with support, etc. Iterating these motivic constructions produces complexes of complexes (or worse), in which case one takes $\oplus$ of terms with equal total degree to obtain a single complex.

Here are some examples of motives:

(a) *Motive of a normal crossing divisor, abbreviated NCD* $Y \subset \mathfrak{X}$. Using the notation above, let $Y^{[l]} := \coprod_{|I|=l+1} Y_I$ (recall that each Y_I is smooth and complete). Let $\langle i \rangle_I :=$ position in which i occurs in I; write $\Delta_I : Y_I \hookrightarrow Y_I \times Y_I$ for the diagonal, and

$$\Delta_I^i := (\mathrm{id} \times i_I^i) \circ \Delta_I : Y_I \hookrightarrow Y_I \times Y_{I \setminus \{i\}} \ .$$

The correspondences

$$Z^{[l]} := \sum_{|I|=l+1} \left(\sum_{i \in I} (-1)^{\langle i \rangle_I} \left[\Delta_I^i(Y_I) \right] \right) \in Z^{n-|I|+1} \left(Y^{[l]} \times Y^{[l-1]} \right)$$

induce pullback morphisms

$$\underset{(\text{deg. } 0)}{\mathbb{Z}_{Y^{[0]}}} \longrightarrow \mathbb{Z}_{Y^{[1]}} \longrightarrow \cdots \longrightarrow \mathbb{Z}_{Y^{[n]}}$$

[17] viewed either as a cycle-class map, or as a 2-term complex with $\mathbb{Z}_{pt}$ in degree 0 ("the motive of $[Z]$").

and the resulting complex defines $\mathbb{Z}_Y$. The relative motive $\mathbb{Z}_{(\mathcal{X},Y)}$ is Cone$\{Z_{\mathcal{X}} \to \mathbb{Z}_Y\}[-1]$, i.e. the complex with $\mathbb{Z}_{\mathcal{X}}$ in degree 0, $\mathbb{Z}_Y$ in degree 1, and morphism ι_Y^*.

(b) *Motive of a NCD complement* $\mathcal{X}\backslash Y =: U$. Since U is quasi-projective this is just $\mathbb{Z}_U$, which for Hodge-theoretic purposes is unsuitable: there is no meaningful Hodge-filtration on the level of currents on U. Instead, we can use the Gysin map $(\iota_Y)_*$ together with the morphisms induced by $Z^{[l]}$ in the *opposite* direction to produce the complex

$$\mathbb{Z}_{Y^{[n]}}(-n-1)[-2n-2] \to \cdots \to \mathbb{Z}_{Y^{[0]}}(-1)[-2] \to \underset{\text{(deg. 0)}}{\mathbb{Z}_{\mathcal{X}}}$$

which is isomorphic to $\mathbb{Z}_U$. Omitting the $\mathbb{Z}_{\mathcal{X}}$ yields a motive isomorphic to $\mathbb{Z}_{\mathcal{X},Y}$, defined as the "motive of $\mathcal{X}$ with support in Y". Here the $Z^{[l]}$-induced maps are essentially sums of (signed) Gysin morphisms.

(c) *The relative n-cube motive.* Consider the "algebraic n-cube" $\square^n := (\mathbb{P}^1\backslash\{1\})^n$ with NCD

$$\partial\square^n := \{(z_1,\ldots,z_n) : z_i = 0 \text{ or } \infty \text{ for some } i\} \subset \square^n .$$

One can think of $(\square;\partial\square)$ as a nodal rational curve minus a point, and $(\square^n;\partial\square^n)$ as $(\square;\partial\square)^{\times n}$. Using (a)[18] to define the relative motive one has

$$\mathbb{Z}_{(\square^n;\partial\square^n)} \cong \mathbb{Z}^{\otimes n}_{(\square;\partial\square)} \cong Z_{\text{pt.}}[-n] . \tag{III.B.6}$$

It is convenient to set $(\partial)\square^n_X := (\partial)\square^n \times X$, and write $(\partial\square^n_X)_{j=1,\ldots,2n}$ for the irreducible components of $\partial\square^n_X$.

(d) *The regulator (or AJ) motive of a relative cycle.* Let X be smooth and complete. Consider $Z \in Z^p(\square^n_X)$ with components meeting all faces $(\partial\square^n_X)_I$ properly and with the intersection condition $Z\cdot(\partial\square^n_X)_i = 0$ for all i. Any class in $H^{2p-n}_{\mathcal{M}}(X,\mathbb{Z}(p)) \cong \mathrm{CH}^p(X,n)$ is represented by such a cycle. Now set

$$U_Z := \square^n_X\backslash|Z|, \qquad \partial U_Z = \partial\square^n_X \cap U ;$$

using (a) and (b) one defines the motive

$$\mathbb{Z}_{(U;\partial U)} .$$

This geometric construction "approximates" (see below) the *ad hoc* regulator motive

$$\mathcal{R}_Z := \text{Cone}\left\{\mathbb{Z}_{\text{pt.}} \xrightarrow{[Z]} \mathbb{Z}_{(\square^n_X;\partial\square^n_X)}(p)[2p]\right\}[-1] .$$

We turn next to the realizations of a motive M, presented as a complex with i^{th} term

$$\mathbb{M}^i = \mathbb{Z}_{X^i}(a_i)[b_i]$$

where each X^i is smooth and *complete*. Fix a cohomology theory K (= motivic, de Rham, Betti) and a theory of cochains (for X smooth and complete) $Z^*_K(X,\mathbb{Z}(p))$

[18] Since $\mathbb{Z}_{\square^m} \cong \mathbb{Z}_{\text{pt.}}$ for all m; we need not use (b) for this, even for Hodge theory.

with cohomology computing $H^p_K(X, \mathbb{Z}(p))$:

C^∞ (co)chains	$C^*_{\text{top}}(X^{an}_{\mathbb{C}}, (2\pi\sqrt{-1})^p\mathbb{Z})$	for $K = \text{B}$
currents	$D^*(X^{an}_{\mathbb{C}})$	for K=dR
higher Chow precycles	$Z^p(X, 2p - *)$	for $K = \mathcal{M}$.

Placing a "stupid weight filtration" directly on cochains

$$W_m Z^Q_K(X, \mathbb{Z}(P)) := \begin{cases} Z^Q_K(X, \mathbb{Z}(P)), & Q - 2P \leq m \\ 0, & \text{otherwise} \end{cases}$$

and a Hodge filtration on de Rham cochains

$$F^n Z^Q_{\text{dR}}(X, \mathbb{Z}(P)) := F^{n+P} D^Q(X) ,$$

we have the following[19]

(III.B.7) **Definition:** The K-cohomology groups of the motive $\mathbb{M}$ are

$$H^q_K(\mathbb{M}, \mathbb{Z}(p)) := H^q \left\{ \underset{i+j=\bullet}{\oplus} Z^{j+b_i}_K(X^i, \mathbb{Z}(p + a_i)) \right\} ,$$

with weight filtration

$$W_m H^q_K(\mathbb{M}, \mathbb{Z}(p)) := \text{Im} \left\{ \begin{array}{c} H^q \left[\underset{i+j=\bullet}{\oplus} W_m Z^{j+b_i}_K(X^i, \mathbb{Z}(p + a_i)) \right] \\ \longrightarrow H^q_K(\mathbb{M}, \mathbb{Z}(p)) \end{array} \right\} .$$

The Hodge filtration $F^n H^q_{\text{dR}}(\mathbb{M}, \mathbb{Z}(p))$ is defined the same way. At least for any "reasonable" motive, the data

$$(W_\bullet H^q_{\mathbb{B}}(\mathbb{M}, \mathbb{Z}(p)), F^* W_\bullet H^q_{\text{dR}}(\mathbb{M}, \mathbb{Z}(p))$$

yields a MHS which we will call a "Hodge realization" of $\mathbb{M}$ and write simply as $H^q(\mathbb{M}, \mathbb{Z}(p))$.

Applied to the motives from (a) and (b), this construction recovers the standard MHS on respectively

$$H^*(Y), \;\; H^*(\mathcal{X}; Y), \;\; H^*(\mathcal{X} \backslash Y), \;\; \text{and} \;\; H^*_Y(\mathcal{X}) .$$

The MHS computation

$$H^p\left((\square^n; \partial\square^n), \mathbb{Z}(0)\right) \cong \begin{cases} \mathbb{Z}(0) & p = n \\ 0 & p \neq n \end{cases}$$

implied by (III.B.5) is also correct. Hence

$$H^{2p}\left((\square^n_X; \partial\square^n_X), \mathbb{Z}(p)\right) \cong H^{2p-n}(X, \mathbb{Z}(p)) ,$$

and one has a short-exact sequence

$$H^{2p-n-1}(X, \mathbb{Z}(p)) \to H^0(\mathcal{R}_Z, \mathbb{Z}(0)) \to \mathbb{Z}(0) ,$$

with extension class $\text{AJ}(Z)$ in the abelian group

(III.B.8) $$\text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H^{2p-n-1}(X, \mathbb{Z}(p))) =: J^{p,n}(X) .$$

To *compute* this class, one uses (III.B.7) and the fact that

$$H^{2p-1}((U; \partial U), \mathbb{Z}(p))$$

[19] taking for granted morphisms $Z^*_K(\mathbb{M}^i, \mathbb{Z}(\cdot)) \to Z^*_K(\mathbb{M}^{i+1}, \mathbb{Z}(i))$ of cochains

(see (d)) has $H^0(\mathcal{R}_Z)$ as a sub-MHS, and actually equals it if $|Z|$ does not have too many irreducible components (see [KLM, §§6–7] where this is carried out). The main point to note here is that regulators of algebraic cycles $/k$ in generalized intermediate Jacobians are just Hodge realizations of motives $/k$.

When $X = \mathrm{Spec}(\mathbb{C})$ and $n = 2p-1$, (III.B.8) becomes

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{Z}(p)) \cong \mathbb{C}/\mathbb{Z}(p) .$$

If we allow (the class of) $Z_s \in \mathrm{CH}^p(\mathbb{C}, 2p-1) \cong_{\otimes\mathbb{Q}} K_p^{\mathrm{alg}}(\mathbb{C})$ to vary, $H^0(\mathcal{R}_{Z_s})$ gives a VMHS. In particular, transversality holds, and so

(III.B.9) $$\mathrm{AJ}(Z_s) \in \mathbb{C}/\mathbb{Z}(p) \text{ is constant if } p > 1 .$$

Moreover, any $Z \in \mathrm{CH}^p(\mathbb{C}, 2p-1)$ is the base change to $\mathbb{C}$ of a cycle defined over a subfield $K \subset \mathbb{C}$ finitely generated over $\mathbb{Q}$, so that $K \cong \bar{\mathbb{Q}}(\mathcal{S})$ for some variety $\mathcal{S}/\bar{\mathbb{Q}}$. Spreading Z out yields $\mathcal{Z} \in \mathrm{CH}^p(U/\bar{\mathbb{Q}}, 2p-1)$, where $U \subset \mathcal{S}$ is a Zariski open, together with $s_0 \in U(K)$ such that $\iota^*_{s_0}\mathcal{Z} = Z$. Taking any $s_1 \in U(\bar{\mathbb{Q}})$, consider $\iota^*_{s_1}\mathcal{Z} \in \mathrm{CH}^p(\bar{\mathbb{Q}}, 2p-1)$; since this is an algebraic cycle in $\square^{2p-1}$ the coefficients of its defining equations generate a number field k. Hence $\iota^*_{s_1}\{Z\} = Z_0 \otimes_k \bar{\mathbb{Q}}$ for some $Z_0 = K_p^{\mathrm{alg}}(k) \otimes \mathbb{Q}$, whose regulator $\mathbb{C}/\mathbb{Z}(p)$ is related to the zeta-value $\zeta_k(p)$ by Borel's theorem. By the above argument, $\mathrm{AJ}(Z) = \mathrm{AJ}(Z_0)$ and so we see for $Z/\mathbb{C}$

(III.B.10) $\mathrm{AJ}(Z) \in \mathbb{C}/\mathbb{Z}(p)$ takes only countably many (and arithmetically meaningful) values if $p > 1$.

Now suppose we have a Tate extension $\mathbb{E}$:

$$\mathbb{Z}(j) \to \mathbb{E} \to \mathbb{Z}(l), \qquad p := l - j \geq 1$$

that has been extracted, by push-pull of extensions as in §I.C, from the LMHS of a SSD$/k$. If one believes (III.B.5), then $\mathbb{E}$ is present in the Hodge-realization $H^q(\mathbb{M}, \mathbb{Z}(p))$ of a motive$/k$. Suppose we believe moreover that such a "motivic Tate extension" is always the regulator of a higher cycle$/k$. *Then for $p > 1$ (III.B.9) applies to $[\mathbb{E}] \in \mathbb{C}/\mathbb{Z}(p)$, while if $p = 1$ then taking $\exp(\cdot)$ of $[\mathbb{E}] \in \mathbb{C}/\mathbb{Z}(1)$ yields an element of $k^*(\cong K_1(k))$.*

Here are two main examples in support of this claim:

Example 1: $(p = 3)$**.** The type I example of §III.A, after performing SSR on the quintic mirror family at $z = 0$, is attached to a SSD$/\mathbb{Q}$. The only nontorsion extension present in the LMHS is of the form $\mathbb{Z}(0) \to \mathbb{E} \to \mathbb{Z}(-3)$, and its class should therefore be a rational multiple of $\zeta(3)$, which it is. It is interesting to note that one can write down a relative cycle $/\mathbb{Q}$ of the form considered in (d) above, so that $H^5((U_Z; \partial U_Z), \mathbb{Z}(3))$ is *precisely* $\mathbb{E}$. Set

$$\begin{aligned} Z_0 \;\; := \;\; & -\left\{\frac{t}{t-1}, \frac{u}{u-1}, 1-tu, t, u\right\}_{u,t\in\mathbb{P}^1\setminus\{0,1,\infty\}} \\ & +\frac{1}{2}\left\{\frac{t}{t-1}, \frac{1}{1-t}, \frac{(u-1)^2}{(u-t)(u-\frac{1}{t})}, tu, \frac{u}{t}\right\}_{\substack{u,t\in\mathbb{P}^1\setminus\{0,1\} \\ u^{\pm 1}\neq t\neq\{\infty\}}} \\ \in \;\; & Z^3(\square^5; \partial\square^5)_{\mathbb{Q}} , \end{aligned}$$

which has $\mathrm{AJ}(Z_0) = \zeta(3)$ by a very direct computation using the currents of [KLM]; then the desired cycle is $Z := -200Z_0$.

Example 2: $(p = 1)$**.** We will give two propositions concerning semistable degenerations of elliptic curves $\mathcal{E} \xrightarrow{t} U \subset \mathbb{P}^1$ $(\{o\} \in U)$.

(III.B.11) **Proposition:** *Let $\mathcal{E}$ be a SSD$/k$ of Kodaira type I_1, so that the normalization $\sigma : \mathbb{P}^1_z \to E_0$ is defined $/k$, with $\sigma(0) = \sigma(\infty) =: q$. Then the attached LMHS is an extension*

$$\mathbb{Z}(0) \to \mathbb{E} \to \mathbb{Z}(-1)$$

with class $\log(a) \in \mathbb{C}/\mathbb{Z}(1)$*, where $a \in k^*$.*

Proof: Passing to $\mathcal{E}^{an}_{\mathbb{C}}$, there exists an analytic neighborhood V of q and analytic functions x, y with the following properties, writing $\Phi = 1 + \mathcal{O}(x) + \mathcal{O}(y)$:

(i) the local equation of E_t is $xy = at\Phi$ for some $a \in k^*$;

(ii) writing $V \cap E_0 = \Delta_0 \cup \Delta_\infty$ and $\sigma_0 := \sigma|_{\Delta_0}$, $\sigma_\infty := \sigma|_{\Delta_\infty}$, we have

$$\sigma_0^* dx = dz, \qquad \sigma_\infty^* dx = 0 \ ,$$

$$\sigma_\infty^* dy = d(\frac{1}{z}), \qquad \sigma_0^* dy = 0 \ ;$$

(iii) there is a family of holomorphic 1-forms

$$\omega(e) \in \Gamma\left(U, \Omega^1_{\mathcal{E}/U} \left\langle \log(E_0/\{0\}) \right\rangle\right)$$

with $\sigma^*\omega(0) = \frac{dz}{z}$, and

$$\omega(t)\big|_{E_t \cap V} = \mathrm{Res}_{E_t \cap V}\left(\frac{dx \wedge dy}{xy - at\Phi}\right) \ .$$

The computation we shall do yields the same result regardless of Φ, so to simplify matters put $\Phi \equiv 1$.

Denoting by $\gamma_{0,t}$ and $\gamma_{1,t}$ cycles on E_t such that $T(\gamma_0) = \gamma_0$, $T(\gamma_1) = \gamma_1 + \gamma_0$, we must compute

$$\text{(III.B.12)} \qquad \lim_{t \to 0} \int_{e_1(t)} \omega(t)$$

where $e_1(\epsilon) =: \gamma_1(t) - l(t)\gamma_0(t)$. This records in

$$\mathbb{C}/\mathbb{Z}(1) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{Z}(-1))$$

the LMHS of the family; we claim that (III.B.12) is just $\log(a)$. Since the constant term of the Taylor series expansion of $\int_{\gamma_0(t)} \omega(t)$ is just $2\pi\sqrt{-1}$, it suffices to compute

$$\text{(III.B.13)} \qquad \lim_{t \to 0} \left\{ \int_{\gamma_1(t)} \omega(t) - \log t \right\} \ .$$

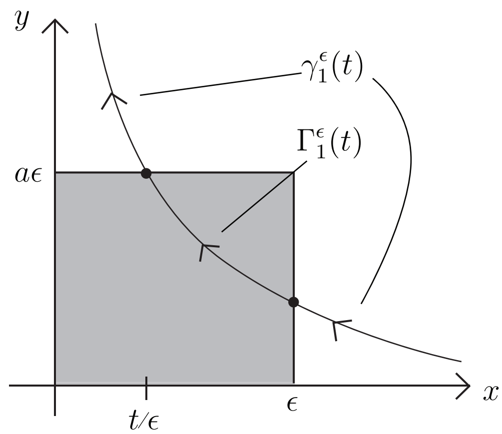
Take $\epsilon > 0$ sufficiently small that

$$B_\epsilon := \{|x| \le \epsilon, |y| \le a\epsilon\} \subset V \ ,$$

and assume for $0 < t < \epsilon^2$ $(t \in \mathbb{R})$ that

$$\gamma_1(t) \cap B_\epsilon = \left\{ \left(x, \frac{at}{\epsilon}\right) : x \in \left[\frac{t}{\epsilon}, \epsilon\right] \right\} =: \Gamma_1^\epsilon(t) \ ,$$

with $\gamma_1^\epsilon(0) = \gamma_1(t)\backslash\Gamma_1^\epsilon(t)$. Hence it makes sense to write $\gamma_1^\epsilon(0)$ for the path on $\mathbb{P}_z^1$ from $z = \frac{1}{y} = \frac{1}{a\epsilon}$ to $z = x = \epsilon$. Here is the picture:



Replacing $\gamma_1(t)$ in (III.B.13) by $\gamma_1^\epsilon(t) + \Gamma_1^\epsilon(t)$ and taking the limit in the first term gives

$$\lim_{t\to 0}\left\{\int_{\gamma_1^\epsilon(0)}\omega(0) + \int_{\Gamma_1^\epsilon(t)}\omega(t) - \log t\right\}$$
$$= \lim_{t\to 0}\left\{\left(\log\epsilon - \log\frac{1}{a\epsilon}\right) + \left(\log\frac{t}{\epsilon} - \log\epsilon\right) - \log t\right\} = \log a\ ,$$

as desired. □

An important corollary of the proof is that while the canonical coordinate

$$s = \exp\left(2\pi\sqrt{-1}\frac{\int_{\gamma_1(t)}\omega(t)}{\int_{\gamma_0(t)}\omega(t)}\right)$$

will not be defined $/k$, the LMHS can still be split using a parameter $/k$, namely $\tilde{t} := at$. More generally one should expect that geometric[20] LMHS can be normalized without enlarging the field of definition.

Next we consider a fairly general SSD of elliptic curves, presented as a family of anticanonical hypersurfaces in a toric Fano surface. Fix a Laurent polynomial $\varphi \in k[x_1^{\pm 1}, x_2^{\pm 1}]$ with reflexive Newton polytope Δ_φ and associated smooth toric variety $\mathbb{P}_{\Delta_\varphi}$, 0btained from the fan on all integer points of the dual $\Delta_\varphi^\vee$. We assume the closure E_t of

$$\{1 - t\cdot\varphi(x_1, x_2) = 0\}$$

in $\mathbb{P}_{\Delta_\varphi}$ is smooth for general t, which implies that it is an elliptic curve.

[20]i.e., arising from a SSD as in (III.B.4) where the field of definition already encompasses (i)–(iv).

Now enumerate the vertices of Δ_φ by $\{\mathbf{v}_i\} \subset \mathbb{Z}^2$, and let

$$\begin{aligned} p &:= \#\{\text{integer points of } \Delta_\varphi^\vee\} - 1, \\ p_i &:= 1 + \#\{\text{integral interior points on the edge of } \Delta_\varphi^\vee \text{ dual to } \mathbf{v}_i\}, \\ a_i &:= \text{coefficient of } \mathbf{x}^{\mathbf{v}_i} \text{ in } \varphi, \text{ and} \\ a &:= \prod a_i^{p_i} \in k^* . \end{aligned}$$

One may then show the following

(III.B.14) **Proposition:** *The family* $\{E_t\}$ *is a* SSD$/k$ *of type* I_p*, with LMHS of class* $\log(a) \in \mathbb{C}/\mathbb{Z}(1)$.

As in Example 1, one can easily write down a relative cycle $/k$ motivating the LMHS $\mathbb{E}$, in either (III.B.11) or (III.B.14); it is just $[a] \in \mathbb{Z}^1(\square/k)$. The corresponding "geometric motive" $(U_Z; \partial U_Z)$ is the motive of a $\mathbb{P}^1_k$ with $\{0\}$ and $\{\infty\}$ identified, and $\{1\}$ and $\{a\}$ removed.

In the end of the last example, swapping "identified" and "removed" yields the dual motive, with the same Hodge realization, which we write as a double complex

(III.B.15)
$$\begin{array}{ccc} \begin{pmatrix} \deg - \\ (0,0) \end{pmatrix} \mathbb{Z}_{\mathbb{P}^1} & \xrightarrow{a^* - 1^*} & \mathbb{Z}_{\text{pt.}} \\ \uparrow {\scriptstyle 0_* - a_*} & & \\ \mathbb{Z}_{\text{pt.}}(-1)[-2] & & \end{array}$$

In the situation of (III.B.11) one would like to think of this as

(III.B.16)
$$\frac{\mathbb{Z}_{\mathcal{E}^*}^{(\deg 0)}}{\mathbb{W}_0(\mathbb{Z}_{\mathcal{E}^*})}[-1] \longrightarrow \frac{\mathbb{Z}_{\mathcal{E}^*}}{\mathbb{W}_1(\mathbb{Z}_{\mathcal{E}^*})}[-2] \, ,$$

whatever this means, with the morphism a "motivic analogue of $\wedge \frac{dt}{t}$". Here is why.

Retaining the notation of (III.B.4), let us recall Steenbrink's construction of the LMHS [Z2]. One considers the double-complex of sheaves on $\mathcal{X}$

(III.B.17)
$$A^{p,q} := \Omega_{\mathcal{X}}^{p+q+1} \langle \log Y \rangle / W_q \Omega_{\mathcal{X}}^{p+q+1} \langle \log Y \rangle \quad (p, q \geq 0)$$

with horizontal and vertical differentials given by exterior derivative, respectively, "$\wedge \frac{dt}{t}$", and with weight and Hodge filtrations

$$\begin{aligned} W_k A^{p,q} &= W_{2q+k+1} \Omega_{\mathcal{X}}^{p+q+1} \langle \log Y \rangle / W_q \Omega_{\mathcal{X}}^{p+q+1} \langle \log Y \rangle \\ F^r A^{\bullet,q} &= A^{\bullet \geq r, q} . \end{aligned}$$

Writing $A^\bullet = \underset{p+q=\bullet}{\oplus} A^{p,q}$

(III.B.18a)
$$H^n(X, A^\bullet)$$

is isomorphic to the cohomology of a general fiber of f, while

(III.B.18b)
$$F^r W_m H^n(X, A^\bullet) := \text{Im}\left\{ H^n(X, F^r W_{m-n} A^\bullet) \to H^n(X, A^\bullet) \right\}$$

recovers the limit Hodge and weight monodromy filtrations.

This begs the question as to the existence of a *motivic Steenbrink complex/k* having (III.B.18) as de Rham-realization, and hence, putting this together with the

Betti realization, motivating the LMHS of a SSD. There are well-definedness issues to be resolved, perhaps in a future work, but the construction should go roughly as follows.

Define a "weight filtration" on the motive

$$\mathbb{Z}_{\mathcal{X}^*} \cong \left\{ \mathbb{Z}_{Y^{[n]}}(-n-1)[-2n-2] \to \cdots \to \mathbb{Z}_{Y^{[0]}}(-1)[-2] \to \underset{(\deg 0)}{\mathbb{Z}_{\mathcal{X}}} \right\}$$

by

$$\mathbb{W}_i(\mathbb{Z}_{\mathcal{X}^*}) := \left\{ \mathbb{Z}_{Y^{[i-1]}(-i)[-2i] \to \cdots \to \mathbb{Z}_{\mathcal{X}}} \right\}$$

so that

$$\mathbb{Z}_{\mathcal{X}^*}/\mathbb{W}_i \cong \left\{ \mathbb{Z}_{Y^{[n]}(-n+i)[-2n+2i] \to \cdots \to \mathbb{Z}_{Y^{[i]}}} \right\} (-i-1)[-2i-2] \; . \tag{III.B.19}$$

One can think of the motivic in braces in (III.B.19) as a "homological" motive of $\cup_{|I|=i+1} Y_I$.

Next, the analogue of "$\wedge \frac{dt}{t}$" should be the Gysin morphism

$$\begin{aligned} \mathbb{Z}_{\mathcal{X}^*} \xrightarrow{\otimes t} \mathbb{Z}_{(\square_{\mathcal{X}^*};\partial\square_{\mathcal{X}^*})}(1)[2] \\ \cong \mathbb{Z}_{\mathcal{X}^*} \otimes \mathbb{Z}_{(\square;\partial\square)}(1)[2] \cong \mathbb{Z}_{\mathcal{X}^*}(1)[1] \end{aligned} \tag{III.B.20}$$

induced by

$$\begin{array}{ccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X}^* \times (\square;\partial\square) \\ x & \mapsto & (x, t(x)) \; . \end{array}$$

In fact, its de Rham realization *is* $\wedge \frac{dt}{t}$ since $[d\log t]$ is the current associated to $\{t\} \in Z_1(\mathcal{X}^*, 1)$. The authors of [BOV] appear to have arrived at a similar morphism in the setting of Voevodsky motives.

Putting everything together, one would seem to get a motive $/k$

$$\underset{(\deg 1)}{\frac{\mathbb{Z}_{\mathcal{X}^*}}{\mathbb{W}_0}} \xrightarrow{\otimes t} \frac{\mathbb{Z}_{\mathcal{X}^*}}{\mathbb{W}_1}(1)[1] \xrightarrow{\otimes t} \frac{\mathbb{Z}_{\mathcal{X}^*}}{\mathbb{W}_2}(2)[2] \to \dots \tag{III.B.21}$$

which does appear to recover (III.B.15) in the curve cases. The weight and Hodge filtrations on the de Rham cochains implied by (III.B.7) are in agreement with Steenbrink's. (Note that in (III.B.21) only the analogue of Steenbrink's *vertical* differential is visible.) The real challenge appears to be recovering a motivic version of the log monodromy operator N, which looks natural in Steenbrink's complex ([Z2, Prop. 4]).

To conclude, we are not claiming to have proved (III.B.5) but hope this can be done by making (III.B.21), or something like it, rigorous.

Part IV: "Frenet framing" of complex variations of quintic mirror Hodge structures

A local complex variation[21] of quintic mirror Hodge structures is given by a homomorphic curve

$$\varphi : S \to \check{D} \tag{IV.1}$$

[21]By a complex VHS we mean that we ignore the real (and therefore the integral and rational) structures.

where S is a disc with coordinate s. We may think (IV.1) as given by a holomorphically varying flag

$$s \to \left\{ F^3(s) \subset F^2(s) \subset F^1(s) \subset F^0(s) = H_{\mathbb{C}} \right\}$$

where $\dim F^i(s)/F^{i-1}(s) = 1$ and

$$\text{(IV.2)} \qquad \begin{cases} \text{(i)} & F^3(s)' \subseteq F^2(s) \\ \text{(ii)} & Q(F^1, F^3) = 0 \\ \text{(iii)} & Q(F^2(s), F^2(s)) = 0 \end{cases} .$$

If we choose a generating vector

$$\varphi(s) \in F^3(s)$$

then the condition

$$\text{(IV.3)} \qquad \varphi(s) \wedge \varphi'(s) \wedge \varphi''(s) \wedge \varphi'''(s) \neq 0$$

is invariant under scaling of φ.

Definition: *The complex variation of quintic mirror Hodge structures is* **non-degenerate** *if (IV.3) is satisfied.*

Now $\check{D} \subset G_{\mathbb{C}}/P$ is a homogeneous space, and there is a long and highly developed story of *differential invariants* for curves in homogeneous spaces (cf. [J], [Gr] and the references cited therein). Such differential invariants arise by determining a canonical lifting of φ in (IV.1) to a map $\tilde{\varphi}$ to $G_{\mathbb{C}}$ in the diagram

$$\text{(IV.4)} \qquad \begin{array}{ccc} & & G_{\mathbb{C}} \\ & \overset{\tilde{\varphi}}{\nearrow} & \downarrow \\ S & \xrightarrow{\ \varphi\ } & \check{D} \end{array}$$

and then pulling back the left-invariant *Maurer-Cartan* form

$$\omega = g^{-1} dg \ .$$

The coefficients of $\tilde{\varphi}^*(\omega)$ then give a complete set of differential invariants of (IV.1) in the sense that

> *two maps (IV.1) differ by a rigid notion — i.e., a transformation by a fixed $g \in G_{\mathbb{C}}$ — if, and only if, they have the same set of differential invariants.*

The classic example is a curve

$$\text{(IV.5)} \qquad \gamma : I \to \mathbb{E}^3$$

in Euclidean 3-space; here I is a connected interval in $\mathbb{R}$ with coordinate t. Such a curve is non-degenerate if it does lie in a 2-plane; i.e., if

$$\gamma \wedge \gamma' \wedge \gamma'' \neq 0 \ .$$

Then we may parametrize the curve by arclength and the Frenet frame gives a lifting

$$\begin{array}{ccc} & & G \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \\ I & \xrightarrow{\gamma} & \mathbb{E}^3 \end{array}$$

where G is the group of rigid motions, which may be identified with the set of position vector with an attached orthonormal frame $(x; e_1, e_2, e_3)$. The independent coefficients in $\tilde{\gamma}^*(\omega_G)$ may then be identified with the curvature and torsion. These may be chosen to be arbitrary functions of one variable,[22] and two non-degenerate curves differ by a rigid motion if, and only if, they have the same curvature and torsion. One says that, *up to rigid motion, non-degenerate curves in* $\mathbb{F}^3$ *depend on two arbitrary functions of one variable.* Analogously, we shall show

(IV.6) **Propostion:** *Non-degenerate complex variations of Hodge structure locally have canonical liftings (IV.4). Up to rigid motion in* $G_{\mathbb{C}}$ *they depend on one arbitrary, non-vanishing function* δ.

The proof will show that there is a canonical choice of parameter (like arc-length). For reasons that will be clear, we shall refer to δ *as the differentially invariant form of the Yukawa coupling.* We will also relate the Frenet lifting to the LMHS when we have a degenerating family of Hodge structures of type I.

Proof: We set

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where the choice of signs is slightly different from what we have used before, but this choice will avoid irrelevant sign distractions in the final formulas.

For any non-zero $\varphi(s) \in F^3(s)$, setting $' = d/ds$ and omitting the dependence on s, we have

(IV.7) $$\left\{ \begin{array}{l} \varphi, \varphi', \varphi'', \varphi''' \textit{ gives a basis for } H_{\mathbb{C}} \textit{ that} \\ \textit{is adapted to } F^3 \subset F^2 \subset F^1 \subset F^0 \end{array} \right\}.$$

We are free to change $\varphi(s)$ by

(IV.8) $$\left\{ \begin{array}{llll} \text{(i)} & \tilde{\varphi}(s) = \lambda(s)\varphi(s), & \lambda \neq 0 & \text{(rescaling)} \\ \text{(ii)} & \hat{\varphi}(t) = \varphi(s(t)), & s' \neq 0 & \text{(reparametrizing).} \end{array} \right.$$

(IV.9) **Lemma:** *Up to a choice of constants representing initial conditions in an ODE, there exist unique changes (IV.8) so that, denoting by* ψ *the result we have*

(IV.10) $$Q\left(\psi^{(i)}, \psi^{(j)}\right) = Q_0$$

where $\psi^{(i)} = (d/dt)^i \psi(t)$, $0 \leqq i,\ j \leqq 3$ *and* $\delta \neq 0$.

[22]Provided of course that the curvature is everywhere non-zero.

Proof: Differentiating $0 = Q(\varphi, \varphi')$ gives

$$Q(\varphi, \varphi'') = 0 \ .$$

Differentiating this relation gives

$$Q(\varphi, \varphi''') + Q(\varphi', \varphi'') = 0 \ .$$

The matrix $Q(\varphi^{(i)}, \varphi^{(j)})$ is thus of the form

$$\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & \beta \\ 0 & \alpha & 0 & \gamma \\ -\alpha & -\beta & -\gamma & 0 \end{pmatrix} .$$

We note that

$$\alpha = Q(\varphi, \varphi''') \neq 0 \ ,$$

and differentiating $-\alpha = Q(\varphi', \varphi'')$ gives

$$\beta = Q(\varphi', \varphi''') = Q(\varphi', \varphi'')' = -\alpha' \ .$$

Under the change in (IV.8.i) we have, with the hopefully self-evident notation

$$\tilde{\alpha} = \lambda^2 \alpha$$

so we can restrict to

$$\alpha = 1 \ , \tag{IV.11}$$

which implies that

$$\beta = 0 \ .$$

We now do both changes in (IV.8) simultaneously

$$\left\{ \begin{aligned} \varphi &\to \lambda\varphi \\ \varphi' &\to \lambda s\varphi' + \lambda'\varphi \\ \varphi'' &\to \lambda(s')^2\varphi'' + (\lambda s'' + 2\lambda' s')\varphi' + \lambda''\varphi \\ \varphi''' &\to \lambda(s')^3\varphi''' + (3\lambda s' s'' + 3\lambda'(s')^2)\varphi'' \\ &\qquad\qquad +(\lambda s''' + 3\lambda' s'' + 3\lambda'' s')\varphi' + \lambda'''\varphi \ . \end{aligned} \right. \tag{IV.12}$$

If we set

$$\psi(t) = \lambda(s(t)) \ \ \varphi(s(t)) \ ,$$

then to maintain the constraint (IV.11) we must have

$$\lambda^2(s')^3 = 1 \ . \tag{IV.13}$$

Differentiating this gives

$$\begin{gathered} 2\lambda\lambda'(s')^3 + 3\lambda^2(s')^2 s'' = 0 \\ \implies 2\lambda' s' + 3\lambda s'' = 0 \ , \end{gathered}$$

and one more differentiation leads to

$$2\lambda'' s' + 5\lambda' s' + 3\lambda s'' = 0 \ . \tag{IV.14}$$

Next, after simplification and using $Q(\varphi', \varphi'') = 1$, (IV.13) and (IV.14) give

$$\begin{aligned} Q(\psi'', \psi''') \quad = \quad & \Big[-4\lambda\lambda''(s')^3 + 6(\lambda')^2(s')^3 + 6\lambda\lambda'(s')^2 s'' \\ & +3\lambda^2 s'(s'')^2 - \lambda^2(s')^2 s''' \Big] + \frac{\gamma}{\lambda^{4/3}} \ . \end{aligned}$$

From (IV.13) the term in brackets turns out to be

$$\frac{4}{3}\left(\frac{\lambda'}{\lambda}\right)^2 - \frac{10}{3}\frac{\lambda''}{\lambda} =: E(\lambda, \lambda, \lambda'') \ .$$

The ODE

$$0 = E(\lambda, \lambda, \lambda'') + \frac{\gamma}{\lambda^{4/3}}$$

may now be solved to obtain $Q(\psi'', \psi''') = 0$. □

Relabeling, at this stage we may assume that the matrix

$$Q(\varphi^{(i)}, \varphi^{(j)}) = Q_0 \ .$$

The desired frame is now

$$F = (\varphi, \varphi', \varphi'', \varphi''') \ .$$

The Picard-Fuchs equation is

$$\varphi'''' = a_0\varphi + a_1\varphi' + a_2\varphi'' + a_3\varphi''' \ .$$

Lemma: *We have*

$$a_1 = a_2 = a_3 = 0 \ .$$

The remaining function $a_0 =: \delta$ may be locally specified arbitrarily.

Proof: We have

$$a_3 = Q(\varphi'''', \varphi) = Q(\varphi''', \varphi)' - Q(\varphi''', \varphi') = 0 \ .$$

Next,

$$-a_2 = Q(\varphi'''', \varphi') = Q(\varphi''', \varphi')' - Q(\varphi''', \varphi'') = 0 \ .$$

Finally,

$$a_1 = Q(\varphi'''', \varphi'') = Q(\varphi''', \varphi'')' = 0 \ .$$ □

It follows that

$$F' = F \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{pmatrix}$$

so that the pulled back Maurer-Cartan form is

$$\omega = F^{-1}F^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{pmatrix} \ . \tag{IV.15}$$

Finally we shall refer to δ as the *different invariant Yukawa coupling* for the following reason: For a degeneration of type I there are canonically determined a parameter s and $\varphi(s) \in F^3(s)$ such that (cf. proposition 5.6.1 in [CoK])

$$\left(\frac{\Phi''}{Y}\right)'' = 0 \ . \tag{IV.16}$$

where Y is the *Yukawa coupling.* Note that s here is noted by q in [CoK], and $' = 2\pi\sqrt{-1}q\frac{d}{dq}$. For the canonically determined frame $\varphi, \varphi', \varphi'', \varphi'''$, it follows from

(IV.16) that the pulled back Maurer-Cartan form is expressed in terms of Y, Y' and Y''. We therefore have the

Conclusion: *The degenerating complex VHS of mirror quintic type and class I is uniquely determined, up to a rigid motion in $G_{\mathbb{C}}$, by the function $Y(s)$. Subject to specifying its singularity at $s = 0$, it may be arbitrarily assigned.*

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