

# Abel-Jacobi maps for higher Chow groups

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0. Review : Cycles + Hodge theory
1. Classical cases : Griffiths + Dirichlet
2. Main tools : Bloch's "higher" cycles  
+ Deligne cohomology
3. Regulator/AJ maps : the KLM formula
4. Examples : dilogarithm + Meijer G-functions

# Cycles and periods

(2)

$X = \text{nonsingular complete algebraic variety} / \mathbb{C}$ ,  $\dim = d$

- The algebraic cycles of codimension  $p$  on  $X$  are elements of the free abelian group on irreducible subvarieties of  $X$  of (complex) codimension  $p$ :

$$Z = \sum_i q_i V_i \in Z^p(X). \quad (q_i \in \mathbb{Q})$$

- The topological cycles of codimension  $n$  on  $X$  are  $\partial$ -closed  $C^\infty$ -chains of (real) codimension  $n$ :

$$C_{\text{top}}^n(X; \mathbb{Q}) \supseteq \ker(\partial) =: Z_{\text{top}}^n(X).$$

We will sometimes take coefficients in  $\mathbb{Q}(m) := (2\pi i)^m \mathbb{Q}$ .

- Sheaves on  $X$ :

$\Omega_X^n := \text{holomorphic forms of degree } n$

$\Omega_{X^\infty}^n := C^\infty \text{ forms of degree } n$

$D_X^n := \text{currents of deg. } n = \text{bounded linear functionals on } \Omega_{X^\infty}^{2d-n}$ .

$$C_{\text{top}}^n(X, \mathbb{Q}) \hookrightarrow D^n(X)$$

$$\Gamma(\Omega_{X^\infty}^n) \hookrightarrow D^n(X)$$

$$\Gamma \longmapsto \int_\Gamma =: "J_\Gamma"$$

$$\Omega \longmapsto \int (\cdot) \wedge \Omega =: "J_\Omega"$$

- Periods:

$$\left. \begin{array}{l} \omega \in \Gamma_{d-cl}(\Omega_{X^{(0)}}^n) \\ \gamma \in Z_{(h)}^{\text{top}}(X) \\ \text{(dimension)} \end{array} \right\} \rightsquigarrow \int_\gamma \omega$$

exterior derivative

$d$ : degree  $n$

$\mapsto$  degree  $n+1$

# Hodge filtration

(ii)

We have

$$H_{dR}^n(X, \mathbb{C}) \underset{ii}{=} \underbrace{H_{sing}^n(X, \mathbb{Q})}_{ii} \otimes \mathbb{C}$$

$$\underset{ii}{H^n\{\Gamma(X, \Omega_{X^{\text{an}}}^\bullet), d\}} \quad H^n\{C_{top}^\bullet(X; \mathbb{Q}), \partial\}$$

$$\underset{ii}{H^n\{D^\bullet(X), d\}}$$

These de Rham cohomology groups admit a decreasing filtration:

$$H_{dR}^n(X, \mathbb{C}) = F^0 H_{dR}^n \supseteq F^1 H_{dR}^n \supseteq \dots \supseteq F^n H_{dR}^n \supseteq \underbrace{F^{n+1} H_{dR}^n = \{0\}}$$

Let

$F^p \Omega_{X^{\text{an}}}^n :=$  sheaf of  $C^\infty$  forms of degree  $n$ ,  
which in a local analytic chart  $(z_1, \dots, z_d)$   
are a sum of terms of the form

$$(*) \quad \underbrace{f(z)}_{C^\infty} dz_{k_1} \wedge \dots \wedge dz_{k_j} \wedge d\bar{z}_{k_{j+1}} \wedge \dots \wedge d\bar{z}_{k_n}$$

where  $j \geq p$ .

Then

$$F^p H_{dR}^n(X, \mathbb{C}) \underset{\text{defn.}}{\cong} H^n\{\Gamma(X, F^p \Omega_{X^{\text{an}}}^\bullet), d\}.$$

Periods yield an embedding

$$\underset{\substack{\text{(topological)} \\ \text{(n-cycle)}}}{H_n(X, \mathbb{Q})} \hookrightarrow \{F^i H^n(X, \mathbb{C})\}^\vee \xrightarrow{\quad} \int_\gamma \quad \underset{\substack{\text{(functional given)} \\ \text{by integration}}}{\quad}$$

# Hodge decomposition

(iii)

$H^{p, n-p}(X) \subseteq H^n_{\text{dR}}(X, \mathbb{C})$  denotes the subspace represented by d-closed forms (\*) with  $j=p$ .

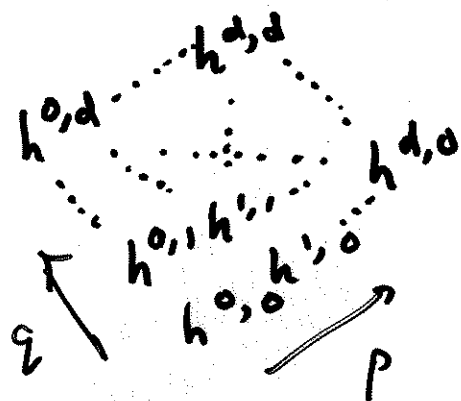
We have

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$$

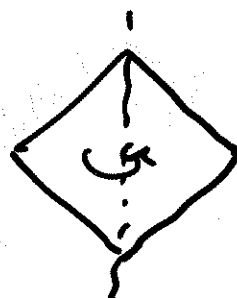
$h^2$   
 $H^2(\mathcal{O}_X^p)$

## The Hodge diamond

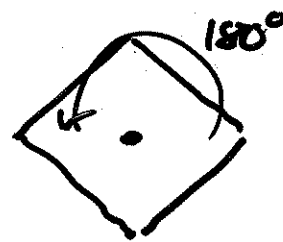
records the  $h^{p,q} := \dim_{\mathbb{C}}(H^{p,q})$ :



Symmetries:



complex conjugation



Serre duality

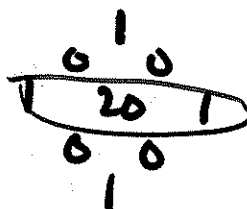
Examples: elliptic curve



(= g for higher genus curve)

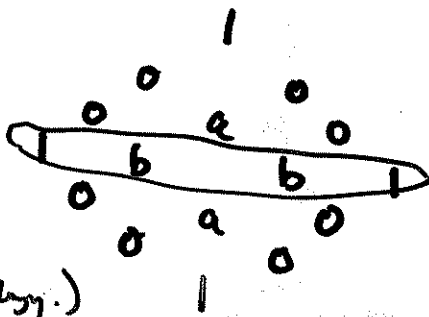
$\mathbb{P}^1$

K3 surface



$\mathbb{P}^2$

Calabi-Yau 3-fold



$\mathbb{P}^3$

(I have circled the "middle-dimension" cohomology.)

# 1. Classical cases

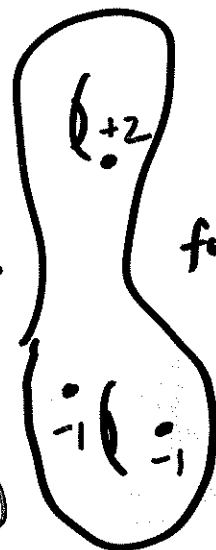
(1)

## 1.1 Griffiths's Abel-Jacobi map

$X/\mathbb{C}$  smooth projective variety,  $\dim_{\mathbb{C}}(X) = m$ .

$Z = \sum q_i V_i \in Z^p(X)$  algebraic cycle on  $X$

(generalizes divisors on a Riemann surface)



e.g.  
for  $m=1$ ,  
 $p=1$ .

Equivalence relations :

$$Z \equiv_{\text{hom}} 0 \stackrel{\text{def}}{\iff} \exists \text{ " } \delta^{-1}Z \text{ " } \in C_{2m-2p+1}^{\text{top}}(X_{\text{an}})$$

$$Z \equiv_{\text{rat}} 0 \stackrel{\text{def}}{\iff} \exists W \in Z^p(X \times \mathbb{P}^1)$$

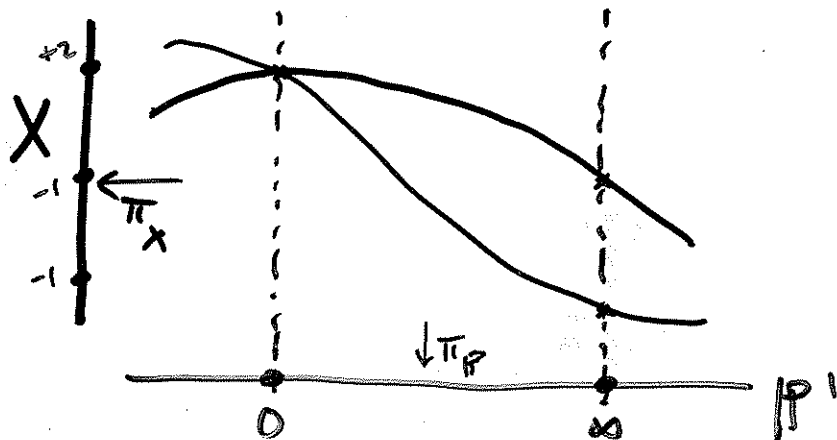
such that

$$Z = \pi_X \{ W \cdot (X \times \{0\}) - W \cdot (X \times \{\infty\}) \}$$



$$\iff \exists Y_i \subseteq X$$

irreducible codim.  $(p-1)$ ,  
 $f_i \in \mathbb{C}(\tilde{Y}_i)^*$ ,  
s.t.  $\sum \iota_{*}((f_i)) = Z$ .



$$\implies Z \equiv_{\text{hom}} 0 : \text{ take } \delta^{-1}Z := \pi_X(\pi_P^{-1}(\infty, 0)).$$

Invariants for cycles modulo rational equivalence:

(2)

$$Z^p(X) \xrightarrow{\sim} \{H^{2m-2p}(X, \mathbb{C})\}^{\vee} \cong H^{2p}(X, \mathbb{C})$$

$\downarrow$

$$Z \longmapsto \int_Z (\cdot) \longmapsto [Z] \quad \text{(fundamental class)}$$

This "cycle-class" map is well-defined by Stokes's theorem

• well-def'd  $\equiv_{\text{rat}}$  by last page

• factors through rational  $(p, p)$  classes

$$H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q}) ;$$

Hodge conjecture says image should span them.

$$Z_{\text{hom}}^p(X) \xrightarrow{\sim} \{F^{m-p+1} H^{2m-2p+1}(X, \mathbb{C})\}^{\vee} / \text{im} \{H_{2m-2p+1}(X, \mathbb{Q})\}$$

$\nwarrow$  periods

well-defined:

$$\int_{\delta^{-1}Z} d\omega = \int_Z \omega = 0$$

modulo  $\equiv_{\text{rat}}$ :

$$Z \equiv_{\text{rat}} 0 \Rightarrow$$

$$\int_{\pi_X(\pi_P^{-1}(\overline{\alpha}, \delta))} \gamma = \int_0^1 (\pi_P)_* \pi_X^* \gamma = 0$$

$d$ -closed  $(1,0)$  current  $\xrightarrow{\text{lemma}} \Omega^1(P') [ = 0 ]$ . (regularity)

$$\left\{ \frac{\int_{\gamma} (F^{m-p+1} \Omega_{X^{\text{an}}}^{2m-2p+1})}{d\omega} \right\}^{\vee} / \text{periods}$$

$$H^{2p-1}(X, \mathbb{Q}) \setminus H^{2p-1}(X, \mathbb{C}) / F^p H^{2p-1}(X, \mathbb{C})$$

(complex torus " $J^p(X)$ " if replace  $\mathbb{Q}$  by  $\mathbb{Z}$ .)

To organize this information, we make some defn's.

- A Hodge structure of weight  $q$  is a vector space  $\mathcal{H}_{\mathbb{Q}}$  with a filtration  $F^{\bullet}$  on  $\mathcal{H}_{\mathbb{C}} := \mathcal{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ , satisfying  $\overline{F^p \mathcal{H}_{\mathbb{C}}} \cap F^{q-p+1} \mathcal{H}_{\mathbb{C}} = \{0\}$ .
- For  $\mathcal{H}$  a HS of weight  $\begin{cases} 2p \\ 2p-1 \end{cases}$  set
 
$$\begin{cases} H_g^p(\mathcal{H}) := \mathcal{H}_{(\mathbb{Q})} \cap F^p \mathcal{H}_{\mathbb{C}} & (\text{Hodge group}) \\ J^p(\mathcal{H}) := \mathcal{H}_{\mathbb{C}} / (F^p \mathcal{H}_{\mathbb{C}} + \mathcal{H}_{(\mathbb{Q})}) & (\text{Jacobi}) \end{cases}$$
- Chow groups:  $CH^p(X) := Z^p(X) / \equiv_{\text{rat}}$ .

Then the maps just constructed are written

$$cl: CH_{\text{cl}}^p(X) \rightarrow H_g^p(H^{2p}(X))$$

$$AJ: CH_{\text{hom}}^p(X) \rightarrow J^p(H^{2p-1}(X))$$

Can collect together into one map:

$$H_D^r(X_{\text{an}}, \mathbb{Q}(p)) := H^r \left\{ \underbrace{C_{2m-p}^{\text{top}}(X, \mathbb{Q}(p)) \oplus \Gamma(F^p D_X^{\bullet}) \oplus \Gamma(D_X^{\bullet-p})}_{\text{Deligne con.}} \right\}$$

$$\text{differential } D: (a, b, c) \mapsto (\partial a, -d[b], d[c] - b + \delta_a)$$

$$C_D: CH^p(X) \xrightarrow{H_g^p(X)} H_D^{2p}(X, \mathbb{Q}(p)) \xrightarrow{J^p(X)}$$

what ultimately gets called AJ

$$C_D(z) := (2\pi i)^p (z, \bar{z}, 0)$$

... will come back to this.

## 1.2 Dirichlet's regulator map

(4)

The term "regulator" classically refers to the covolume of a "thickening" of this map.

$k = \# \text{ field}$

$$[k:\mathbb{Q}] = n = \underbrace{|\text{Hom}(k, \mathbb{C})|}_{\text{real}} = r_1 + 2r_2$$

conj. pairs of ex.

Write embeddings:

$$\sigma_1, \dots, \sigma_{r_1}; \sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}; \sigma_{r_1+r_2+1}, \dots, \sigma_n$$

conjugation

$\mathcal{O}_k = \text{ring of } \underline{\text{integers}}$  (= solutions of algebraic equations with integer coefficients)  
v1

$\mathcal{O}_k^* = \text{units}$  (= invertibles in  $\mathcal{O}_k$ )

One has

$$\mathcal{O}_k^* \otimes \mathbb{Q} \cong \mathbb{Q}^{r_1+r_2-1}$$

v1

$$\left\{ \alpha_j \right\}_{j=1}^{r_1+r_2-1} \text{ basis}$$



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$$\underbrace{(i.e. \alpha \otimes 1)}_{\alpha} \longrightarrow (\log \sigma_1(\alpha), \dots, \log \sigma_n(\alpha))$$

$$k^* \otimes \mathbb{Q} \xrightarrow{"A_T"} (\mathbb{C}/\mathbb{Q}(1))^n \xrightarrow{\pi_R} \mathbb{R}^n$$



$$\mathbb{Q}^{r_1+r_2-1} \cong \mathbb{Q}_k^* \otimes \mathbb{Q} \xrightarrow{"r"} \mathbb{R}^{r_1+r_2}$$

$$\begin{aligned} \text{thickening} \left\{ \begin{aligned} \oplus \quad \alpha &\longrightarrow (\log |\sigma_1(\cdot)|, \dots, \log |\sigma_{r_1}(\cdot)|; 2 \log |\sigma_{r_1+1}(\cdot)|, \dots, 2 \log |\sigma_{r_1+r_2}(\cdot)|) \\ \mathbb{Q} \ni 1 &\longrightarrow (1, \dots, 1; 1, \dots, 1) \end{aligned} \right. \end{aligned}$$

$\text{Im}(\hat{\gamma}^{\text{thickened}})$  has a "covolume" defined up to  $\mathbb{Q}^*$ ;

writing  $d_{r_1+r_2}$  for the "1",

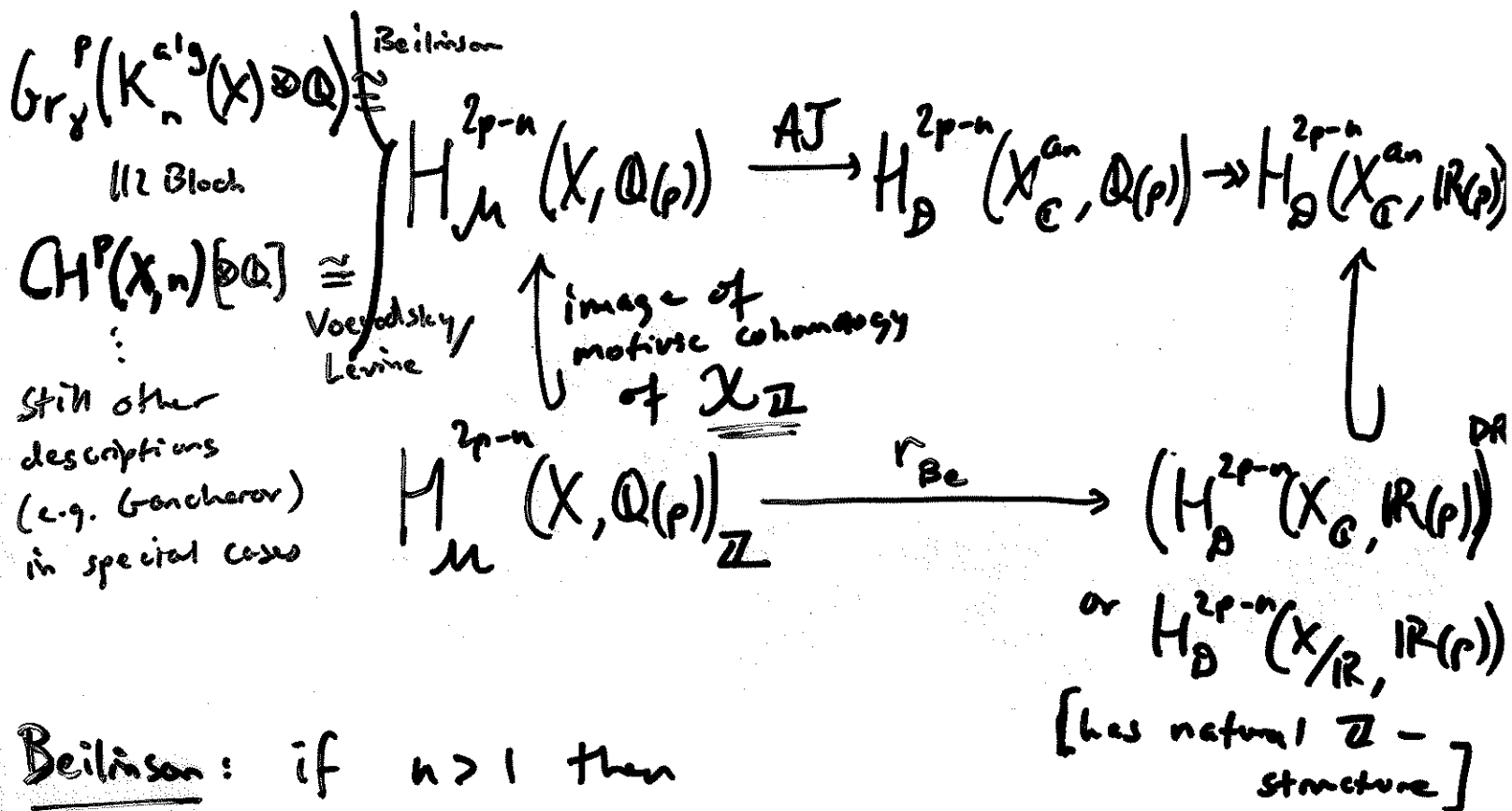
$$\frac{1}{r_1+r_2} \det [\text{matrix w./ rows } r(\alpha_j)]$$

$\xrightarrow{\text{Dirichlet regulator}} \mathbb{R} \xrightarrow{\cdot \mathbb{Q}^*} \sum_k^* (0)$

$1^{\text{st}}$  non zero coefficient in Taylor expansion about 0.

# 1.3 General picture

$X = \text{smooth quasi-projective, defined } / \mathbb{Q}$



Beilinson: if  $n > 1$  then

$$\text{cokernel}(\text{im}(r_{Be})) \xrightarrow[\mathbb{Q}^*]{\text{conj.}} L(H^{2p-n-1}(X), p)$$

( $\Rightarrow$  also a conjecture about rank of image)

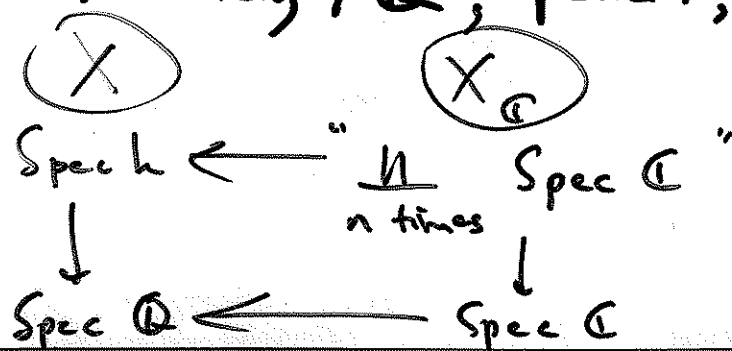
This is a sort of global  $\sim$  local statement.  
(at  $\infty$ ) (FF<sub>primes</sub>)

Dirichlet:  $X = \text{Spec } k$  viewed as variety  $/ \mathbb{Q}$ ,  $p=n=1$ ,

using  $K_1(k) \cong k^*$

$$X_{\mathbb{Z}} = \text{Spec } \mathcal{O}_k$$

$$K_1(\mathcal{O}_k) = \mathcal{O}_k^*$$



New

$$AJ: CH^p(X, n) \rightarrow H_D^{2p-n}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}(p))$$

(7)  
or even  $\mathbb{Z}(p)$   
(but we use  $\mathbb{Q}(p)$ )

is defined for  $X$ /any field, e.g.  $\mathbb{C}$ , and carries more information than  $r_{Be}$ .

Griffiths: Case  $n=0$ , i.e.

$$Gr^p_y K_0^{\text{alg}}(X)_{\mathbb{Q}} \cong CH^p(X) \cong CH^p(X, 0).$$

higher Chow groups are set up in this way

For  $n > 0$ , one can view  $AJ$  as splitting into 2 parts (as with  $C_D$  for  $n=0$  above).

$$\mathbb{Q}_D(p)^{\bullet} := \left\{ \mathbb{Q}(p) \xrightarrow{\text{deg. } 0} \mathcal{O}_X \xrightarrow{1} \Omega_X^1 \xrightarrow{2} \dots \xrightarrow{p} \Omega_X^{p-1} \right\}$$

$$\cong \underline{\text{Cone}(\mathbb{Q}(p) \oplus F^p \Omega_X^{\bullet} \rightarrow \Omega_X^{\bullet})}[-1]$$

$$H_D^m(X, \mathbb{Q}(p)) // := H^m(X, \mathbb{Q}_D(p))$$

Deligne cohomology

automatically get short-exact sequence

$$\frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Q}(p))} \rightarrow H_D^{2p-n}(X, \mathbb{Q}(p)) \rightarrow F^p H^{2p-n}(X, \mathbb{C}) \cap H^{2p-n}(X, \mathbb{Q}(p))$$

$$\cong J^{p,n}(X)$$

Generalized Intermediate Jacobian

$$\cong Hg^{p,n}(X)$$

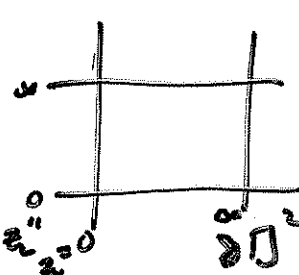
$\times$  projective and  $n \geq 1$   
 $\Rightarrow \{0\}$ .

## 2. Tools for a geometric description of AJ

(8)

### 2.1 Higher Chow groups

$\square^n = (\mathbb{P}^1 \setminus \{1\})^n$  — "algebraic n-cube" — weird version of affine space  
 $UI \leftarrow \text{facets}$

$$\partial \square^n = \bigcup_i \{ (z_1, \dots, z_n) \in \square^n \mid z_i \in \{0, \infty\} \}$$


$$\partial^k \square^n = \bigcup_{|I|=k} \{ (z_1, \dots, z_n) \in \square^n \mid z_i \in \{0, \infty\} \forall i \in I \}$$

faces  $\rho_i^0, \rho_i^\infty : \square^{n-1} \hookrightarrow \square^n$

$$Z^p(X \times \square^n) \geq c^p(X, n) \geq d^p(X, n)$$

cycles

irreducible components intersect all  $X \times \partial^k \square^n$  properly

"degenerate" pullbacks from faces

$$Z^p(X, n) := \frac{c^p(X, n)}{d^p(X, n)}$$

"admissible"

Bloch differentiation

$$\partial_B : Z^p(X, n) \rightarrow Z^p(X, n-1)$$

$$\mathbb{Z} \hookrightarrow \sum_{i=1}^n (-1)^{i-1} (\rho_i^0)^* - (\rho_i^\infty)^* \mathbb{Z}$$

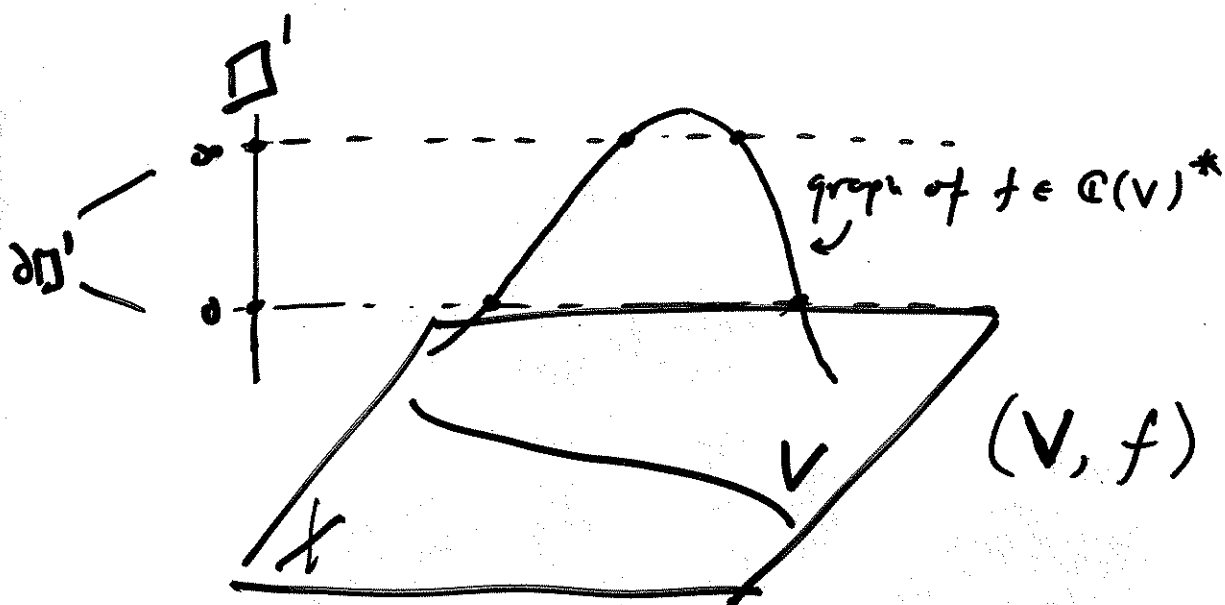
collapsing intersections of  $Z$  with the facets to one  $\square^{n-1}$ .

$$CH^p(X, n) := H_n(Z^p(X, \cdot); \partial_B)$$

# Some examples

(9)

$n=0$ :  $X$  any dimension,  $V \subseteq X$  codim  $(p-1)$

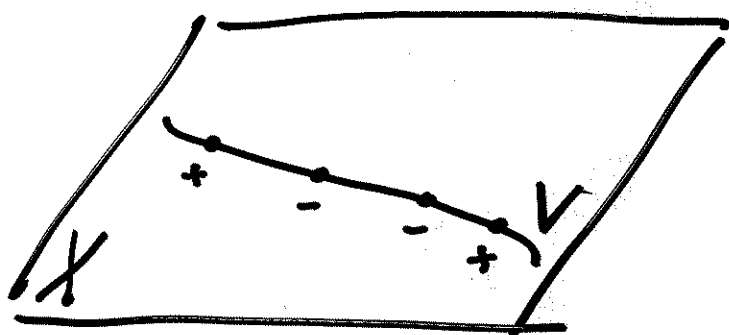


$$(V, f) \in \mathcal{Z}^p(X, 1)$$

$n=1$ :  $CH^1(X, 1) \cong \Gamma(X, \mathcal{O}_X^*)$ .

More generally,  $CH^p(X, 1)$  is "empty rational equivalences" — the divisor push-forwards cancel.

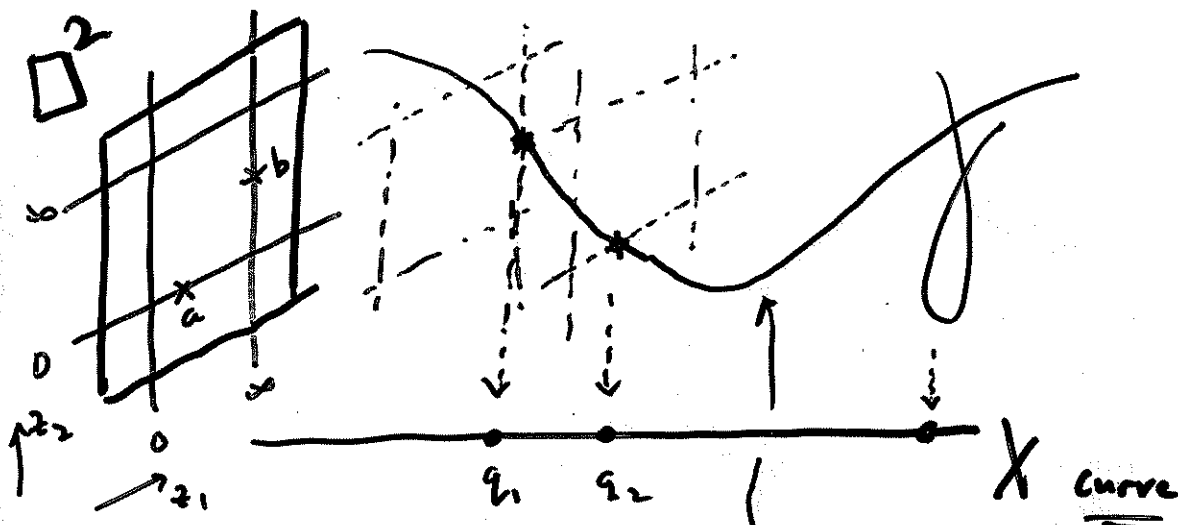
$\partial_B$



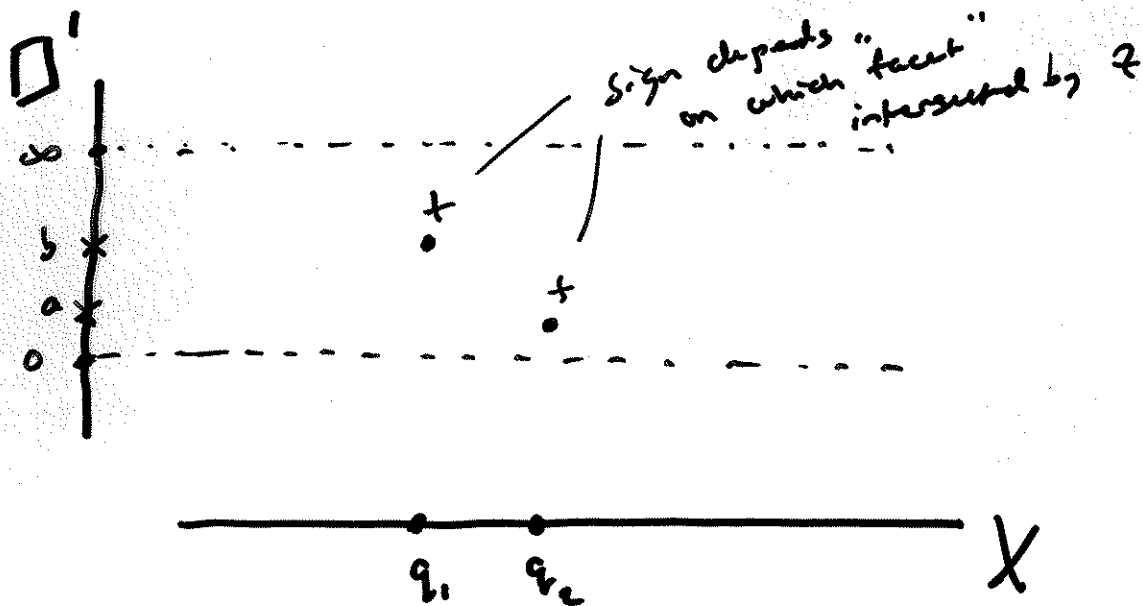
$$(l_V)_* \underbrace{(f)}_{\text{divisor of } f} \in \mathcal{Z}^p(X)$$

$\Rightarrow$  Image  $(\partial_B)$  produces usual rational equivalences,  
so  $CH^p(X, 0) \cong CH^p(X)$ .

$n=2$ :  $X \cong \text{curve}$



$$Z := \left\{ \begin{array}{l} \text{graph of } (f, g) \\ (f, g \in \mathbb{C}(X)^*) \end{array} \right\} \in Z^2(X, 2)$$

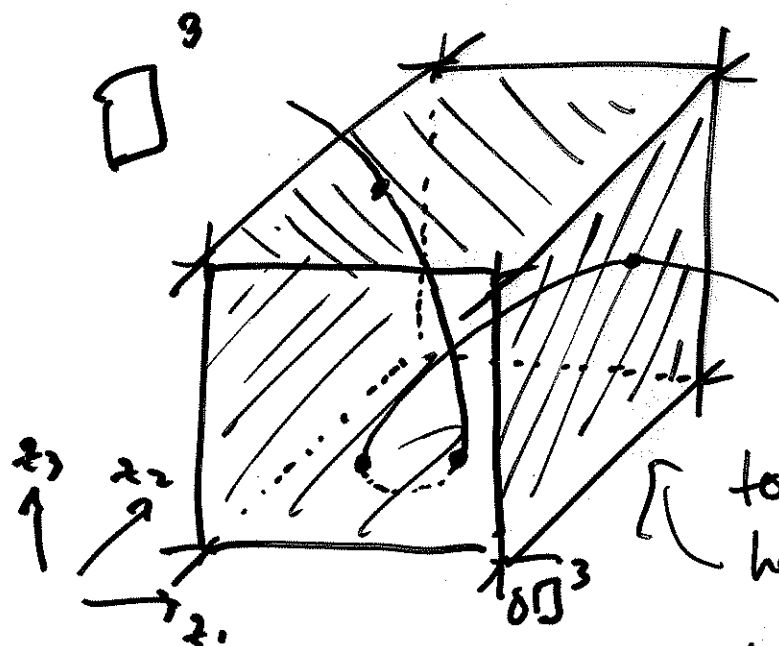


$$d_B Z := \sum_{q \in X} \begin{pmatrix} v_q(g) f(q) \\ -v_q(f) g(q) \end{pmatrix}_q \in Z^2(X, 1)$$

$Z$  will be " $d_B$ -closed" — a higher Chow cycle — if say  $f, g \equiv 1$  at each others' poles & zeroes, since " $1 \notin \square$ ".

(11)

$n=3$ :  $X = \text{point}$

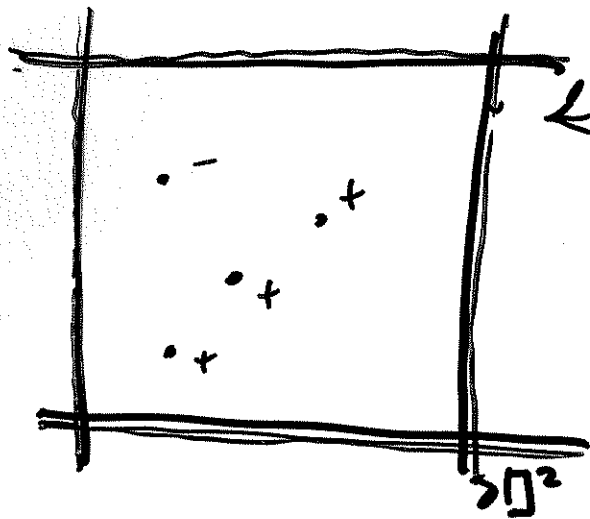


to get a cycle in here, points must cancel in here

$Z^2(\text{pt.}, 3)$

$\partial_B$

$\square^2$



$Z^2(\text{pt.}, 2)$

Example of  $\partial_B$ : first, need an element of  $Z^2(\text{pt.}, 3)$ .

$z \in \mathbb{R}^3 \mapsto (1 - \frac{\epsilon}{2}, z, z) ?$  NO.

This hits the point  $(\infty, 0, 0)$  in  $\partial^3 \square^3$ , so is inadmissible

(12)

Here is an admissible one:

$$\left[ z \xrightarrow{\epsilon P'} \right] \left( 1 - \frac{\epsilon}{z}, 1 - z, z \right)$$

omit this  
henceforth

 $\downarrow d_B$ 

$$\boxed{(1-a, a)}$$

(only facet intersection  
is with  $z_1 = 0$ , when  $\underline{z} = a$ ).

We can use such relations to compute  $CH^2(\mathbb{G}, 2)$ :

$$\left( t, \frac{(t-\alpha)(t-\beta)}{(t-\alpha\beta)(t-1)}, \gamma \right) \xrightarrow{d_B} \boxed{(\alpha\beta, \gamma) - (\alpha, \gamma) - (\beta, \gamma)}$$

$$- \left( t, \frac{(t-\alpha)(t-\beta)}{(t-\alpha\beta)(t-1)}, \frac{\alpha\beta}{t} \right) \xrightarrow{d_B} \boxed{(\alpha, \beta) + (\beta, \alpha)}$$

Using modulo relations of these types gives

$$CH^2(\mathbb{G}, 2) := \frac{\mathbb{Z}[C^* \times C^*]}{\text{im}(d_B)} \otimes \mathbb{Q} \cong \frac{C^* \otimes_{\mathbb{Q}} C^*}{\langle a, 1-a \rangle}$$

(or any field)

$$\cong: K_2^M(\mathbb{G}) \otimes \mathbb{Q}$$

(Milnor K-theory)

more generally,

$$CH^n(\mathbb{G}, n) \cong K_n^M(\mathbb{G})$$

defined  
similarly

higher n:

Recall that the regulator map on  $CH^1(\mathbb{F}, 1) \cong \mathbb{F}^*$  was essentially  $\log(\cdot)$ . The "generalization" of this is

$$CH^p(\mathbb{F}, 2p-1) \cong_{\text{conj.}} B_p(\mathbb{F}) \quad [\text{Bloch group}]$$

and AJ is given by a modification of  $p^{\text{th}}$  polylogarithm.



## 2.2 Deligne cohomology and currents

$Y = \text{smooth proj. } / \mathbb{C}, \quad \dim = M.$

$D_Y^i = i\text{-currents on } Y$  (These admit a Hodge filtration and behave like  $i$ -forms with discontinuous — even  $\delta$ -function — coeffs.)

Ex/  $V \subseteq Y$  subvariety of (complex) codimension  $p$ .

$\delta_V \in \Gamma(Y, D_Y^{2p})$

$$\int_Y \omega \wedge \delta_V := \int_V \iota_V^* \omega$$

Can extend this (by linearity) to cycles  $\sum q_i V_i$ .

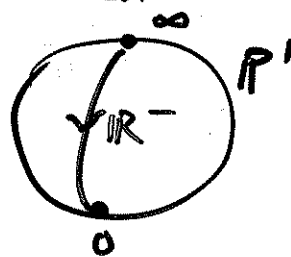
Ex/ Given  $Y \xrightarrow{f} \mathbb{P}^1$  dominant morphism,

$T_f := f^{-1}(\mathbb{R}^-)$  ( $\mathbb{R}^- = \text{negative reals oriented from } \infty \text{ to } 0$ )

$\log(f) := \text{branch with } \arg \in (-\pi, \pi)$

$\in \Gamma(D_Y^0)$

(discontinuous function w. / "jump" along  $T_f$ )



$$d[\log(f)] = \overset{0\text{-form}}{d \log f} - 2\pi i \overset{1\text{-current}}{\delta_{T_f}}$$

$\frac{df}{f}$

Recall Deligne cohomology  $H_D^*(Y, \mathbb{Q}(p))$  computed by  $C_D(Y; \mathbb{Q}(p)) :=$

$$C_{2M-0}^{top}(Y, \mathbb{Q}(p)) \oplus \Gamma(Y, F^p D_Y^\bullet) \oplus \Gamma(Y, D_Y^{\bullet-1}),$$

with  $D(T, \Omega, R) = (-\partial T, -d[\Omega], d[R] - \Omega + \delta_T)$ .

A natural  $d$ -closed element in

$$C_D'((\mathbb{C}^*, \{1\}), \mathbb{Q}(1))$$

Lefschetz dual  
of  $(\square', \partial \square')$

is  $\Theta_{(1)} := (2\pi i \underset{T_1''}{T_z}, \underset{\Omega_1''}{d \log z}, \underset{R_1''}{\log z})$

$$D \downarrow \text{mod } \{0, \infty\}$$

$$(0, 0, \underbrace{d[\log z] - d \log z + 2\pi i \delta_{T_z}}_0)$$

0 by 2<sup>nd</sup> example above

Using the Deligne cup-product formula

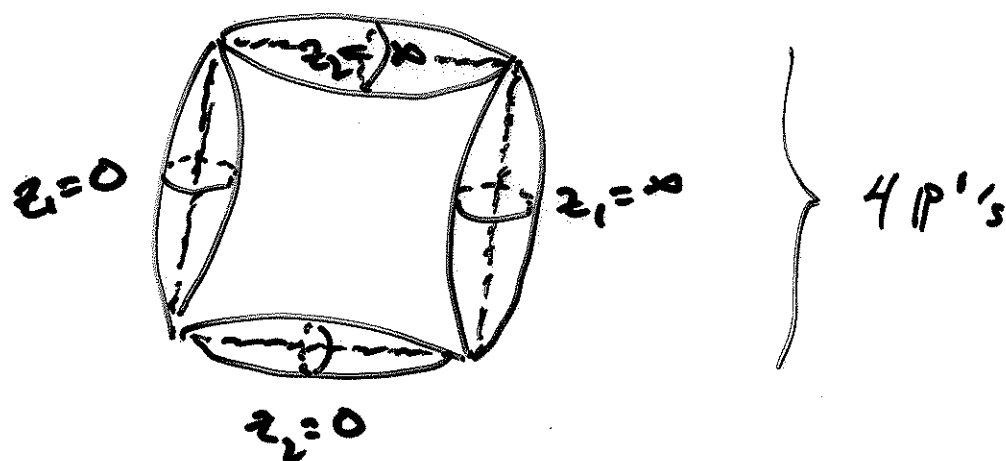
$$(A, B, C) \cup (a, b, c) = (A \cap a, B \wedge b, C \wedge b \pm c \wedge \delta_A),$$

$$\Theta_2 := \Theta \cup \Theta = \left( \underbrace{(2\pi i)^2 T_z \wedge T_z}_{T_2}, \underbrace{d \log z \wedge d \log z}_{\Omega_2}, \underbrace{\log z, d \log z, -2\pi i \log z \delta_{T_z}}_{R_2} \right)$$

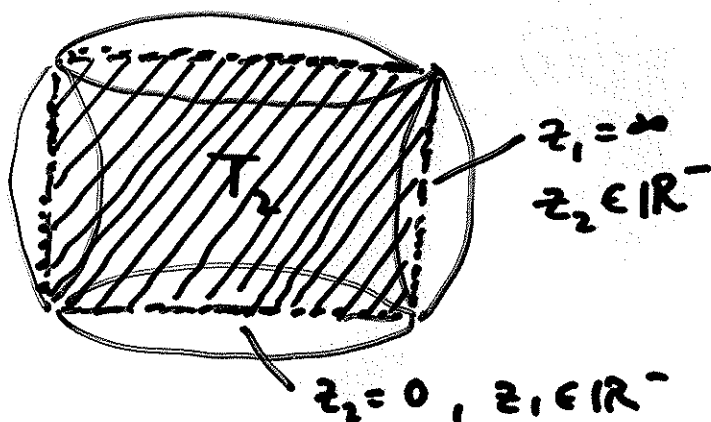
$$\in H_D^2((\mathbb{C}^*, \{1\})^2; \mathbb{Q}(2)).$$

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Here if we draw  $\partial \square^2 \subset \square^2$  as



then  $T_2$  is the "membrane" in real:



We complete one more:

$$\Theta_3 = \Theta_2 \cup \Theta_1 = ((2\pi)^3 T_3, \Omega_3, \underbrace{R_2 \wedge d\log z_3 + \log z_3 \delta_{T_2}}_{\text{"}})$$

$$\log z_1 d\log z_2 \wedge d\log z_3 + 2\pi i \log z_2 d\log z_3 \delta_{T_2}, \\ - 4\pi^2 \log z_3 \delta_{T_2} \wedge T_{\mathbb{R}^2}$$

In general, we have

$$\Theta_n = ((2\pi i)^n T_n, \Omega_n, R_n) \in H^n_D((\mathbb{C}^*, \{1\})^n; \mathbb{Q}(n)).$$

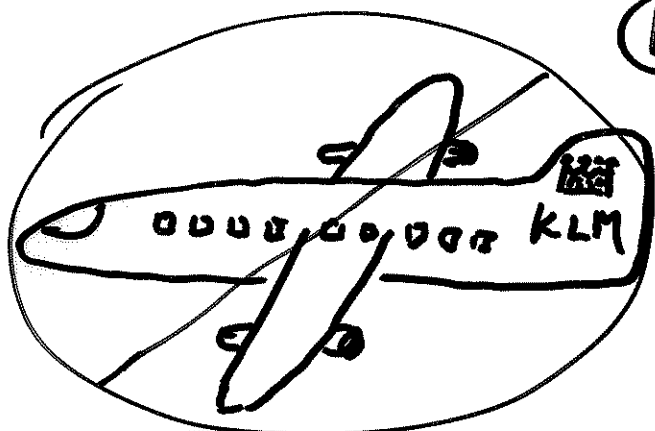
currents on  $\square^n$  essentially

# 3. The AJ map

## 3.1 KLM formula

What the formula is  
going to do:

- under identification of higher Chow with  $gr; K^q(\mathbb{P}^n)$ , provides a lift (from real to rational Deligne coh.) of the Beilinson regulator maps. (Goncharenko wrote formulas for the real regulators.)
- under identification of higher Chow with the relative Chow groups  $CH^p(X \times \mathbb{A}^n, X \times \partial \mathbb{A}^n)$ , gives a generalization of Griffiths - style AJ maps by membrane integrals:  
given  $[z] \in CH^p(X, n)$ , one can modify  $z$  by  $im(\partial_B)$  to get a cycle whose facet intersections are zero,  
$$z_{rel} \in z^p(X \times \mathbb{A}^n, X \times \partial \mathbb{A}^n).$$



K —, Lewis, Möller-Stadel  
Compositio 2006  
K —, Lewis  
rec. arXiv posting

This  $z_{rel}$  leads to a class

$$\langle z \rangle \in H_D^{2p}(X \times (\square^n, \partial \square^n), \mathbb{Q}(p))$$

given by  $((2\pi i)^p z, (2\pi i)^p \delta_z, 0)$ .

The idea of the KLM map is equivalent to setting

$$\begin{array}{ccc} \square^n & \xleftarrow{\pi_0} & X \times \square^n \\ & & \downarrow \pi_x \\ & & X \end{array}$$

$$\begin{aligned} AJ^{p,n}(z) &:= (2\pi i)^{-n} (\pi_x)_* \{ (\pi_0)^* \theta_n \cup \langle z \rangle \} \\ &\in H_D^{2p-n}(X, \mathbb{Q}(p)), \end{aligned}$$

but is more explicit, given by a map of complexes

$$\mathcal{Z}_{\mathbb{R}}^p(X, -\cdot) \xrightarrow{\mathcal{R}} C_D^{2p+\cdot}(X, \mathbb{Q}(p))$$

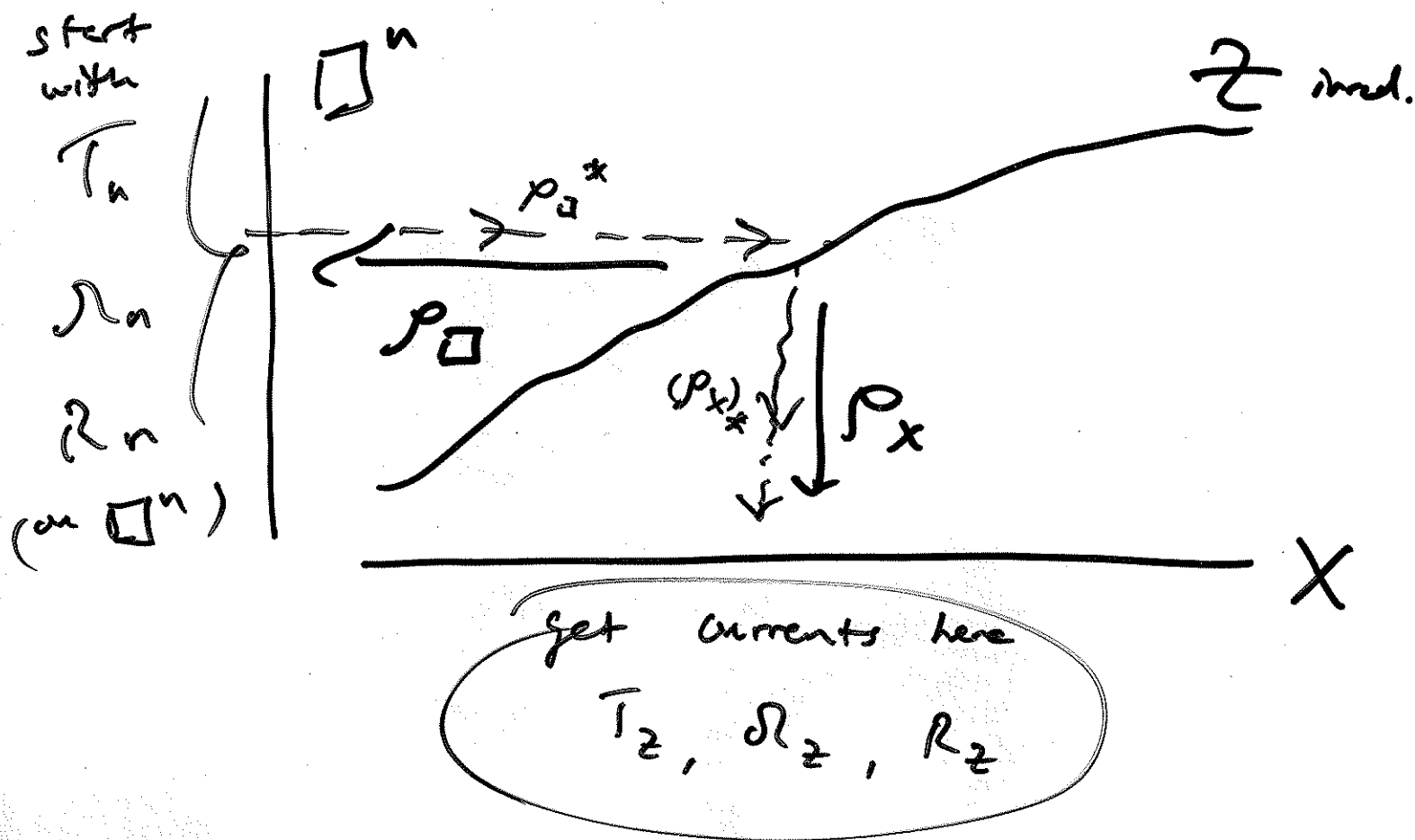
quasi-isomorphic /

$$\mathcal{Z}^p(X, -\cdot)$$

this means the higher cycles must be in good position w.r.t. some real analytic subsets of  $X \times \square^n$

$$\begin{array}{ccc} \mathcal{Z} & \longmapsto & (-2\pi i)^{p-n} \left( (2\pi i)^n T_z, \overset{\text{degrees}}{\underbrace{\Omega_z, R_z}_{2p-n, 2p-n-1}} \right) \\ \text{irred.} & & \downarrow \\ & & R(z) \\ & & \text{+ in good real pos'n.} \end{array}$$

# Approximate idea of " $R(z)$ ":



$$T_z := \rho_x(z \wedge (x \wedge T_n))$$

$$\int_x \left\{ \begin{matrix} \Omega_z \\ R_z \end{matrix} \right\} \wedge \omega := \int_z \rho_{\square}^* \left\{ \begin{matrix} \Omega^n \\ R^n \end{matrix} \right\} \wedge \rho_x^* \omega$$

test form  
on  $x$

of course, must  
show these converge  
in order to have  
currents.

## 3.2 Simplification of KLM

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- Suppose  $n \geq 1$  and  $X$  is projective:

$$\begin{array}{c}
 \textcircled{??} \dashrightarrow [(\hat{T}_z, \hat{\Omega}_z, \hat{R}_z)] \longrightarrow [(\hat{T}_z, \hat{\Omega}_z)] \\
 \text{Currents associated to higher cycle } z \\
 0 \rightarrow J^{p,n}(X) \rightarrow H_D^{2p-n}(X, \mathbb{Q}(p)) \longrightarrow H_g^{p,n}(X) \rightarrow 0 \\
 \parallel \quad \parallel \rightsquigarrow \text{ says } T_z, \Omega_z \text{ are (co)boundaries.} \\
 \underline{H^{2p-n-1}(X, \mathbb{Q})} \\
 \text{stuff}
 \end{array}$$

$$\begin{array}{l}
 \text{So } \textcircled{R_z} := R_z + (2\pi i)^n \Delta_\Gamma + \Xi \\
 \left. \begin{array}{l} T_z = \partial \Gamma \\ \Omega_z = d[\Xi] \end{array} \right\} \\
 \text{is a closed } (2p-n-1)\text{-current representing the} \\
 \text{"pullback" of } \pi^* J(z) \text{ to } J^{p,n}. \quad \leftarrow \text{after mult. by } -(2\pi i)^{p-n}
 \end{array}$$

- Suppose  $n \geq p$  or  $p \geq d$ ; then

$$\Omega_z \in F^p D^{2p-n}(X) = \{0\}$$

vanishes and there is no need for  $\Xi$ .

Hence  $R_z$  "simplifies to"

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$$R_z' := R_z + (2\pi i)^n \int_{\gamma^{-1}} T_z$$

giving a class in

$$J^{p,n}(X) \xrightarrow[\text{to}]{\text{simplify}} H^{2p-n-1}(X, \mathbb{C}/\mathbb{Q}(p))$$

$$\cong \text{Hom}(H_{2n-(2p-1)}(X, \mathbb{Z}), \mathbb{C}/\mathbb{Q}(p))$$

$$\gamma \mapsto \frac{1}{(2\pi i)^{n-p}} \int_{\gamma} R_z$$

These integrals are called "regulator periods".

(no  $\int_{\gamma^{-1}} T_z$  needed)  
since  $\int_{\gamma^{-1}} T_z \cdot \gamma \in \mathbb{Q}$

$$\bullet \text{CH}^p(\mathbb{C}, 2p-1) \xrightarrow{AJ} J^{p,2p-1}(pt.) = H^0(pt., \mathbb{C}/\mathbb{Q}(p))$$

$\downarrow$

$z$

$$z \mapsto \frac{1}{(2\pi i)^{p-1}} \int_z R_{2p-1}$$

That is,  $AJ(z)$  is a

number, related to polylogarithms. this is  $R_z = \rho_x \rho_{\eta}^* R_{2p-1}$



# 4. Examples

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## 4.1 Dilogarithms and $CH^2(\mathbb{C})^3$

really  $\mathbb{Q}(\alpha)$ ,  
 $\alpha = \text{primitive } l^{\text{th}}$   
 $\text{root of unity.}$

Consider  $Z_\alpha := \left(1 - \frac{\alpha}{t}, 1-t, t\right) - \frac{1}{l} \left(1-\alpha, \frac{(z-\alpha)^l}{(z-1)^l}, z\right)$

each component parametrized by  $t \in \mathbb{P}^1$

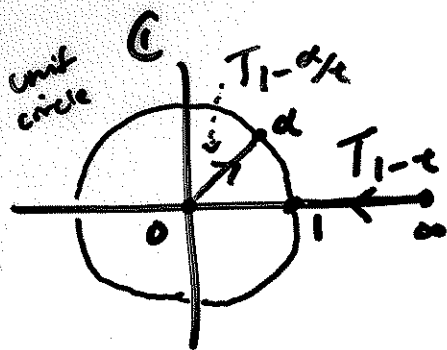
$$-(1-\alpha, \alpha) + \frac{1}{l} \cdot l(1-\alpha, \alpha) = 0.$$

$$AJ(z) = \frac{-1}{2\pi i} \int_{Z_\alpha} R_3 \left[ \begin{aligned} &= \log z_1 d\log z_2 \wedge d\log z_3 + 2\pi i \log z_2 d\log z_3 d\log z_1 \\ &- 4\pi^2 \log z_3 \delta_{T_{z_1} \cap T_{z_2}} \end{aligned} \right]$$

turns out OK  
to ignore 2nd term

$$= \frac{-1}{2\pi i} \int_{\mathbb{P}_t^1} \left( \begin{aligned} &\log\left(1 - \frac{\alpha}{t}\right) d\log(1-t) \wedge d\log t \xrightarrow{\text{on } \mathbb{P}^1} 0 \text{ (by type)} \\ &+ 2\pi i \log(1-t) d\log t \delta_{T_{(1-\frac{\alpha}{t})}} \\ &- 4\pi^2 \log t \delta_{T_{(1-\frac{\alpha}{t})} \cap T_{(1-t)}} \end{aligned} \right)$$

$\emptyset$



$$= - \int_{\mathbb{P}^1} \log(1-t) d\log t \delta_{T_{1-\frac{\alpha}{t}}}$$

$$= - \int_0^\alpha \log(1-t) d\log t = \text{Li}_2(\alpha) = \boxed{\sum_{k \geq 1} \frac{\alpha^k}{k^2}}.$$

So

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$$AJ(z_\alpha - z_{\bar{\alpha}}) = \begin{cases} \sqrt{-3} L(X_{-3}, 2) & l=3 (\alpha = \rho_3) \\ 2i L(X_{-4}, 2) & l=4 (\alpha = i) \\ \vdots & \vdots \end{cases}$$

$0, 1, -1; 0, 1, -1, \dots$   
 $0, 1, 0, -1; 0, 1, 0, -1, \dots$

(Here  $L(X, s) := \sum_{k \geq 1} \frac{X(k)}{k^s}$  for  $\text{Re}(s) > 1$ )

which are related to  $L(H^0(\text{Spec } \mathbb{Q}(\alpha)), 2)$  in line



w./ Beilinson conj.

Note that  $L(X_{-4}, 2) = \sum_{m \geq 0} \frac{(-1)^m}{(2m+1)^2} = \text{Catalan's Constant}$

Another cycle that yields  $4i \cdot 6$  is

$$Z := \frac{1}{4} \left( \tau^4, \underbrace{\frac{1 + \frac{1}{\tau}}{1 - \frac{1}{\tau}}}_{f(\tau)}, \underbrace{\frac{1 - i\tau}{1 + i\tau}}_{g(\tau)} \right);$$

there are similar examples in  $CH^3(\mathbb{Q}, 5)$  and  $CH^4(\mathbb{Q}, 7)$  which are explicit but have multiple terms.

Notice that

$$(f-1)(g-1) + i(f+1)(g+1) = 0.$$

## 4.2 Hypergeometric functions and $CH^2(E_\lambda, \mathbb{Z})$

(23)

Here  $E_\lambda$  is an elliptic curve  $\subseteq \mathbb{P}^1 \times \mathbb{P}^1$ , varying in  $\lambda$ :  
 $\mathbb{P}^1_{x:w} \times \mathbb{P}^1_{y:z} \supseteq E_\lambda := \{ \lambda X Y Z W = (X-W)^2 (Y-Z)^2 \}.$

Set  $x = \frac{x}{w}, y = \frac{y}{z} \in \mathbb{C}(E_\lambda)^*$ .

Notice

$$x = 0 \text{ or } \infty \Rightarrow x \text{ or } w = 0 \Rightarrow y = z$$

$$\Rightarrow y = 1, \text{ and}$$

$$y = 0 \text{ or } \infty \xRightarrow{\text{similarly}} x = 1. \quad (\text{Recall } 1 \notin \square. \text{ So there are no facet } \cap \text{'s.})$$

So their graph over  $E_\lambda$  in  $E_\lambda \times \square^2$  gives a higher Chow cycle:

$$CH^2(E_\lambda, \mathbb{Z}) \xrightarrow{\hookrightarrow} J^{3,2}(E_\lambda) \cong \text{Hom}(H_1(E_\lambda, \mathbb{Z}), \mathbb{C}/\mathbb{Q}(2))$$

$$\mathbb{Z}_\lambda = \Gamma_{\{x,y\}} \longmapsto \{ [\gamma_\lambda] \mapsto \int_{\gamma_\lambda} R_{\mathbb{Z}_\lambda} \},$$

where  $R_{\mathbb{Z}_\lambda} = \log x \, d \log y - 2\pi i \log y \, \delta_{T_\lambda}$

is the pullback of  $R_2$  via  $x$  &  $y$ .

A similar example:

$$E_\lambda := \left\{ \lambda - \overbrace{(x^2+1)(y^2+1)/xy}^{F(x,y)} = 0 \right\}$$

$P_\lambda(x,y)$   
(take zeta function in  $P^1 \times P^1$ )

We want to compute  $\int_{\gamma_\lambda} R_{E_\lambda} =: R(\lambda) \in \mathbb{C}/\mathbb{Q}(2)$

for  $\gamma_\lambda := E_\lambda \cap \{ |x| \leq 1 \} \cap \{ |y| = 1 \}$ . This is

a cycle for  $|\lambda| \geq 4$  since then  $\partial \gamma_\lambda = E_\lambda \cap \partial P$   
 $= E_\lambda \cap \{ |x|=|y|=1 \}$   
 $= \emptyset. \quad =: \pi_2$

With some work, one finds that

$$R(\lambda) \equiv \frac{1}{2\pi i} \int_{\pi_2} (\log P_\lambda) d \log x + d \log y$$

$\log \lambda + \log(1 - \lambda^{-1} F(x,y))$

Cauchy  
residue

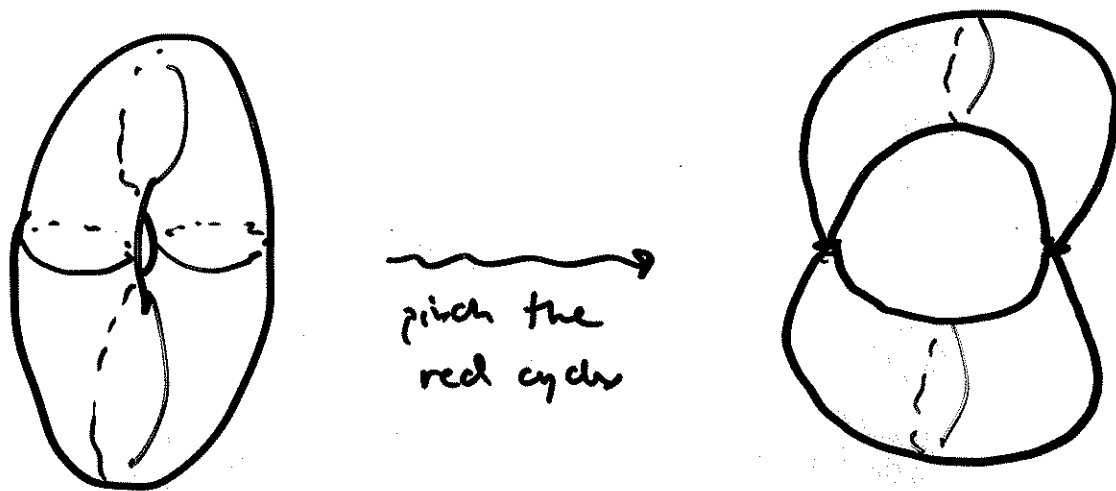
$$\equiv 2\pi i \left( \log \lambda - \sum_{k>0} \frac{\{\text{constant term of } F^k\}}{\lambda^k k} \right)$$

$$= 2\pi i \left( \log \lambda - \sum_{m>0} \frac{\binom{2m}{m}^2}{\lambda^{2m} (2m)} \right).$$

$$\begin{aligned} [F^k]_0 &= [(x+x^{-1})(y+y^{-1})^k]_0 \\ &= \begin{cases} 0 & k \text{ odd} \\ \binom{2m}{m}^2 & k=2m \text{ even} \end{cases} \end{aligned}$$

The last expression is a hypergeometric integral known as a "Meijer G-function" (rarely).

Now if we take  $\lambda \rightarrow 4$ , then  $E_\lambda$  degenerates into 2 rational curves:



$$(x^2+1)(y^2+1) - 4xy =$$

$$\frac{1}{2} \{ (x-1)(y-1) - i(x+1)(y+1) \}$$

$$\rightarrow \{ (x-1)(y-1) + i(x+1)(y+1) \}.$$

Notice this is the same relation as between  $f$  &  $g$  in the parametrization of the  $CH^2(\mathbb{C}, 3)$  example in §4.1.

In fact, the  $CH^2(E, 2)$  elements essentially degenerate to two  $CH^2(\mathbb{C}, 3)$  elements via the  $P' \times \square^2 \leftrightarrow \square^3$  correspondence; one can show

$$2. AJ^{2,3}(\textcircled{2}) \equiv \lim_{\lambda \rightarrow 4} R(\lambda) \text{ in } \mathbb{C}/\mathbb{Q}(2)$$

from 3.4.1

$$2 \cdot 4; 6 \equiv \underset{\substack{\text{mod } \mathbb{Q}(2) \\ \downarrow \cdot \frac{2}{2\pi i}}}{2\pi i} \left( \log 4 - \sum_{m \geq 0} \frac{\binom{2m}{m}^2}{16^m 2m} \right)$$

$$\frac{8}{\pi} 6 \equiv \underset{\text{mod } \mathbb{Q}(1)}{\log(16) - \sum_{m \geq 0} \frac{\binom{2m}{m}^2}{16^m m}}$$

these are real #'s,  
so they are equal!

which is a series identity due essentially to Ramanujan.