Admissible Nilpotent Cones[†]

Matt Kerr (WUSTL)

Stony Brook, August 15, 2017

[†] joint with Pearlstein & Robles

PART I :

Degenerations of Hodge Structure

(I. A) Hodge structures

(finite-dim'l $\mathbb{Q}$-vector space)

A Hodge structure of weight n on V is:

(a)

(b)

(c)

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A Hodge structure of weight n on V is:

(a) a decreasing filtration $F^{\bullet}$ on $V_{\mathbb{C}}$ s.t. $V_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$

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$F^j = \bigoplus_{p \geq j} V^{p,n-p}$

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(b) a decomposition $V_\mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$ w/ $\overline{V^{p,q}} = V^{q,p}$

$V^{p,q} = F^p \cap \overline{F^{n-p}}$

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$\updownarrow$ $V^{p,q} = z^{p-q}$-eigenspace

(c) a homomorphism $\varphi : S^1 \longrightarrow SL(V_{\mathbb{R}})$ s.t. $-1 \longmapsto (-1)^n \cdot \mathrm{id}_V$

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Sub-HS: $U \subset V$ s.t. φ stabilizes $V_{\mathbb{R}}$

$$\Updownarrow$$

$$U_{\mathbb{C}} = \bigoplus U_{\mathbb{C}} \cap V^{p,q}$$

$\Bigg\}$ $h^{p,q} := \dim_{\mathbb{C}} V^{p,q}$

(I. B) Polarization

(I. B) **Polarization** = nondegenerate, $(-1)^n$-symmetric, bilinear form

$$Q : V \times V \longrightarrow \mathbb{Q}$$

satisfying

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(HR I) $Q_{\mathbb{C}}(V^{p,q}, V^{p',q'}) = 0$ unless $p+p' = n = q+q'$

(HR II) $\sqrt{-1}^{\,p-q} Q(v, \bar{v}) > 0 \quad \forall\ 0 \neq v \in V^{p,q}$

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Example / X smooth projective (algebraic) of dimension d, h = hyperplane

$$\Longrightarrow H^n(X) \underset{\cong}{\xrightarrow{\cup h^{d-n}}} H^{2d-n}(X), \quad \text{and}$$

$P^n(X,\mathbb{Q}) := \ker(\cup h^{d-n+1})$ is polarized by $Q(\alpha,\beta) := (-1)^{\binom{n}{2}} \int_X \alpha \wedge \beta \wedge h^{d-n}$. //

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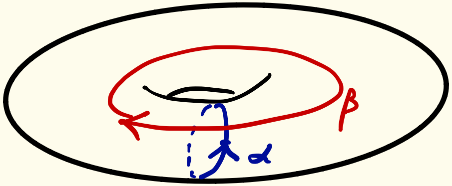
Example / $E = \mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle \rightsquigarrow V = H^1(E,\mathbb{Q}) = \mathbb{Q}\langle \alpha^*, \beta^*\rangle$

$Q(\alpha^*, \beta^*) = 1$

$V^{1,0} = \mathbb{C}\langle dz\rangle = \mathbb{C}\langle \alpha^* + \tau\beta^*\rangle$



Then $0 < \sqrt{-1}\, Q(dz, d\bar{z}) = \dfrac{\tau - \bar{\tau}}{\sqrt{-1}} \iff \tau \in \mathfrak{h}$. //

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$\mathcal{D}_{\underline{h}} := \mathcal{G}_{\mathbb{R}}^{+} \cdot \varphi$ acts by conjugation (or by translation on $V^{p,q}$, $F^{\bullet}$)

$\cap$ connected component (= if n odd)

$\tilde{\mathcal{D}}_{\underline{h}} := \{$decompositions $V = \oplus V^{p,q}$ (s.t. $\dim_{\mathbb{C}} V^{p,q} = h^{p,q}$) on which Q satisfies HR I & II$\}$

$\cap$ analytic open

$\check{\mathcal{D}}_{\underline{h}} := \mathcal{G}_{\mathbb{C}} \cdot \varphi = \{$flags $F^{\bullet}V$ (w/ $\dim_{\mathbb{C}} Gr_F^p = h^{p,n-p}$) satisfying HR I : $Q(F^p, F^{n-p+1}) = 0$ $(\forall p)\}$

flag manifold

Ex/ $\mathfrak{h} \subset \mathbb{P}^1$ //

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- a local system $\mathbb{V} \longrightarrow \mathcal{S}$ with $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}$ (nondegenerate, $(-1)^n$-symmetric)

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- a decreasing filtration of $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_{\mathcal{S}}$ by holomorphic subbundles $\mathcal{F}^p$

  s.t. (a) restricts to PHS on fibers

  (b) $\nabla : \mathcal{V} \to \Omega^1_{\mathcal{S}} \otimes \mathcal{V}$ def'd. by $\nabla \mathbb{V}_{\mathbb{C}} = 0$

  satisfies $\nabla \mathcal{F}^p \subset \Omega^1_{\mathcal{S}} \otimes \mathcal{F}^{p-1}$ (IPR/horizontality/Griffiths transversality)

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Example/ $\mathcal{X} \to \mathcal{S}$ holomorphic family of smooth projective varieties $\rightsquigarrow$

"Z" $:= d_{\mathcal{X}/\mathcal{S}}$-closed sections

$$\begin{tikzcd} Z(\mathcal{S}, F^p A^n_{\mathcal{X}/\mathcal{S}}) \arrow[r, dashed, bend left] \arrow[rrr, dashed, bend left=30, "\nabla"] & Z(\mathcal{X}, F^p A^n_{\mathcal{X}}) \arrow[l] \arrow[r, "d_{\mathcal{X}}"] & Z(\mathcal{X}, F^p A^{n+1}_{\mathcal{X}}) \arrow[r] & Z(\mathcal{S}, \Omega^1_{\mathcal{S}} \otimes F^{p-1} A^n_{\mathcal{X}/\mathcal{S}}) \end{tikzcd}$$

//

A VHS gives rise to a period map $\Phi : S \to \Gamma \backslash \mathcal{D}_{\underline{h}}$, which is locally liftable, holomorphic, and horizontal:

$$\mathrm{im}(d\Phi) \subset \underset{\text{(horizontal distribution)}}{F^{-1}\mathrm{End}_{\mathbb{C}}(V,Q)/F^0} \subset \mathrm{End}_{\mathbb{C}}(V,Q)/F^0 = T\mathcal{D}$$

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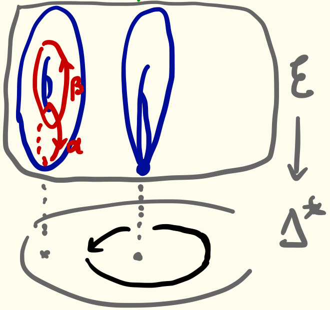
Example/ $\mathbb{V} = H^1(\mathcal{E}_s)$ locally generated by α^*, β^*, with monodromy sending

$$\alpha^* \mapsto \alpha^* - \beta^*, \quad \beta^* \mapsto \beta^*.$$



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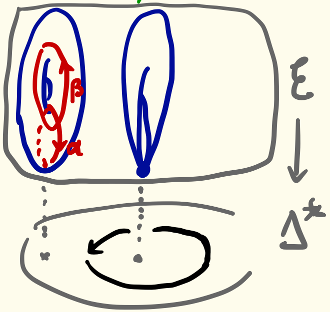

$\mathcal{F}^1_s = \mathcal{O}\langle\omega_s\rangle$ where

$$\underbrace{\omega_s}_{\text{single-valued}} = \underbrace{f(s)}_{\text{holo.}/\Delta}\alpha^* + \Big(\underbrace{g(s)}_{\text{holo.}/\Delta} + \frac{\log(s)}{2\pi i}f(s)\Big)\beta^*$$

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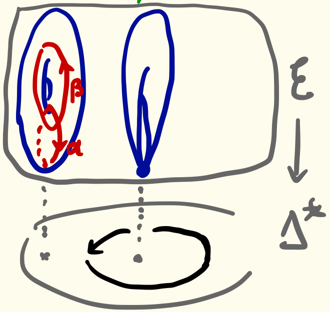

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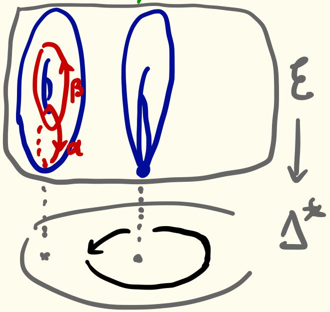

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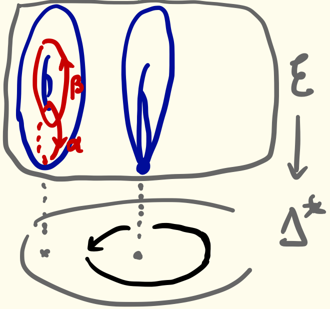

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Monodromy Theorem: Given a VHS $/\Delta^*$, T is quasi-unipotent:

$$(T^M - I)^{m+1} = 0 \text{ for some } M, m \in \mathbb{N} \ (m \le n).$$

$$\Longrightarrow T = \underbrace{T_{ss}\,T_{un}}_{\text{commuting}}, \quad \text{w/ } T_{ss}^M = I .$$

(I.E) Unipotent degenerations

For VHS $/\Delta^*$ with $M = 1$, set

$$N := \log(T) = \log(I - (I - T)) = \sum_{k>0} \frac{(-1)^{k-1}}{k} (T - I)^k \in \mathrm{End}(V, \mathbb{Q}).$$

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Given a basis $\{\gamma_i\} \subset V$, $\tilde{\gamma}_i := e^{\frac{\log(s)}{2\pi i}N}\gamma_i$ is a $\left\{\begin{array}{l}\text{single-valued}\\ \text{holomorphic}\end{array}\right.$ basis of $\underline{\underline{\mathcal{V}}}_s$ yielding the canonical extension $\mathcal{V}_e \to \Delta$.

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\Delta^* \arrow[r, "\Phi"] & \langle T\rangle \backslash D \\
\mathfrak{h} \arrow[u] \arrow[r, "\tilde{\Phi}"] & D \arrow[u]
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SCHMID: Ψ extends across the origin. Set $F_\infty^\bullet := \Psi(0) \in \check{D}$.

This is NOT the coarser "naive limit" $\lim_{s\to 0} \tilde{\Phi}\left(\frac{\log(s)}{2\pi i}\right) =: F_{\lim}^\bullet \in \overline{D}$.

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$\omega_s = \underbrace{f(s)}_{}\tilde{\alpha}^* + \underbrace{g(s)}_{}\tilde{\beta}^*$ ⇝ $F^1_\infty = \mathbb{C}\langle\omega_0\rangle$ //

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$\omega_s = f(s)\tilde{\alpha}^* + g(s)\tilde{\beta}^* \rightsquigarrow F^1_\infty = \mathbb{C}\langle\omega_0\rangle$ //

The associated nilpotent orbit $\Theta(z) := e^{zN}F^\bullet_\infty$ $\left(z = \frac{\log(s)}{2\pi i}\right)$ descends to a VHS (over a possibly smaller disk) approximating Φ.

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$F^1_{nilp,s} =$

$\mathbb{C}\left\langle f(0)\alpha^* + \left(g(0) + f(0)\frac{\log(s)}{2\pi i}\right)\beta^*\right\rangle$ //

(I. F) Limiting mixed Hodge structure (LMHS)

A **MHS** on V consists of :

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$\Big\}$ inducing a HS of weight n on each $Gr_n^W V$.

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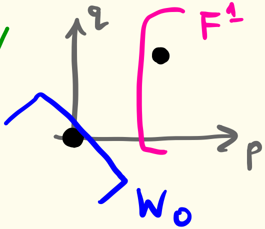
Example/



$$\begin{tikzcd} & & \gamma_1 \arrow[r, mapsto] & (2\pi i)^{-1} & \\ 0 \arrow[r] & \mathbb{Q}(0) \arrow[r] & V \arrow[r] & \mathbb{Q}(-1) \arrow[r] & 0 \\ & 1 \arrow[r, mapsto] & \gamma_0 & & \end{tikzcd}$$

$$F^1 = \mathbb{C}\langle \omega \rangle = \mathbb{C}\langle \gamma_1 + a\gamma_0 \rangle$$

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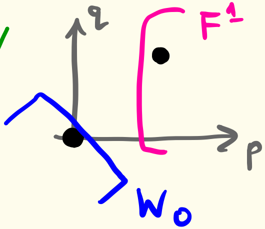

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"extension class": info. beyond the HS on the associated graded //

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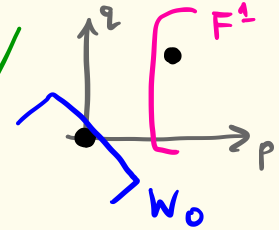

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Where might $W_\bullet$ come from? Given $N \in \mathrm{End}(V)$ nilpotent, $\exists!$ increasing filtration $W(N)_\bullet$ on V s.t.

$$\left.\begin{array}{l} \bullet\ N(W_k) \subset W_{k-2} \\ \bullet\ N^k : Gr^W_k \xrightarrow[\cong]{} Gr^W_{-k} \end{array}\right\} (\forall k).$$

Now for $\mathcal{V} \to \Delta^*$ (of weight n), base-change by $t \mapsto t^M (=s)$ to obtain a unipotent VHS, hence $F_\infty^\bullet$, N, and $W_\bullet := W(N)[-n]_\bullet$.

monodromy weight filtration

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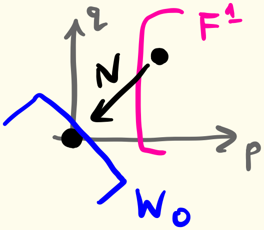

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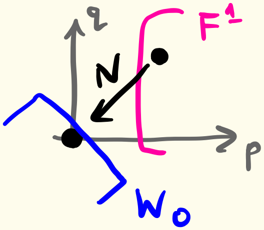

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Corollary: If $V_{\lim}$ = limit of $\Phi : \Delta^* \to \langle T \rangle \backslash \mathcal{D}_h$, then

- $\sum_q i^{p,q} = h^{p,n-p}$ $(\forall p)$ $\Longleftarrow$ Nilpotent orbit thm. (rk Gr_F^i remain constant)
- $i^{p,q} = i^{q,p} = i^{n-q,n-p}$
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These conditions on the Hodge–Deligne #s are also sufficient.

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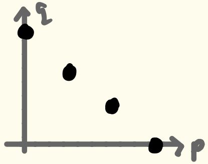
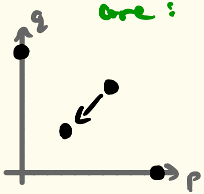
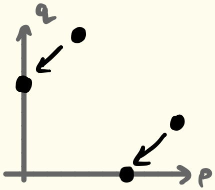
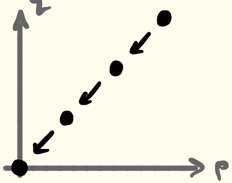
Example / weight 3, $\underline{h} = (1,1,1,1)$

The possible Hodge-Deligne diagrams (w/action of N) are:









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(I.G) Geometric degenerations

In the setting shown,

$N := \log(T_{un})$ acts on

the LMHS $H^n_{\lim}(X_{\circ t})$.
$\quad\nwarrow$ general

$$\begin{tikzcd}
\text{(singular fibers)} & X_0 \arrow[r, equal] \arrow[d, phantom, "\cap"] & X_0 \arrow[d, phantom, "\cap"] & X_0' = \underset{\text{NCD}}{\tilde{X}_0 \cup E} \arrow[l] \arrow[d, phantom, "\cap"] \\
\mathcal{X}^* \arrow[r, phantom, "\subset"] \arrow[d, "\text{sm. proj.}"'] & \mathcal{X} \arrow[d] & \hat{\mathcal{X}} \arrow[l, "\text{base change}"'] \arrow[d] & \mathcal{X}' \arrow[l] \arrow[dl, "\text{SSD}"] \\
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There are two important exact sequences of MHS (interesting mainly for $n=d$):

(i) $(\mathcal{X}, X_t)$-exact sequence $+$ $H^*(\mathcal{X}) \cong H^*(X_0)$ $\Longrightarrow$

$$\to H^n(X_0) \xrightarrow[\rho^*]{} H^n_{\lim}(X_t) \to \underbrace{H^{n+1}(\mathcal{X}, X_t)}_{\text{can be ``localized'' along sing}(X_0)} \to H^{n+1}(X_0) \to$$

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↺

T_{ss} acts on the sequence through automorphisms of MHS (action on $H^*(X_0)$ is trivial)

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(ii) "Generalized Clemens-Schmid"

$$H_{2d-n+2}(X_0)(-d-1) \xrightarrow[\iota^*\iota_*]{} H^n(X_0) \xrightarrow[\rho^*]{} H^n_{inv}(X_t) \longrightarrow 0$$

$\uparrow$ $d = \dim(X_t)$

$$H^n_{inv}(X_t) = \ker(T-I) \subset H^n_{lim}(X_t) \qquad \left(\begin{array}{c} = \ker(N) \\ \Updownarrow \\ T \text{ unipotent} \end{array} \right)$$

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(ii) $\quad H_{2d-n+2}(X_0)(-d-1) \xrightarrow[\iota^*\iota_*]{} H^n(X_0) \xrightarrow[\rho^*]{} H^n_{inv}(X_t) \to 0$

Suppose $\mathrm{sing}(X_0) = \{p_1, \ldots, p_r\}$, with Milnor fibers $Y_j := B_\epsilon(p_j) \cap X_t$.

Then $H^{d+1}(\mathcal{X}, X_t) = \bigoplus_{j=1}^{r} H^d(Y_j)$, and if $B_\epsilon(p_j) \cap X_0$ is defined by f_j (hypersurface singularity) then $\dim H^d(Y_j) = \mu_{f_j} := \dim(\mathcal{O}/J_{f_j})$ [Milnor number], while there is an algorithm to compute eigenvalues of T_{ss} on $Gr^a_F H^d(Y_j)$ [spectrum].

(I.G) Geometric degenerations

d=1, r=1 (one singularity)

$\mu_f = \dim(\mathcal{O}/J_f) = \dim H^1(Y)$

σ_f = "spectrum" of T_{ss} on $Gr_F^{\bullet} H^1(Y)$

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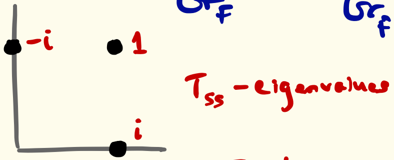
\Delta^* \arrow[r, phantom, "\subset"] & \Delta & \Delta \arrow[l, "t^M \mapsfrom t"'] &
\end{tikzcd}$$

$$0 \to H^1(X_0) \to H^1_{\lim}(X_t) \to H^1(Y) \to 0$$

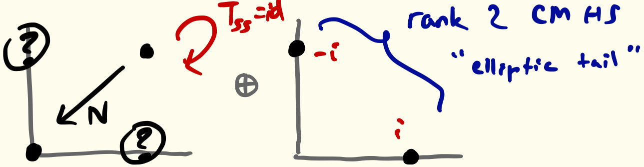
Example/ $\mathrm{Sing}(X_0) = \{p\} = A_3$ curve singularity: $\underbrace{x^2+y^4}_{f}+t=0$

$\Rightarrow \mu_f = 3, \quad \sigma_f = \underbrace{-\frac{1}{4}, 0}_{Gr_F^1}, \underbrace{\frac{1}{4}}_{Gr_F^0}$

$\Rightarrow H^1(Y) =$



$\Rightarrow H^1_{\lim}(X_t) =$



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PART II :

Polarized Relations

(II.A) Classification questions

Let $\mathcal{V} \to \mathcal{S}$ be a VHS of weight n, $\bar{\mathcal{S}} \supset \mathcal{S}$ a good compactification.

(II.A) Classification questions

$$\begin{array}{ccccc}\Delta^r & \supset & (\Delta^*)^r & \xrightarrow{\Phi} & \Gamma\backslash D_h \\ \cap & & \cap & & \text{(assume unipotent)}\end{array}$$

Let $\mathcal{V} \to S$ be a VHS of weight n, $\bar{S} \supset S$ a good compactification.

Q0 Can we "classify" the possible asymptotic behaviors?

(II.A) Classification questions

$$\begin{tikzcd} \Delta \arrow[r, phantom, "\supset"] & \Delta^* \arrow[r, "\Phi"] & \Gamma \backslash D_{\underline{h}} \\ {} \arrow[u, phantom, "\cap"] & {} \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

(assume unipotent)

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Q0 Can we "classify" the possible asymptotic behaviors?

r=1: LMHS $(F_\infty^\bullet, W_\bullet)$ $\rightsquigarrow$ boundary components classify possible $F_\infty^\bullet$ for fixed $W_\bullet$

(II.A) Classification questions

$$\begin{tikzcd} \Delta \arrow[r, phantom, "\supset"] & \Delta^* \arrow[r, "\Phi"] & \Gamma\backslash D_{\underline{h}} \\ \bar{\mathcal{S}} \arrow[u, phantom, "\cap"] \arrow[r, phantom, "\supset"] & \mathcal{S} \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

(assume unipotent)

Let $\mathcal{V} \to \mathcal{S}$ be a VHS of weight n, $\bar{\mathcal{S}} \supset \mathcal{S}$ a good compactification.

Q0 Can we "classify" the possible asymptotic behaviors?

$r=1$: LMHS $(F^\bullet_\infty, W_\bullet)$ $\rightsquigarrow$ boundary components classify possible $F^\bullet_\infty$ for fixed $W_\bullet$

$\uparrow W(N)[-n]_\bullet$

PMHS $(F^\bullet_\infty, N)$: $NF^\bullet \subset F^{\bullet-1}$

$F^\bullet Gr^W_m V$ = weight m HS $(\forall m)$

$Q_\ell(u,v) := Q(u, N^\ell v)$ polarizes each $P_\ell = (Gr^W_{n+\ell} V)_{prim}$

(II.A) Classification questions

$$\begin{tikzcd} s \in \Delta \supset \Delta^* \arrow[r, "\Phi"] & \Gamma\backslash \mathcal{D}_{\underline{h}} \\ \mathcal{H} \arrow[u] \arrow[r, "\tilde{\Phi}"] & \mathcal{D}_{\underline{h}} \arrow[u] \end{tikzcd}$$

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Naive limit $F^\bullet_{\lim} = \lim_{s\to 0} \tilde{\Phi}\left(\frac{\log(s)}{2\pi i}\right) \in \overline{\mathcal{D}}^{an}_{\underline{h}}$.

(II.A) Classification questions

$$\begin{tikzcd}
\Delta^r \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cap"] & (\Delta^*)^r \arrow[r, "\Phi"] \arrow[d, phantom, "\cap"] & \Gamma\backslash D_h \\
\bar{S} \arrow[r, phantom, "\supset"] & S &
\end{tikzcd}$$

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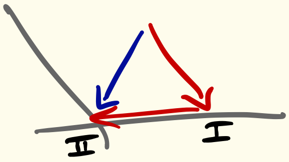

$\underline{r \geq 1}$: can consider "degenerating degenerations"

(in this scenario the red & blue limits coincide)

↖ "LMHS of LMHS"

(II.A) Classification questions

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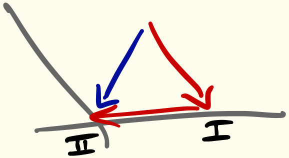

Ⓠ1 (a) What LMHS types arise? ($r=1$ question)

(b) How are they related? ($r=2$ question, about what kinds of pairs I/II can appear)

(c) In what combinations can they appear on all "faces"? ($r \geq 2$)

Motivations: Understand how boundary components fit together, what sorts of multiparameter degenerations of algebraic varieties are possible, etc.

(II.B) Mumford-Tate domains (& a further refinement)

$\tilde{\Phi}$ may factor through a smaller Hodge-theoretic classifying space D:

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given (V, Q, φ) very general in image$(\tilde{\Phi})$, set

$G :=$ $\mathbb{Q}$-algebraic group closure of $\varphi(S^1)$ Mumford-Tate group

$=$ subgroup of $\mathrm{Aut}(V,Q)$ fixing all Hodge classes in all $V^{\otimes k} \otimes (V^*)^{\otimes l}$

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$$= \text{subgroup of } \mathrm{Aut}(V,Q) \text{ fixing all Hodge classes in all } V^{\otimes k} \otimes (V^*)^{\otimes \ell}$$

and

$$\begin{array}{ccl}
\mathcal{D}_{\underline{h}} & \supset & G^+_{\mathbb{R}}.\varphi =: D \cong G^+_{\mathbb{R}} / \underset{\text{compact}}{G^0_{\mathbb{R}}} \\
\cap & & \quad \overset{\text{analytic}}{\underset{\text{open}}{}} \cap \\
\check{\mathcal{D}}_{\underline{h}} & \supset & G_{\mathbb{C}}.F_{\varphi} =: \check{D} \cong G_{\mathbb{C}} / \underset{\text{parabolic}}{P}
\end{array}$$

$\leftarrow$ notation reflects $\mathfrak{g}_{\mathbb{C}} = \oplus\, \mathfrak{g}^{j,-j}$, ($\mathfrak{g}^0 =$ isotropy Lie algebra)

Mumford-Tate domain

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\end{tikzcd}$$

$\underline{r=1}$: $T \in G_{\mathbb{Q}} \Rightarrow N \in \mathfrak{g}_{\mathbb{Q}}$. Define boundary components

$$\tilde{B}(N) := \left\{ F^\bullet \in \check{D} \;\middle|\; e^{zN}F^\bullet \in D \text{ for } \mathrm{Im}(z) \gg 0 \text{ and } NF^\bullet \subset F^{\bullet-1} \right\}$$

$$\cup$$

$$\tilde{B}_{\mathbb{R}}(N) := \left\{ F^\bullet \in \check{D} \;\middle|\; \begin{array}{l} z \mapsto e^{zN}F^\bullet =: \theta(z) \text{ gives an } SL_2(\mathbb{R})\text{-equivariant} \\ \text{embedding } \mathfrak{h} \hookrightarrow D \; (\rho : SL_2(\mathbb{R}) \to G_{\mathbb{R}} \text{ s.t. } \theta(\gamma\langle i \rangle) = \rho(\gamma)\theta(i)) \end{array} \right\}$$

$$= \left\{ F^\bullet \in \check{D} \;\middle|\; (F^\bullet, W(N)[-n]_\bullet) \; \mathbb{R}\text{-split} \right\}$$

parametrizing nilpotent orbits. (For partial compactifications of $\Gamma \backslash D$, really need $\overline{B}(N) = \Gamma_N \backslash \tilde{B}(N)/e^{\mathbb{C}N}$, whose structure can be read off of LMHS on $\mathfrak{g}$ itself.)

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$\overline{B}(N) = \Gamma_N \backslash \tilde{B}(N) / e^{\mathbb{C}N}$

Example / Study of HS w/ additional symmetries goes back to Picard's curves

$y \mapsto \zeta_3 y \circlearrowright C := \{ y^3 = x(x-1)(x-\alpha)(x-\beta) \} \subset \mathbb{P}^2$ (genus 3)

$\Longrightarrow \mathbb{Q}(\zeta_3) =: \mathbb{F} \overset{\mu}{\hookrightarrow} \mathrm{End}(V, \varphi)$, $V_{\mathbb{F}} = \underset{(2,1)}{V_+} \oplus \underset{(1,2)}{V_-}$, $G_{\mathbb{R}} \cong U(2,1)$,

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Example/ (classical) $\underset{(1,20-\rho,1)}{D_\Lambda} \subset \underset{(1,19,1)}{D_{K3}}$; $\mathbb{B}_{10} \subset D_{(1,20,1)}$; $I_{n,n} \subset \mathfrak{h}_{2n\,(\geq 4)}$

w/ cubic auto. ↩ cubic 4-folds

Weil abelian 2n-folds

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(nonclassical) $\underset{\text{HSD}}{D_{FL}} \subset D_{(1,a,a,1)}$; $\underset{\text{3-tensor}}{D_{G_2}} \subset D_{(2,3,2)}$ //

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Ⓠ1′ How do the extra Hodge tensors affect the answers to Q1(a,b,c)?

(II.C) A general framework

(henceforth ignore rational structure!)

Begin with classifying spaces $\check{D} \supset \overline{D}^{an} \supset D$ for HS.

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$$\tilde{B}_{\mathbb{R}}(D) := \coprod_{N \in \mathrm{Nilp}(\mathfrak{g}_{\mathbb{R}})} \tilde{B}_{\mathbb{R}}(N) \qquad \text{“horizontal } SL(2)_s\text{”}$$

$$\begin{tikzcd}
 & \tilde{B}_{\mathbb{R}}(D) \arrow[dl, swap, "\text{factors thru } \overline{D}^{an}"] \arrow[dr] & \\
\check{D} & & \mathrm{Nilp}(\mathfrak{g}_{\mathbb{R}})
\end{tikzcd}$$

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Begin with classifying spaces $\check{D} \supset \overline{D}^{an} \supset D$ for HS.

$$\begin{tikzcd}
 & & (F,N) \arrow[dll, mapsto] \arrow[drr, mapsto] & \in \tilde{B}_{\mathbb{R}}(D) \arrow[dll] \arrow[drr] & & \\
\lim\limits_{y\to\infty} e^{iyN}F^{\bullet} & \in \check{D} & & & N & \in \mathrm{Nilp}(\mathfrak{g}_{\mathbb{R}})
\end{tikzcd}$$

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"horizontal $SL(2)_s$"

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\end{tikzcd}$$

both maps are $G_{\mathbb{R}}^{+}$-equivariant: so take quotients

$$\begin{tikzcd}
& \Psi_D \arrow[dl] \arrow[dr] & \\
\mathcal{O} & & \mathcal{N}
\end{tikzcd}$$

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$$\begin{tikzcd}
 & & \Psi_D \arrow[dll, two heads, "\phi_{\lim}"'] \arrow[drr, "\pi"] \arrow[ddl] \arrow[ddr] & & \\
\mathcal{O}_D \arrow[dr, phantom, "\subset"] & & & & \mathcal{N}_D \\
 & \mathcal{O} & & \mathcal{N} \arrow[ur, phantom, "\supset"] &
\end{tikzcd}$$

"Hodge-theoretically accessible"

(polarizable boundary strata

(D is one element; rest are in ∂D)

polarizable nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$

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Begin with classifying spaces $\check{D} \supset \overline{D}^{an} \supset D$ for HS.

$$\begin{tikzcd}[column sep=small]
 & & (F,N) \arrow[dll] \arrow[dr] & \in & \tilde{B}_{\mathbb{R}}(D) := \coprod_{N \in Nilp(\mathfrak{g}_{\mathbb{R}})} \tilde{B}_{\mathbb{R}}(N) \arrow[dll] \arrow[dr] & \\
\lim_{y \to \infty} e^{iyN} F^{\bullet} & \in & \check{D} & N & \in & Nilp(\mathfrak{g}_{\mathbb{R}})
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"horizontal $SL(2)$s"

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\mathcal{O}_D \arrow[dr, hook] & & & & \mathcal{N}_D \arrow[dl, hook] \\
 & \mathcal{O} & & \mathcal{N} &
\end{tikzcd}$$

polarizable boundary strata

polarizable nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$

These are finite sets, with natural poset structures (Bruhat order) on $\mathcal{O}_D$ $(\mathcal{O}_1 \leq \mathcal{O}_2 \Leftrightarrow \mathcal{O}_2 \subseteq \overline{\mathcal{O}_1})$ and $\mathcal{N}_D$ $(\mathcal{N}_1 \leq \mathcal{N}_2 \Leftrightarrow \mathcal{N}_1 \subseteq \overline{\mathcal{N}_2})$.

(II.C) A general framework

$$\begin{tikzcd} \text{polarizable boundary strata} & (\mathcal{O}_D, \leq) & \overset{G_{\mathbb{R}}^{+}\text{-ccls of horizontal } SL(2)\text{s}}{\Psi_D} \arrow[l, two heads, "\phi_{\lim}"'] \arrow[r, two heads, "\pi"] & (\mathcal{N}_D, \leq) & \text{polarizable nilpotent orbits in } \mathfrak{g}_{\mathbb{R}} \end{tikzcd}$$

Q2 (a) How close are $\phi_{\lim}$, π to being $\underline{\text{bi}}$jective?

(b) Is there a related <u>intrinsic</u> poset structure on Ψ_D?

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ⓆⒶ Q2 (a) How close are $\phi_{\lim}$, π to being bijective?

(b) Is there a related intrinsic poset structure on Ψ_D?

Motivation: Elucidate "combinatorial" structure of polarizable nilpotent cones, i.e. abelian $\sigma = \mathbb{R}_{>0}\langle N_1, \ldots, N_r \rangle \subset \mathrm{Nilp}(\mathfrak{g}_{\mathbb{R}})$ for which $\exists$ σ-nilpotent orbit $\theta : \mathfrak{h}^r \to \check{D}$

$$\underline{z} \longmapsto e^{\sum z_j N_j} F^{\bullet} \quad \text{where} \quad \begin{cases} F^{\bullet} \in \check{D} \\ N_j F^{\bullet} \subset F^{\bullet - 1} \ (\forall j) \\ \theta(\underline{z}) \in D \text{ if } \mathrm{Im}(z_j) >> 0 \ (\forall j) \end{cases}$$

(II.C) A general framework

$$\begin{tikzcd} \text{polarizable boundary strata}\ (\mathcal{O}_D,\le) & \text{$G_{\mathbb{R}}^+$-ccls of horizontal $SL(2)$s}\ \Psi_D \arrow[l, two heads, "\phi_{\lim}"'] \arrow[r, "\pi"] & (\mathcal{N}_D,\le)\ \text{polarizable nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$} \end{tikzcd}$$

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$\Updownarrow$

$\exists$ VHS$/(\Delta^*)^r$ with monodromies $\{e^{N_j}\}$, and LMHS

$V_{\mathbb{C}} = \oplus I^{p,q} \leftsquigarrow$ $(F^\bullet, W_\bullet)$ where $W_\bullet = \underset{\text{independent of choice}}{W(N)[-n]_\bullet}$ ($N \in \sigma$ arbitrary)

(II.C) A general framework

polarizable boundary strata — $G_{\mathbb{R}}^{+}$-ccls of horizontal SL(2)s — polarizable nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$

$$\begin{tikzcd} (\mathcal{O}_D, \leq) & \Psi_D \arrow[l, two heads, "\phi_{\lim}"'] \arrow[r, two heads, "\pi"] & (\mathcal{N}_D, \leq) \end{tikzcd}$$

Write $\delta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$(p, q) \longmapsto i^{p,q} \left(= \dim_{\mathbb{C}} I^{p,q}\right)$

"limiting Hodge diamond"

and let $\Diamond_D :=$ set of all $\{\delta\}$ arising as LMHS of $\Delta^* \to \Gamma \backslash D$.

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$G_{\mathbb{R}}^{+}$-ccls of horizontal SL(2)s Ψ_D

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We have

$$\begin{tikzcd} \Psi_D \arrow[r, two heads] & \mathcal{O}_D \arrow[r, two heads] & \Diamond_D \arrow[r, phantom, "\subset"] & \Diamond_{\mathcal{D}_{\underline{h}}} \\ (F^{\bullet}_{\infty}, N) \arrow[r, mapsto] & F^{\bullet}_{\lim} \arrow[r, mapsto] & \{i^{p,q}\} & \end{tikzcd}$$

$D \subset \mathcal{D}_{\underline{h}}$ by picking a representation V of G

flip $I^{p,q}$s about $p+q=n$

recover $\{i^{p,q}\}$ as $\dim\left(\dfrac{F^p \cap \overline{F^q}}{F^p \cap \overline{F^{q+1}} + F^{p+1} \cap \overline{F^q}}\right)$

(II.D) Period domains $\quad (D = \mathcal{D}_{\underline{h}})$

(II.D) Period domains $(D = \mathcal{D}_{\underline{h}})$

2 connected components if n even

Modify the above construction by starting with $\tilde{\mathcal{B}}_{\mathbb{R}}(\tilde{\mathcal{D}}_{\underline{h}})$, and quotienting by $G_{\mathbb{R}}$ (not $G_{\mathbb{R}}^{+}$). This replaces our earlier diagram by

$$\begin{tikzcd}
 & \overline{\Psi}_D \arrow[dl, two heads] \arrow[dr, two heads, bend left] & \\
\overline{\Theta}_D & & \overline{\eta}_D
\end{tikzcd}.$$

(II.D) Period domains $(\mathcal{D} = \mathcal{D}_{\underline{h}})$

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Write

$$\Diamond_{\underline{h}} := \left\{ \delta \;\middle|\; \begin{array}{l} \delta(p,q) = \delta(q,p) = \delta(n-q, n-p) \\ \sum_q \delta(p,q) = h^{p,n-p} \\ \delta(p-1, q-1) \leq \delta(p,q) \text{ if } p+q \leq n \end{array} \right\}.$$

$\boxed{\text{Theorem 1}}$ $\overline{\Psi}_D = \overline{\Theta}_D = \Diamond_{\mathcal{D}_{\underline{h}}} = \Diamond_{\underline{h}}$. (Answers Q1(a).)

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$$\begin{tikzcd} & \overline{\Psi}_D \arrow[dl, two heads] \arrow[dr, two heads] & \\ \overline{\Theta}_D & & \overline{\eta}_D . \end{tikzcd}$$

Write

$$\Diamond_{\underline{h}} := \left\{ \delta \;\middle|\; \begin{array}{l} \delta(p,q) = \delta(q,p) = \delta(n-q,n-p) \\ \sum_q \delta(p,q) = h^{p,n-p} \\ \delta(p-1,q-1) \leq \delta(p,q) \text{ if } p+q \leq n \end{array} \right\}.$$

Theorem 1 $\overline{\Psi}_D \overbrace{= \overline{\Theta}_D =}^{(a)} \Diamond_{\mathcal{D}_{\underline{h}}} \overbrace{=}^{(b)} \Diamond_{\underline{h}}$. (Answers Q1(a).)

Proof: (a) can be seen by using H^* of products of elliptic curves (some degenerating)

(b) uses the inclusion $\prod_{k\geq 0} \mathrm{Aut}(P_k, Q_k) \subset \mathrm{Aut}(V,Q)$ to get transitive action on MHS with same δ. $\square$

(II.D) Period domains $(D = \mathcal{D}_{\underline{h}})$

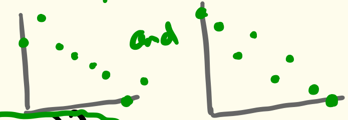
2 connected components if n even

Modify the above construction by starting with $\tilde{B}_{\mathbb{R}}(\tilde{\mathcal{D}}_{\underline{h}})$, and quotienting by $G_{\mathbb{R}}$ (not $G_{\mathbb{R}}^{+}$). This replaces our earlier diagram by

$$\begin{tikzcd} & \overline{\Psi}_D \arrow[dl, two heads] \arrow[dr, two heads] & \\ \overline{\Theta}_D & & \overline{\eta}_D \end{tikzcd}$$

NOT bijective in general: $\underline{h} = (1,\ldots,1)$ wt. 7

same N polarizes LMHS types



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Theorem 1 $\overline{\Psi}_D = \overset{\text{ⓐ}}{\overline{\Theta}_D} = \Diamond_{\mathcal{D}_{\underline{h}}} \overset{\text{ⓑ}}{=} \Diamond_{\underline{h}}$. (Answers Q1(a).)

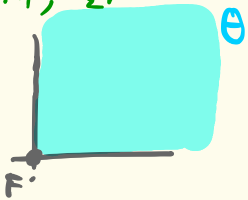
Proof: ⓐ can be seen by using H^* of products of elliptic curves (some degenerating)

ⓑ uses the inclusion $\prod_{k\ge 0} \mathrm{Aut}(P_k, Q_k) \subset \mathrm{Aut}(V,Q)$ to get transitive action on MHS with same δ. □

Suppose $\Theta(z_1, z_2) = e^{z_1 N_1 + z_2 N_2} F^\bullet$ is a σ - nilpotent orbit,

$\sigma = \mathbb{R}_{>0}\langle N_1, N_2\rangle$

with $(F^\bullet, N)$ $\mathbb{R}$-split. Then



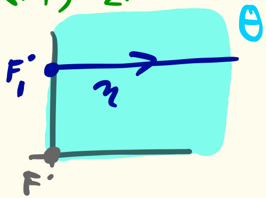
Suppose $\Theta(z_1, z_2) = e^{z_1 N_1 + z_2 N_2} F^\bullet$ is a σ-nilpotent orbit,

$\hookleftarrow \mathbb{R}_{>0}\langle N_1, N_2 \rangle$

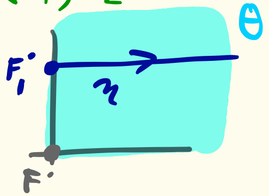
with $(F^\bullet, N)$ $\mathbb{R}$-split. Then

$\swarrow \mathbb{R}_{>0}\langle N_1 \rangle$

- $\eta(z_1) = e^{z_1 N_1}\big(\underbrace{e^{iN_2} F^\bullet}_{=: F_1^\bullet}\big)$ is a τ-nilpotent orbit



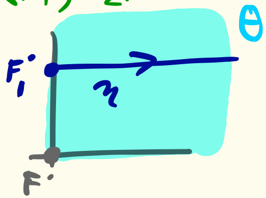
Suppose $\Theta(z_1,z_2) = e^{z_1N_1+z_2N_2}F^\bullet$ is a $\underset{\uparrow\ \mathbb{R}_{>0}\langle N_1, N_2\rangle}{\sigma}$-nilpotent orbit,
with $(F^\bullet, N)$ $\mathbb{R}$-split. Then



- $\eta(z_1) = e^{z_1N_1}(\underbrace{e^{iN_2}F^\bullet}_{=:F_1^\bullet})$ is a $\overset{\mathbb{R}_{>0}\langle N_1\rangle}{\tau}$-nilpotent orbit
- $(F_1^\bullet, \overset{N_1}{W_\bullet^1})$ degenerates to $(F^\bullet, \overset{N=N_1+N_2}{W_\bullet})$ $\Longrightarrow$ write $\delta_{(F,W)} \precneq \delta_{(F_1,W^1)}$.

"is less degenerate than"

Suppose $\Theta(z_1,z_2) = e^{z_1 N_1 + z_2 N_2} F^\bullet$ is a $\underset{\uparrow\ \mathbb{R}_{>0}\langle N_1, N_2\rangle}{\sigma}$-nilpotent orbit, with $(F^\bullet, N)$ $\mathbb{R}$-split. Then



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Letting σ, Θ vary, we obtain a polarized relation "$\precsim$" on $\Diamond_{\underline{h}}$.

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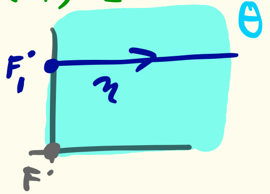


$\mathbb{R}_{>0}\langle N_1\rangle$

- $\eta(z_1) = e^{z_1 N_1}(\underbrace{e^{iN_2}F^\bullet}_{=:F_1^\bullet})$ is a τ-nilpotent orbit

N_1 $\qquad$ $N = N_1 + N_2$

- $(F_1^\bullet, W'_\bullet)$ degenerates to $(F^\bullet, W_\bullet)$ $\Longrightarrow$ write $\delta_{(F,W)} \precnsim \delta_{(F_1,W')}$.

Letting σ, Θ vary, we obtain a polarized relation "$\precnsim$" on $\Diamond_{\underline{h}}$.

Theorem 2 Writing $\delta_\alpha \in \Diamond_{\underline{h}}$ in primitive parts

$$\delta_\alpha(p,q) = \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \underbrace{P_\ell}(p+j, q+j)$$

$\leftrightarrow$ HS of weight $n+\ell$ Hodge #s $\underline{h}(\ell)$

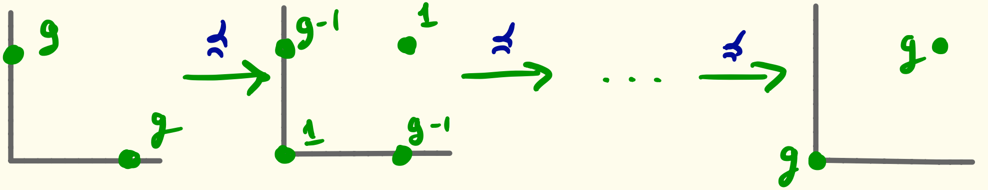
we have

$$\delta_\alpha \precnsim \delta_\beta \iff \delta_\beta = \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \Delta_\ell(p+j, q+j) \text{ with } \Delta_\ell \in \Diamond_{\underline{h}(\ell)} \ (\forall \ell).$$

(Answers Q1(b); proof again "by elliptic curves".)

$$\delta_\alpha \precnsim \delta_\beta \iff \delta_\beta = \sum_{l=0}^{n} \sum_{j=0}^{l} \Delta_l(p+j, q+j) \quad \text{with } \Delta_l \in \Diamond_{\underline{h}(l)} \ (\forall l)$$

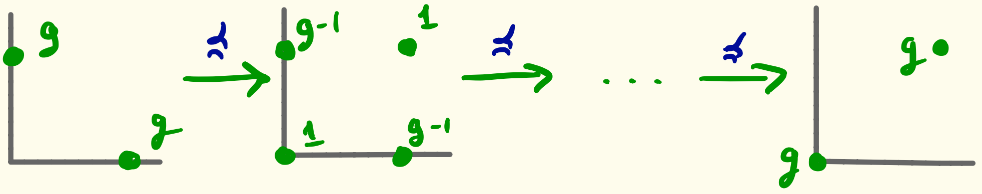
Example ($\underline{h} = (g,g)$) /



linear order //

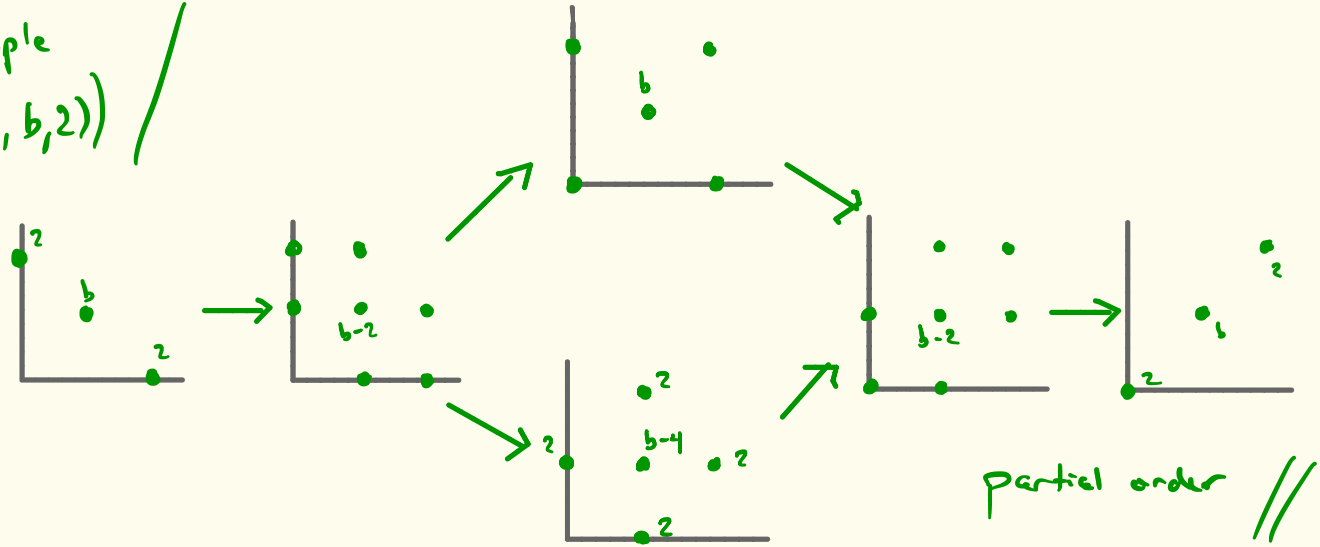
$$\delta_\alpha \precsim \delta_\beta \iff \delta_\beta = \sum_{l=0}^{n} \sum_{j=0}^{l} \Delta_l(p+j, q+j) \quad \text{with} \quad \Delta_l \in \Diamond_{\underline{h}(l)} \quad (\forall l)$$

Example $(\underline{h} = (g,g))$ /



linear order //

Example $(\underline{h} = (2, b, 2))$ /

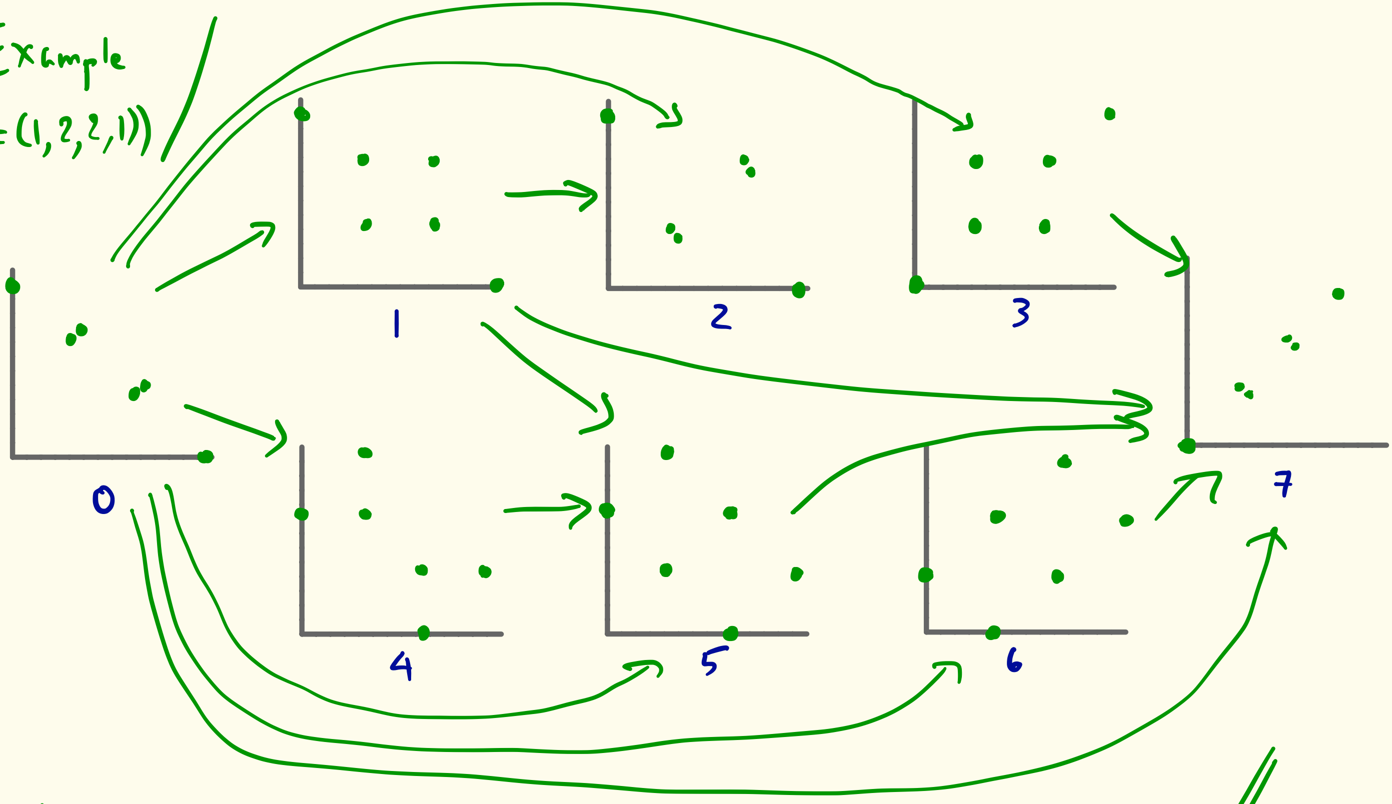


partial order //

$$\delta_\alpha \precsim \delta_\beta \iff \delta_\beta = \sum_{l=0}^{n}\sum_{j=0}^{l} \Delta_l(p+j, q+j) \quad \text{with} \quad \Delta_l \in \Diamond_{\underline{h(l)}} \; (\forall l)$$

Example

$(\underline{h} = (1,2,2,1))$



transitivity fails: $4 \precsim 5 \precsim 7$ but $4 \not\precsim 7$

//

(II.E) General (Mumford-Tate domain) case

Fix $D = G_{\mathbb{R}}^{+}/G_{\mathbb{R}}^{0} \ni \varphi$, $\varphi'(1) = E \in \sqrt{-1}\,\mathfrak{t}_{\mathbb{R}}$

$\rightsquigarrow$ E-grading $\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^{p}$.

type $(p,-p)$

(II.E) General (Mumford-Tate domain) case

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Set $\mathcal{L}(D) := \left\{ \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ Levi} \supset \mathfrak{t}_{\mathbb{R}} \;\middle|\; \overset{\mathcal{Z}}{\overset{/\!/:}{2\pi_{\mathfrak{l}^{ss}}(E)}} \text{ is distinguished in } \mathfrak{l}^{ss}_{\mathbb{C}} \right\}$

Robles: $\mathcal{L}(D)/\underset{\uparrow\, W(\mathfrak{g}^0,\mathfrak{t})}{W_0} \xrightarrow[\cong]{} \Psi_D$ (answers Q1(a))

(II.E) General (Mumford-Tate domain) case

Fix $D = G_{\mathbb{R}}^{+}/G_{\mathbb{R}}^{\circ} \ni \varphi$, $\varphi'(1) = E \in \sqrt{-1}\,\mathfrak{t}_{\mathbb{R}}$

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type $(p,-p)$

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Robles: $\mathcal{L}(D)/W_0 \xrightarrow[\cong]{} \Psi_D$

$[\mathfrak{l}] \longmapsto [(\rho^{-1}F^{\bullet}_{\varphi}, \mathrm{Ad}(\rho^{-1})\mathcal{E})] =: [(F^{\bullet}, N)]$

$\{\mathcal{E}, \mathcal{Z}, \overline{\mathcal{E}}\} \subset \mathfrak{l}^{ss}_{\mathbb{C}}$ $\mathfrak{sl}_2$-triple with $\mathcal{E} \in \mathfrak{g}^{-1}$. Write $\rho := e^{i\frac{\pi}{4}(\mathcal{E}+\overline{\mathcal{E}})}$

Set $\tilde{\mathfrak{l}} := \oplus\, \mathfrak{g}^{p,p}_{(F^{\bullet}, W_{\bullet})}$ $(\supset \mathfrak{l})$

(Levi, well-def'd up to action of W_0)

(III.E) General (Mumford-Tate domain) case

Fix $D = G^+_{\mathbb{R}}/G^0_{\mathbb{R}} \ni \varphi$, $\varphi'(1) = E \in \sqrt{-1}\,\mathfrak{t}_{\mathbb{R}}$

$\rightsquigarrow$ E-grading $\mathfrak{g}_{\mathbb{C}} = \oplus\, \mathfrak{g}^p$.

type $(p,-p)$

Set $\mathcal{L}(D) := \{ \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ Levi} \supset \mathfrak{t}_{\mathbb{R}} \mid \overset{/\!/:\,\mathcal{Z}}{2\pi_{\mathfrak{l}^{ss}}(E)} \text{ is distinguished in } \mathfrak{l}^{ss}_{\mathbb{C}} \}$

Robles: $\mathcal{L}(D)/W_0 \xrightarrow{\cong} \Psi_D$ $\qquad \mathfrak{l} \subset \tilde{\mathfrak{l}} = \bigoplus_p \mathfrak{g}^{p,p}_{(F^\bullet, W_\cdot)}$

$[\mathfrak{l}] \longmapsto [(\rho^{-1}F^\bullet_\varphi, \mathrm{Ad}(\rho^{-1})\mathcal{E})] =: [(F^\bullet, N)]$

Write $[\mathfrak{l}_1] \leq [\mathfrak{l}_2] \underset{\text{def.}}{\iff} \mathfrak{l}_1 \subset w\tilde{\mathfrak{l}}_2$ for some $w \in W_0$.

not a partial order in general

(II.E) General (Mumford-Tate domain) case

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Robles:

$$\begin{tikzcd}
\mathcal{L}(D)/W_0 \arrow[r, "\cong"'] & \Psi_D & \tilde{\mathfrak{l}} = \bigoplus_p \mathfrak{g}^{p,p}_{(F^\bullet, W_\bullet)} \\
{[\mathfrak{l}]} \arrow[r, mapsto] & {[(\rho^{-1}F^\bullet_\varphi, \mathrm{Ad}(\rho^{-1})\mathcal{E})] =: [(F^\bullet, N)]} \arrow[ur, bend right] &
\end{tikzcd}$$

Write $[\mathfrak{l}_1] \leq [\mathfrak{l}_2] \underset{\text{def.}}{\iff} \mathfrak{l}_1 \subset w\tilde{\mathfrak{l}}_2$ for some $w \in W_0$.

Now let Θ be a $\sigma = \mathbb{R}_{>0}\langle N_1, \ldots, N_r \rangle$-nilpotent orbit,

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Applying $\phi_{\lim}$, π to the sub-nilpotent orbits in faces of σ yields morphisms of posets

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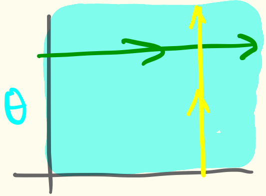

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Set $\mathcal{L}(D) := \{ \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ Levi} \supset \mathfrak{t}_{\mathbb{R}} \mid \underset{\substack{\shortparallel \\ \vdots \\ \mathfrak{z}}}{2\pi_{\mathfrak{l}^{ss}}(E)} \text{ is distinguished in } \mathfrak{l}^{ss}_{\mathbb{C}} \}$

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$\boxed{\text{Theorem 3}}$ (i) $\exists$ p^{Θ}_{Ψ} factoring these maps, such that all $\leq$'s are compatible under π, $\phi_{\lim}$, p^{Θ}_{Ψ}.

(Answers Q2 in general)

(ii) Moreover, $\phi_{\lim}$ is a bijection of sets.

(II. E) General (Mumford-Tate domain) case

Set $\mathcal{L}(D) := \{ \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ Levi} \supset \mathfrak{t}_{\mathbb{R}} \mid \underset{\overset{\shortparallel}{Z}}{2\pi_{\mathfrak{l}^{ss}}(E)} \text{ is distinguished in } \mathfrak{l}_{\mathbb{C}}^{ss} \}$

Robles: $\mathcal{L}(D)/W_0 \xrightarrow[\cong]{} \Psi_D$ $\qquad \tilde{\mathfrak{l}} = \bigoplus_p \mathfrak{g}^{p,p}_{(F^\bullet_{\varphi}, W_\cdot)}$

$[\mathfrak{l}] \longmapsto [(\wp^{-1}F^\bullet_{\wp}, \mathrm{Ad}(\wp^{-1})\mathcal{E})] =: [(F^\bullet, N)]$

$\boxed{\text{Thm. } 3}$ (i) $\exists$ p^Θ_Ψ factoring these maps, such that all $\leq$'s are compatible.

(ii) Moreover, $\phi_{\lim}$ is a bijection of sets.

$$\begin{tikzcd}
 & C_r \arrow[dl, "p^\Theta_\mathcal{O}"'] \arrow[d, "p^\Theta_\Psi"] \arrow[dr, "p^\Theta_\eta"] & \\
\mathcal{O}_D & \Psi_D \arrow[l, "\cong", "\phi_{\lim}"'] \arrow[r, two heads, "\pi"'] & \eta_D
\end{tikzcd}$$

Idea of proof of (ii): If naive limits $\lim\limits_{y\to\infty} e^{iyN_1}F^\bullet_1$ & $\lim\limits_{y\to\infty} e^{iyN_2}F^\bullet_2$ are $G^+_{\mathbb{R}}$-conjugate, so are $\mathfrak{g}_i^{p,q} \Rightarrow \tilde{\mathfrak{l}}_i \Rightarrow \mathfrak{z}_i \Rightarrow N_i^+ - N_i \Rightarrow (N_i, Y, N_i^+) \Rightarrow (\mathcal{E}_i, \mathfrak{z}_i, \mathcal{E}_i^+) \Rightarrow (F_i, N_i)$.

(II.E) General (Mumford-Tate domain) case

Set $\mathcal{L}(D) := \{ \mathfrak{l}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}} \text{ Levi} \supset \mathfrak{t}_{\mathbb{R}} \mid \underset{\substack{\shortparallel \\ \vdots \\ Z}}{2\pi_{\mathfrak{l}^{ss}}(E)} \text{ is distinguished in } \mathfrak{l}^{ss}_{\mathbb{C}} \}$

Robles: $\mathcal{L}(D)/W_0 \xrightarrow[\cong]{} \Psi_D$ $\tilde{\mathfrak{l}} = \bigoplus_p \mathfrak{g}^{p,p}_{(F;W.)}$

$$[\mathfrak{l}] \longmapsto [(g^{-1}F^\bullet_\varphi, \mathrm{Ad}(g^{-1})\mathcal{E})] =: [(F^\bullet, N)]$$

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(ii) Moreover, $\phi_{\lim}$ is a bijection of sets.

$$\begin{tikzcd} & \mathcal{C}_r \arrow[dl, "p^\theta_{\mathcal{O}}"'] \arrow[d, "p^\theta_\psi"] \arrow[dr, "p^\theta_{\mathcal{N}}"] & \\ \mathcal{O}_D & \Psi_D \arrow[l, "\cong"', "\phi_{\lim}"] \arrow[r, two heads, "\pi"'] & \mathcal{N}_D \end{tikzcd}$$

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Remark: As before, we may define polarized relations "$\preccurlyeq$" on $\mathcal{N}_D, \mathcal{O}_D, \Psi_D$ by all p^θ-images of "$\leq$" on $\mathcal{C}_r$ (for all θ).

Though general nilpotent orbits / cones are far from being $SL_2^{\times 2}$-orbits (much more so than in one variable), we have

Theorem 4 All polarized relations are realized & so $(\Psi_D, \precsim)$ is computed by horizontal $SL_2^{\times 2}$'s, which may be classified by applying Robles's identification inductively.

(Answer to Q1(b) in general.)

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Sketch: Given a $\precsim$, have $\sigma = \mathbb{R}_{>0}\langle N_1, N_2 \rangle$, $N = N_1 + N_2 \rightsquigarrow W_\bullet$.

$(F^\bullet, W_\bullet)$ $\mathbb{R}$-split

and

$$\Theta(z_1, z_2) = e^{z_1 N_1 + z_2 N_2} F^\bullet$$

$$\cup$$

$$\Theta(z) = e^{z N_1} (\underbrace{e^{i N_2} F^\bullet}_{=: F_1^\bullet})$$

$\rightsquigarrow \phi_{\lim}(F^\bullet, N)$ & $\phi_{\lim}(F_1^\bullet, N_1)$.

$\boxed{\text{Theorem 4}}$ All polarized relations are realized by horizontal $SL_2^{\times 2}$'s, which may be classified by applying Robles's identification inductively.

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$$\Theta(z_1,z_2) = e^{z_1N_1+z_2N_2}F^\bullet$$
$$\cup$$
$$\Theta(z) = e^{zN_1}(\underbrace{e^{iN_2}F^\bullet}_{=:F_1^\bullet})$$

$\rightsquigarrow \phi_{\lim}(F^\bullet, N)$ & $\phi_{\lim}(F_1^\bullet, N_1)$.

Apply CKS multi-SL_2-splitting: writing $\hat{N}_2 = \pi_{\ker(\mathrm{ad}\,Y_1)}N_2$,

$SL_2^{\times 2}$-orbit

$$\hat{\Theta}(z_1,z_2) := e^{z_1N_1+z_2\hat{N}_2}\boxed{\hat{F}}$$

SL_2-splitting of $(F^\bullet, W_\bullet)$

$$\cup$$

$$\hat{\Theta}_1(z) := e^{zN_1}\boxed{\hat{F}_1} = e^{zN_1}(e^{i\hat{N}_2}\hat{F}^\bullet)$$

SL_2-splitting of $(F_1^\bullet, W^1_\bullet)$

$\boxed{\text{Theorem 4}}$ All polarized relations are realized by horizontal $SL_2^{\times 2}$'s, which may be classified by applying Robles's identification inductively.

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$(F^\bullet, W_\bullet)$ $\mathbb{R}$-split

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$\cup$

$\Theta(z) = e^{zN_1}(\underbrace{e^{iN_2}F^\bullet}_{=:F_1^\bullet})$ $\rightsquigarrow \phi_{\lim}(F^\bullet, N)$ & $\phi_{\lim}(F_1^\bullet, N_1)$.

Apply CKS multi-SL_2-splitting: writing $\hat{N}_2 = \pi_{\ker(\operatorname{ad} Y_1)} N_2$,

$SL_2^{\times 2}$-orbit

$\hat{\Theta}(z_1,z_2) := e^{z_1 N_1 + z_2 \hat{N}_2}\boxed{\hat{F}}$

SL_2-splitting of $(F^\bullet, W_\bullet)$

$\cup$

$\hat{\Theta}_1(z) := e^{zN_1}\boxed{\hat{F}_1} = e^{zN_1}(e^{i\hat{N}_2}\hat{F}^\bullet)$

SL_2-splitting of $(F_1^\bullet, W^1_\bullet)$

By our Mumford-Tate refinement of the CKS SL_2-orbit theorem,

$\hat{\Theta}(iy_1, iy_2) \in G_{\mathbb{R}}^+ \cdot \Theta(iy_1, iy_2)$ and $\hat{\Theta}_1(iy) \in G_{\mathbb{R}}^+ \cdot \Theta_1(iy)$.

Taking limits, $\phi_{\lim}(F, N) = \phi_{\lim}(\hat{F}, \hat{N})$ $(\hat{N} = N_1 + \hat{N}_2)$ and $\phi_{\lim}(F_1, N_1) = \phi_{\lim}(\hat{F}_1, N_1)$ with the "$\preceq$" visibly given by the $\cup$ above.

Since $\phi_{\lim}$ is bijective, we are done. $\square$

PART III :

The secondary poset

(III. A) The "Chemistry" of degenerations

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Theorem 4 says that $(\Psi_D, \precsim)$ can be computed by classifying SL_2- and $SL_2 \times SL_2$-orbits; while by Theorem 3, morphisms

$$(\mathcal{C}_r, \leq) \longrightarrow (\Psi_D, \precsim)$$

determine all candidates for the configurations of interest.

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determine all candidates for the configurations of interest.

This is fascinating given that there are polarizable nilpotent cones σ of dimension larger than the biggest multi-SL_2:

Example/ for $D=$ the F_4 adjoint variety, max rk $\sigma = 7$, while max rk (multi-SL_2) $= 4$. //

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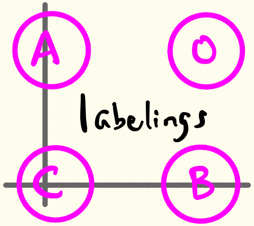
determine all candidates for the configurations of interest.

WAIT! Why aren't we just classifying nilpotent cones/orbits up to $G_{\mathbb{R}}^+$-conjugacy? (Because that would be stupid?)

Example / $D = F_4$-adjoint variety:

The space of $G_{\mathbb{R}}^+$-cc's of polarizable nilp. cones has dim ≥ 7. //

(III.B) 2-cubes for period domains (Degenerations / $(\Delta^*)^2$)



labelings $\longleftrightarrow$ morphisms from $(\mathcal{P}_2, \leq) \to (\Diamond_{\underline{h}}, \preccurlyeq)$ $\longleftrightarrow$ notation $\langle A | C | B \rangle$

power set of $\{1,2\}$

sending $\odot \mapsto 0$

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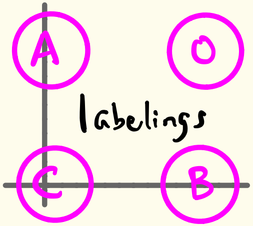

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3 types of 2-cubes:

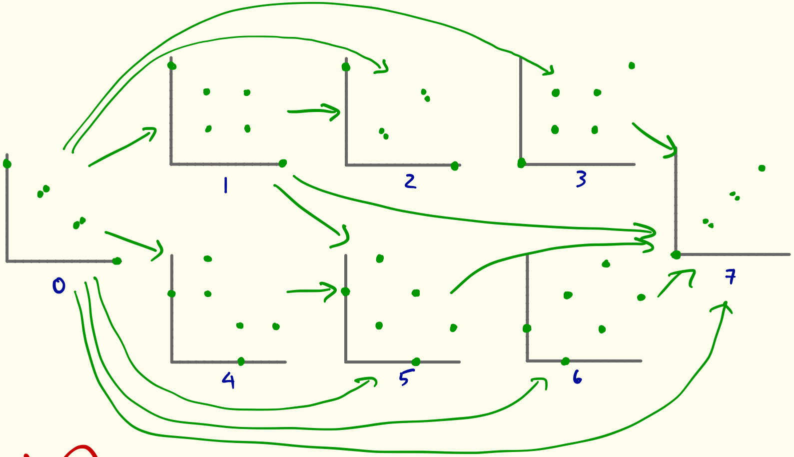
always polarizable

(i) $B = C$ (for any $A \precsim B$)

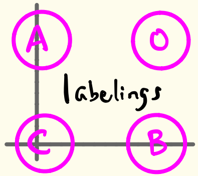
(ii) $A, B \neq C$ but represented by $SL_2 \times SL_2$-orbit (N_1, N_2 extend to commuting sl_2-triples)

(iii) $A, B \neq C$ but NOT represented by $SL_2 \times SL_2$-orbit

(III.B) 2-cubes for period domains (Degenerations / $(\Delta^*)^2$)



$\langle A | C | B \rangle$



$\underline{h} = (1,2,2,1)$ example

3 types of 2-cubes:

always polarizable

(i) $B = C$ (for any $A \precsim B$)

(ii) $A, B \neq C$ but represented by $SL_2 \times SL_2$-orbit

e.g. $\langle 1|5|4 \rangle$, $\langle 1|2|1 \rangle$, $\langle 1|7|3 \rangle$

(iii) $A, B \neq C$ but NOT represented by $SL_2 \times SL_2$-orbit

e.g. $\langle 1|7|6 \rangle$, $\langle 3|7|6 \rangle$ polarizable (geometrically realizable by mirror symmetry)

$\langle 1|7|1 \rangle$, $\langle 3|7|5 \rangle$, $\langle 1|5|1 \rangle$ not polarizable

how to filter out this garbage?

III. C) Admissible n-cubes

filter out bad candidates!

III. C) Admissible n-cubes (general NT domain setting)

Given $\ell \in \mathcal{I}(D)$, with $\tilde{\ell}$ as above. Denoting its roots by

$$R_{\tilde\ell} = \underset{\substack{\cup \\ \text{roots of } \mathcal{I}^{p,p} \\ (p>0)}}{R^+_{\tilde\ell}} \cup R^-_{\tilde\ell} ,$$

we define the capacity

$$\mathrm{Cap}(\ell) := \max_{\left\{\substack{w \in W_{\tilde\ell} \\ w(R^+_{\tilde\ell}) \supset R^+_{\tilde\ell^{0,0}}}\right\}} \left| w(R^+_{\tilde\ell}) \cap R^-_{\tilde\ell} \right| .$$

This gives a function $\mathcal{I}_D \to \mathbb{Z}_{\geq 0}$.

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$\cup$ roots of $\tilde{l}^{\rho,\rho}$ ($\rho > 0$)

$$\mathrm{cap}(l) := \max_{\left\{\substack{w \in W_{\tilde{l}} \\ w(R^+_{\tilde{l}}) \supset R^+_{\tilde{l}^{0,0}}}\right\}} \left| w(R^+_{\tilde{l}}) \cap R^-_{\tilde{l}} \right| .$$

This gives a function $\Psi_D \to \mathbb{Z}_{\geq 0}$.

Definition: An n-cube is a map $\mu : (C_n, \leq) \to (\Psi_D, \precsim)$ with $\mu^{-1}(\{0\}) = \emptyset$.

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- **admissible** if $|I| \leq \mathrm{cap}(\mu(I)) \ \forall I \in \mathcal{C}_n$

(III.C) Admissible n-cubes

(general MT domain setting)

Given $\mathfrak{l} \in \mathcal{I}(D)$, with $\tilde{\mathfrak{l}}$ as above. Denoting its roots by

$$R_{\tilde{\mathfrak{l}}} = \underset{\substack{\cup \\ \text{roots of } \mathfrak{l}^{p,p} \\ (p>0)}}{R^+_{\tilde{\mathfrak{l}}}} \cup R^-_{\tilde{\mathfrak{l}}},$$

we define the capacity

$$\mathrm{cap}(\mathfrak{l}) := \max_{\left\{\substack{w \in W_{\tilde{\mathfrak{l}}} \\ w(R^+_{\tilde{\mathfrak{l}}}) \supset R^+_{\tilde{\mathfrak{l}}^{0,0}}}\right\}} \left| w(R^+_{\tilde{\mathfrak{l}}}) \cap R^-_{\tilde{\mathfrak{l}}} \right|.$$

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Definition: An n-cube is a map $\mu : (\mathcal{C}_n, \leq) \to (\Psi_D, \precsim)$ with $\mu^{-1}(\{0\}) = \emptyset$. It is

{ arrange $\mathfrak{l}$'s so that $\tilde{\mathfrak{l}}_I \subset \tilde{\mathfrak{l}}_{I'}$ if $I \subset I'$ }

- admissible if $|I| \leq \mathrm{cap}(\mu(I))$ $\forall I \in \mathcal{C}_n$
- strongly admissible if $\exists$ independent commuting $\tilde{N}_i \in (\tilde{L}^{0,0}_{\mathbb{R}})^+ . N_i$ s.t. $\mathbb{R}_{>0}\langle\{N_i\}_{i\in I}\rangle \subset (\tilde{L}^{0,0}_{\mathbb{R}})^+ . N_I$

Open orbit in $\mathfrak{l}^{-1,-1}_{i,\mathbb{R}}$ $\leftrightarrow$

(III.C) Admissible n-cubes (general MT domain setting)

Given $\mathfrak{l}\in\mathcal{I}(D)$, with $\tilde{\mathfrak{l}}$ as above. Denoting its roots by

$R_{\tilde{\mathfrak{l}}} = R^+_{\tilde{\mathfrak{l}}} \cup R^-_{\tilde{\mathfrak{l}}}$, we define the **capacity**

$\cup$ roots of $\tilde{\mathfrak{l}}^{p,p}$ $(p>0)$

$$\mathrm{cap}(\mathfrak{l}) := \max_{\left\{\substack{w\in W_{\tilde{\mathfrak{l}}} \\ w(R^+_{\tilde{\mathfrak{l}}})\supset R^+_{\tilde{\mathfrak{l}}^{0,0}}}\right\}} \left|w(R^+_{\tilde{\mathfrak{l}}})\cap R^-_{\tilde{\mathfrak{l}}}\right| .$$

This gives a function $\Psi_D \to \mathbb{Z}_{\geq 0}$.

Definition: An **n-cube** is a map $\mu:(\mathcal{C}_n,\leq)\to(\Psi_D,\precsim)$ with $\mu^{-1}(\{0\})=\emptyset$. It is

Arrange $\mathfrak{l}$'s so that $\tilde{\mathfrak{l}}_I\subset\tilde{\mathfrak{l}}_{I'}$ if $I\subset I'$

- **admissible** if $|I|\leq \mathrm{cap}(\mu(I))\ \forall I\in\mathcal{C}_n$
- **strongly admissible** if $\exists$ independent commuting $\tilde{N}_i\in(\tilde{L}^{0,0}_{\mathbb{R}})^+.N_i$ s.t. $\mathbb{R}_{>0}\langle\{N_i\}_{i\in I}\rangle\subset(\tilde{L}^{0,0}_{\mathbb{R}})^+.N_I$

  $\leftrightarrow$ Open orbit in $\mathfrak{l}^{-1,-1}_{i,\mathbb{R}}$
- **polarizable** if it arises from an actual nilpotent orbit.

$\boxed{\text{Theorem 5}}$ Polarizability $\Longrightarrow$ strong admissibility $\Longrightarrow$ admissibility.

used these to rule
out some 2-cubes
in the example

Theorem 5 Polarizability $\Longrightarrow$ strong admissibility $\Longrightarrow$ admissibility.

The sets of n-cubes with these properties inherit partial orders via inclusions of maps μ, yielding (finite!) secondary posets

$$\widetilde{\Psi}^{adm}_{D} \supseteq \widetilde{\Psi}^{str}_{D} \supseteq \widetilde{\Psi}^{pol}_{D}$$

surjecting onto Ψ_D.

$\boxed{\text{Theorem 5}}$ Polarizability $\Longrightarrow$ strong admissibility $\Longrightarrow$ admissibility.

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$$\tilde{\Psi}_D^{adm} \supseteq \tilde{\Psi}_D^{str} \supseteq \tilde{\Psi}_D^{pol}$$

surjecting onto Ψ_D. If one can compute $\tilde{\Psi}_D^{pol}$, then one has maps of posets

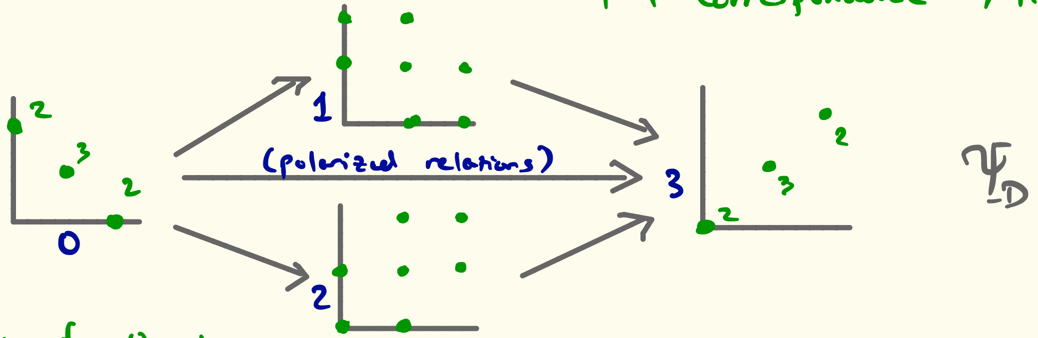
$$\begin{tikzcd}
 & (\tilde{\Psi}_D^{pol}, \leq) \arrow[dl, two heads] \arrow[dr, two heads] & \\
(\Theta_D, \preceq) & & (\eta_D, \preceq)
\end{tikzcd}$$

by first sending $\mu \longmapsto \mu(\{1, \dots, n\}) \in \Psi_D$. This may be regarded as a "full solution" to Q1(c) (insofar as one is possible with finite sets).

Example/ D = MT domain for HS with $\underline{h} = (2,3,2)$ and $G = G_2$.

$\check{D} = G_2/P_2$.

Ψ_D is actually a poset & in 1-1 correspondence w/ Hodge diamonds.



Write $\begin{cases} \mu_0 & \text{for 0-cube} \\ \mu_i & \text{for 1-cubes} \\ \mu_{ij} & \text{for } \langle i\,|3|\,j\rangle \text{ 2-cube} \end{cases}$; then $\tilde{\Psi}^{adm}$ is

$$\begin{tikzcd}
 & & \mu_{11} \\
 & \mu_1 \arrow[ur, red] \arrow[r] \arrow[dr] & \mu_{13} \\
\mu_0 \arrow[ur] \arrow[r] \arrow[dr] & \mu_2 \arrow[ur] \arrow[dr] & \mu_{12} \\
 & \mu_3 \arrow[ur] \arrow[r] \arrow[dr] & \mu_{23} \\
 & & \mu_{22}
\end{tikzcd}$$

and $\tilde{\Psi}^{st} = \tilde{\Psi}^{pol}$ is the stuff in blue. //

(III.D) Collored n-cubes ? (or n-mollecubes?)

There are ways to refine the secondary poset while still keeping things finite.

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There are ways to refine the secondary poset while still keeping things finite. For the simplest such refinement:

- Given $\mu \in \tilde{\Psi}_D$, choose a system of representatives $\ell_{\mathcal{I}} \in \mathcal{I}(D)$ with $[\ell_{\mathcal{I}}] = \mu(\mathcal{I}) \in \Psi_D$ and $\mathcal{I} \subseteq \mathcal{J} \Rightarrow \tilde{\ell}_{\mathcal{I}} \leq \tilde{\ell}_{\mathcal{J}}$.

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- Write l, N, F for the data associated to $J = \{1, \dots, n\}$, and restrict the bigrading associated to (F, N) to all $l_I, \tilde{l}_I$.

(III.D) Collored n-cubes ? (or n-mollecubes?)

There are ways to refine the secondary poset while still keeping things finite. For the simplest such refinement:

- Given $\mu \in \tilde{\Psi}_D$, choose a system of representatives $\ell_I \in \mathcal{L}(D)$ with $[\ell_I] = \mu(I) \in \Psi_D$ and $I \subseteq J \Rightarrow \tilde{\ell}_I \subseteq \tilde{\ell}_J$.
- Write ℓ, N, F for the data associated to $J = \{1, \ldots, n\}$, and restrict the bigrading associated to (F, N) to all ℓ_I, $\tilde{\ell}_I$.
- Write $Y \in \mathfrak{t}$ for the element giving this bigrading, and $(\forall i = 1, \ldots, n)$ N_i for a nilpotent belonging to an open orbit of $L^{0,0}_{\{i\}}(\mathbb{R})^+$ on $\ell^{-1,-1}_{\{i\}}$.

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- Write ℓ, N, F for the data associated to $J = \{1, \ldots, n\}$, and restrict the bigrading associated to (F, N) to all $\ell_I, \tilde{\ell}_I$.
- Write $Y \in \mathfrak{t}$ for the element giving this bigrading, and $(\forall i = 1, \ldots, n)$ N_i for a nilpotent belonging to an open orbit of $L^{0,0}_{\{i\}}(\mathbb{R})^+$ on $\ell^{-1,-1}_{\{i\}}$.
- Take $\mathfrak{h} \subset \tilde{\mathfrak{l}}$ any semisimple subalgebra containing $Y, N_1, \ldots, N_n$, and $\underline{h}$ the (palindromic) list of numbers $h^i := \dim_{\mathbb{C}}(\mathfrak{h} \cap \tilde{\mathfrak{l}}^{i,i})$.

Denoting by $\underline{\mathcal{H}}$ the set of possible $\underline{h}$'s, we define

$\hat{\Psi}_D :=$ pairs $(\mu, \underline{h}) \in \tilde{\Psi}_D \times \underline{\mathcal{H}}$ which arise as above.

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$$\hat{\Phi}_D := \text{pairs } (\mu, \underline{h}) \in \tilde{\Phi}_D \times \underline{\mathcal{H}} \text{ which arise as above.}$$

The "polarizable" object is

$$\hat{\Phi}_D^{pol} := \text{pairs } (\mu, \underline{h}) \in \tilde{\Phi}_D^{pol} \times \underline{\mathcal{H}} \text{ where } h \text{ is the}$$

Looijenga - Lunts algebra of some σ-nilpotent orbit with class μ.

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What is this? Given a polarizable nilpotent cone σ, and $F_0^\bullet \in \tilde{B}(\sigma)^\alpha$ a $\mathbb{Q}$-split CM base point, $e^{\sum z_i N_i} F_0^\bullet$ descends to a VHS Φ_0 on $(\Delta^*)^n$, and the LL group is simply the <u>derived Mumford-Tate group of Φ_0</u>.

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Looijenga - Lunts algebra of some σ-nilpotent orbit with class μ.

$$\begin{tikzcd}
\hat{\Psi}_D \arrow[r, phantom, "\supset"] \arrow[d, "\rho"] & \hat{\Psi}_D^{pol} \arrow[d, "\rho_{pol}"] \\
\tilde{\Psi}_D \arrow[r, phantom, "\supset"] & \tilde{\Psi}_D^{pol}
\end{tikzcd}$$

We say that $\underline{h}$ is Hermitian if only h^{-1}, h^{0}, and h^{1} are nonzero.

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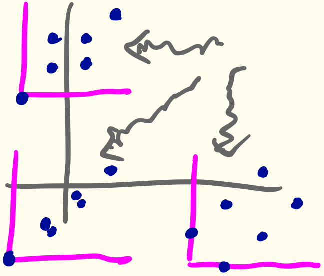
Proposition: If $\rho^{-1}(\mu)$ contains no Hermitian $\underline{h}$'s, then no period map $\Phi : (\Delta^*)^n \longrightarrow \Gamma \backslash D$ of class μ factors through a Shimura variety.

Example/ VHS with $\underline{h} = (1,2,2,1)$ over $(\Delta^*)^2$, with

$\mu = \langle 3 \,|\, 7 \,|\, 6 \rangle$:



(This was not just polarizable, but geometric!) Such VHS cannot even locally factor through a Shimura variety. //

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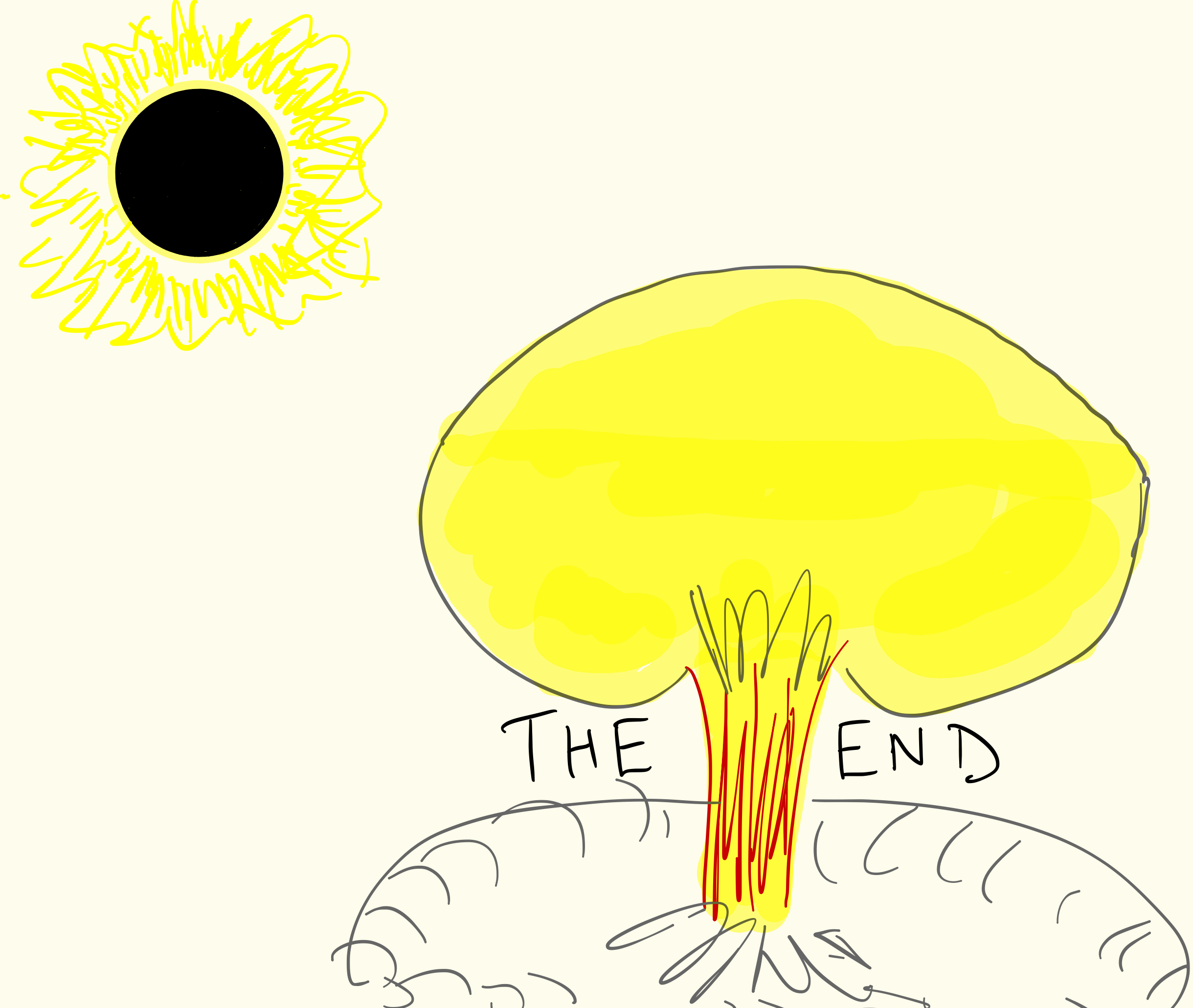
THE END