MUMFORD-TATE DOMAINS

MARK GREEN, PHILLIP GRIFFITHS, AND MATT KERR

For Aldo Andreotti, a mathematician of impeccable taste whose work added further luster to the extraordinary Italian tradition in geometry.

Outline

Introduction

Mumford-Tate groups have emerged as the principal symmetry groups in Hodge theory. Their significance is both geometric and arithmetic. Geometrically, this is due in part to their relation with monodromy and Noether-Lefschetz loci. Arithmetically, they arise in the study of the endomorphism algebra of a Hodge structure and of the fields of definition associated to Noether-Lefschetz loci. The theory of Mumford-Tate groups is relatively much more highly developed in the classical case of weight one Hodge structures.

A *Mumford-Tate domain* D_M is, by definition, the orbit under a Mumford-Tate group M of a point in the period domain D classifying polarized Hodge structures with given Hodge numbers. In the classical weight one case, the quotient of a Mumford-Tate domain by an arithmetic group are the complex points of a *Shimura variety*. These have been the object of extensive and deep study over the years. In contrast, in the non-classical case the study of Mumford-Tate domains is relatively unexplored. In this largely expository paper we will describe some of the main properties of Mumford-Tate domains, with special emphasis on those aspects that are different from the classical case. Very roughly speaking, we may say that many of the geometric aspects of Shimura domains carry over, in some ways in a richer form, to general Mumford-Tate domains. But the arithmetic theory is much less developed, perhaps due at least in part to the absence thus far of any connection between automorphic representations in $L_2(M(\mathbb{Q}) \setminus M(\mathbb{A}))$ and variations of Hodge structure in quotients of Mumford-Tate domains by arithmetic groups.

As mentioned, this paper is largely expository and is intended to be an introduction to and overview of some aspects of Mumford-Tate domains. Very few complete proofs of stated results will be given. For these we refer to the monograph [GGK] as well as to specific references cited in the paper. As general references to the theory of Mumford-Tate groups we suggest the original paper [Mu], [DMOS], the very useful notes [Mo], and the relevant section in the recent book [PS].

We conclude this brief introduction with the

Convention. Throughout this paper, unless mentioned otherwise, Hodge structures will always refer to ones that are *polarized*. Some of the aspects of Mumford-Tate groups are present for general Hodge structures, but their deeper properties seem to require a polarization.

I. Definition of Mumford-Tate domains

I.A. **Hodge structures.** Throughout this paper, V will denote a $\mathbb{Q}$-vector space and

$$Q : V \otimes V \to \mathbb{Q}$$

a non-degenerate bilinear form satisfying

$$Q(u,v) = (-1)^n Q(v,u) \qquad u,v \in V$$

where n will be the weight of the Hodge structure. We denote by

$$\begin{cases} T^{a,b} = (\otimes^a \check{V}) \otimes (\otimes^b V) \\ T^{\bullet,\bullet} = \bigoplus\limits_{a,b} T^{a,b} \end{cases}$$

the tensor algebra associated to V where $\check{V}$ is the dual of V. Thus $Q \in T^{1,1}$. We also set $V_{\mathbb{R}} = V \otimes \mathbb{R}$ and $V_{\mathbb{C}} = V \otimes \mathbb{C}$. We will use the terminology of algebraic groups [Bo]; unless otherwise noted they will be defined over $\mathbb{Q}$. We set

$$G = \mathrm{Aut}(V, Q)$$

and denote by $G(\mathbb{R})$ and $G(\mathbb{C})$ its real and complex points.

In keeping with general conventions, we set

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}}$$

where "Res" denotes the restriction of scalars à la Weil. This is an algebraic group defined over $\mathbb{R}$. For $k = \mathbb{R}$ or $\mathbb{C}$ we have

$$\begin{aligned} \mathbb{S}(k) &= \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \begin{array}{c} a^2 + b^2 \neq 0 \\ a, b \in k \end{array} \right\} \\ \mathbb{U}(k) &= \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \begin{array}{c} a^2 + b^2 = 1 \\ a, b \in k \end{array} \right\}. \end{aligned}$$

Then

$$\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^* \text{ via } \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \to a + ib,$$

and $\mathbb{U}(\mathbb{R}) \cong S^1$ is a maximal compact subgroup of $\mathbb{S}(\mathbb{R})$.

Setting $t = a + ib$, a representation

$$\varphi : \mathbb{U}(\mathbb{R}) \to G(\mathbb{R}) \tag{I.A.1}$$

decomposes over $\mathbb{C}$ into eigenspaces $V^{p,q}$ such that

$$\left\{ \begin{array}{ll} \varphi(t)u = t^p \bar{t}^q u & \qquad u \in V^{p,q} \\ V^{p,q} = \overline{V}^{q,p}. & \end{array} \right.$$

The *Weil operator* associated to (I.A.1) is

$$C = \varphi(i),$$

so that $C = i^{p-q}$ on $V^{p,q}$. We denote by $S^1_\varphi \subset G(\mathbb{R})$ the circle given by the image $\varphi(\mathbb{U}(\mathbb{R}))$.

Definition. A *Hodge structure* of weight n is given by a representation (I.A.1) where all non-zero eigenspaces have $p + q = n$ and where the Hodge-Riemann bilinear relations

$$\left\{ \begin{array}{lrl} \text{(I)} & Q(V^{p,q}, V^{p',q'}) = 0 & \quad p' \neq n - p \\ \text{(II)} & Q(u, C\bar{u}) > 0 & \quad 0 \neq u \in V_{\mathbb{C}} \end{array} \right.$$

are satisfied.

We will denote a Hodge structure by (V, Q, φ), or sometimes simply by V_φ, and we set $V_{\varphi,\mathbb{C}} = \bigoplus_p V^{p,q}_\varphi$.

Remark. In general, a not-necessarily-polarized $\mathbb{R}$-split mixed Hodge structure is given by V as above together with a representation

$$\mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V_{\mathbb{R}}).$$

Then $V_{\mathbb{R}}$ decomposes into a direct sum of weight spaces $V_{\mathbb{R},n}$ on which $t \in \mathbb{R}^* \subset \mathbb{C}^*$ acts by t^n, and then under the action of S^1, $V_{\mathbb{C},n}$ decomposes as above into a direct sum of $V_{\mathbb{C},n}^{p,q}$'s, $p+q=n$.

A Hodge structure induces Hodge structures $T_{\varphi}^{a,b}$ on the tensor spaces $T^{a,b}$.

Definition. The *Hodge tensors* $\mathrm{Hg}_{\varphi}^{a,b}$ are given by the subspace of $T^{a,b}$ on which $\mathbb{U}(\mathbb{R})$ acts trivially.

Since $T_{\varphi}^{a,b}$ is a Hodge structure of weight $n(a-b)$, for $\mathrm{Hg}_{\varphi}^{a,b}$ to be non-zero we must have $n(a-b)=2m$ and then $\mathrm{Hg}_{\varphi}^{a,b}$ are the rational tensors of Hodge type (m,m).

We will denote by $\mathrm{Hg}_{\varphi}^{\bullet,\bullet} = \underset{a,b}{\oplus} \mathrm{Hg}_{\varphi}^{a,b}$ the *algebra of Hodge tensors.*

Definition. A *sub-Hodge structure* of (V,Q,φ) is given by a linear subspace $V' \subset V$ such that $V'_{\mathbb{R}}$ is invariant under $\varphi(\mathbb{U}(\mathbb{R}))$.

It follows that $Q' := Q \mid_{V}$ is non-singular and polarizes the (p,q)-decomposition of $V'_{\mathbb{C}}$ given by the eigenspaces of $\varphi(\mathbb{U}(\mathbb{R}))$ acting on $V'_{\mathbb{C}}$. Moreover, the Q-orthogonal complement ${V'}^{\perp} := V''$ is again a sub-Hodge structure and $V = V' \oplus V''$. Briefly, the category of Hodge structures is semi-simple.

I.B. **Period domains and their compact duals** (cf. [C-MS-P] and [CGG])**.** Let (V,Q) be as in section I.A above and $h^{p,q}=h^{q,p}$, $p+q=n$ with $\Sigma h^{p,q} = \dim V$, a set of *Hodge numbers.*

Definition. The *period domain* D associated to the above data is the set of polarized Hodge structures $\varphi : \mathbb{U}(\mathbb{R}) \to G(\mathbb{R})$ with the given Hodge numbers.

The real Lie group $G(\mathbb{R})$ acts transitively on D by conjugation, and because for $\varphi \in D$ the polarizing forms are definite on the $V_{\varphi}^{p,q}$, the isotropy group H_{φ} of φ is a compact subgroup of $G(\mathbb{R})$. It is clear that $S_{\varphi}^1 \subset H_{\varphi}$, and that

$$H_{\varphi} = Z(S_{\varphi}^1) \textit{ is the centralizer of } S_{\varphi}^1 \textit{ in } G(\mathbb{R}).$$

Fixing a reference point $\varphi_0 \in D$, there is an identification

$$D = G(\mathbb{R})/H_{\varphi_0}.$$

This gives D the structure of a manifold. As will be seen shortly, it is in fact a homogeneous complex manifold. A useful equivalent set-theoretic identification is

$$D = \left\{ \begin{array}{c} \text{set of conjugacy classes} \\ g^{-1}H_{\varphi_0}g \text{ of } H_{\varphi_0} \text{ in } G(\mathbb{R}) \end{array} \right\}.$$

The Hodge structure associated to the conjugacy class $g^{-1}H_{\varphi_0}g$ is given by $\varphi = g^{-1} \circ \varphi_0 \circ g : \mathbb{U}(\mathbb{R}) \to G(\mathbb{R})$.

The Lie algebra $\mathfrak{g}$ of the simple algebraic group G is a $\mathbb{Q}$-linear subspace of $\mathrm{Hom}(V, V)$, and the form Q induces on $\mathfrak{g}$ a non-degenerate, symmetric bilinear form

$$B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C},$$

which is (up to scale) just the Cartan-Killing form. For each point $\varphi \in D$

$$\mathrm{Ad}\,\varphi : \mathbb{U}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$$

induces a Hodge structure of weight zero on $\mathfrak{g}$. This Hodge structure is polarized by B, and it is a sub-Hodge structure of $\check{V} \otimes V$. We have the description

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \oplus \mathfrak{g}_{\varphi}^{-i,i} \\ \mathfrak{g}_{\varphi}^{-i,i} &= \left\{ \begin{array}{l} X \in \mathfrak{g}_{\mathbb{C}} \text{ satisfying} \\ X : V_{\varphi}^{p,q} \to V_{\varphi}^{p-i,q+i} \end{array} \right\}. \end{aligned}$$

The complexified Lie algebra of the isotropy group is

$$\mathfrak{h}_{\varphi,\mathbb{C}} = \mathfrak{g}_{\varphi}^{0,0}.$$

We note that

(I.B.1) $$\left[\mathfrak{g}_{\varphi}^{-i,i}, \mathfrak{g}_{\varphi}^{-j,j}\right] \subseteq \mathfrak{g}_{\varphi}^{-(i+j),i+j}.$$

Setting

$$\mathfrak{g}_{\varphi}^{-} = \bigoplus_{i>0} \mathfrak{g}_{\varphi}^{-i,i}$$

we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\varphi}^{-} \oplus \overline{\mathfrak{g}_{\varphi}^{-}} \oplus \mathfrak{h}_{\varphi,\mathbb{C}}.$$

The complexification of the real tangent space $T_{\varphi,\mathbb{R}}(D)$ to D at φ is

$$T_{\varphi,\mathbb{R}}(D) \otimes \mathbb{C} \cong \mathfrak{g}_{\varphi}^{-} \oplus \overline{\mathfrak{g}_{\varphi}^{-}},$$

and setting

$$T_{\varphi}D = \mathfrak{g}_{\varphi}^{-}$$

defines a $G(\mathbb{R})$-invariant almost complex structure. By (I.B.1) this almost complex structure is integrable, verifying that D is a homogeneous complex manifold.

We are denoting by $T_\varphi D$ the usual holomorphic or $(1,0)$ tangent space at φ to the complex manifold D, and by TD the holomorphic tangent bundle.

To define the compact dual $\check{D}$ of D we need to give the

Alternate definition of a Hodge structure. This is given by a Hodge filtration $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ satisfying

$$\begin{cases} Q(F^p, F^{n-p+1}) = 0 & \\ \qquad Q(u, C\bar{u}) > 0 & \qquad 0 \neq u \in V_{\mathbb{C}}. \end{cases}$$

Here we set

$$V^{p,q} = F^p \cap \overline{F}^q$$

and define C to be i^{p-q} on $V^{p,q}$. The relation to the previous definition is given by defining the subspaces F^p_φ in the Hodge filtration associated to $\varphi \subset D$ by

$$F^p_\varphi = \bigoplus_{p' \geqq p} V^{p',q'}_\varphi.$$

We set

$$f^p = \sum_{p' \geqq p} h^{p',q'}.$$

Definition. The *compact dual* $\check{D}$ of D is defined to be the set of flags $F^\bullet = \{F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}\}$ where $\dim F^p = f^p$ and where the first Hodge-Riemann bilinear relation

$$Q(F^p, F^{n-p+1}) = 0$$

is satisfied.

We note that by the non-degeneracy of Q

$$F^p = (F^{n-p+1})^\perp.$$

The complex group $G(\mathbb{C})$ acts transitively on $\check{D}$ with isotropy group $P_{F^\bullet}$ a parabolic subgroup of $G(\mathbb{C})$. The identification

$$\check{D} = G(\mathbb{C})/P_{F^\bullet}$$

gives $\check{D}$ the structure of a homogeneous, rational projective algebraic variety. The above $\mathbb{Q}$-bilinear relations and the inclusions

$$\check{D} \hookrightarrow \prod_p \operatorname{Grass}(f^p, V_{\mathbb{C}}),$$

together with the standard Plücker embedding of the Grassmanians, show that $\check{D}$ *is defined over* $\mathbb{Q}$. Thus, we may talk about the field of definition of a point $F^\bullet \in \check{D}$.

For a point $\varphi \in D$ we denote by $F^\bullet_\varphi$ the corresponding point in $\check{D}$. Since $G(\mathbb{R})$ acts transitively on D, it follows that D *is the open* $G(\mathbb{R})$*-orbit of a point* $F^\bullet_\varphi \in D$. This provides an alternate description of the complex structure on D.

We denote by $\check{D}^0 \subset \check{D}$ the (real-Zariski-) open set of all filtrations $F^\bullet$ that define Hodge structures; i.e. that satisfy the open conditions

$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$$

for all p. Filtrations in some of the topological components of $\check{D}^0$ outside of D define Hodge structures that satisfy Hodge-Riemann I and where the Hermitian forms in Hodge-Riemann II are non-singular but indefinite. These components are orbits of $G(\mathbb{R})$ where the isotropy groups are non-compact.

Among the interesting points in $\check{D}$ are those corresponding to the limiting Hodge filtrations that arise when a family of Hodge structures arising from a variation of Hodge structure degenerates.

I.C. **Mumford-Tate groups.**

Definition. Given a Hodge structure (V, Q, φ), the *Mumford-Tate group* M_φ is the smallest $\mathbb{Q}$-algebraic subgroup of G with the property that

$$\varphi(\mathbb{U}(\mathbb{R})) \subset M_\varphi(\mathbb{R}).$$

In other words, M_φ is the intersection of all $\mathbb{Q}$-algebraic subgroups $M' \subset G$ such that $M'(\mathbb{R})$ contains the circle S^1_φ. The following is the *basic property* of Mumford-Tate groups (cf. [Mu], [DMOS], [Mo]):

(I.C.1) *M_φ is the subgroup of G that fixes pointwise the algebra of Hodge tensors.*

We will deduce (I.C.1) from the following more general result:

(I.C.2) *M_φ is the subgroup of G with the property that the M_φ-stable subspaces $W \subset T^{a,b}_\varphi$ are exactly the sub-Hodge structures of these tensor spaces.*

Proof of (I.C.2). If $M_\varphi(W) \subseteq W$ then $M_\varphi(\mathbb{R})(W_{\mathbb{R}}) \subseteq W_{\mathbb{R}}$, and since the circle $S^1_\varphi \subset M_\varphi(\mathbb{R})$ it follows that W is a sub-Hodge structure.

Conversely, if $W \subset T^{a,b}_\varphi$ is a sub-Hodge structure, then the form Q_W induced by Q is non-degenerate on W and we have

$$T^{a,b}_\varphi = W \oplus W^\perp.$$

Moreover, the image $\varphi(\mathbb{U}(\mathbb{R}))$ respects this decomposition. Thus, if $M_W \subset G$ is the $\mathbb{Q}$-algebraic subgroup of G stabilizing the subspace $W \subset T^{a,b}_\varphi$, it follows that $\varphi(\mathbb{U}(\mathbb{R})) \subset M_W(\mathbb{R})$ and thus $M_\varphi \subseteq M_W$. $\square$

Let M'_φ be the subgroup of G that pointwise fixes the algebra of Hodge tensors. We first show that $M_\varphi \subset M'_\varphi$. Since $\mathrm{Hg}^{a,b}_\varphi \subset T^{a,b}_\varphi$ is a sub-Hodge structure and M_φ is faithfully represented acting on $T^{a,b}_\varphi$, and since $\varphi(\mathbb{U}(\mathbb{R}))$ acts trivially on $\mathrm{Hg}^{a,b}_{\varphi,\mathbb{R}}$, it follows that the image of M_φ acting on $T^{a,b}_\varphi$ splits according to $T^{a,b}_\varphi = \mathrm{Hg}^{a,b}_\varphi \oplus (\mathrm{Hg}^{a,b}_\varphi)^\perp$ and acts as the identity on the first factor.

To show that $M'_\varphi \subseteq M_\varphi$, we let $W \subset T^{a,b}_\varphi$ be a sub-Hodge structure and we have to show that $M'_\varphi(W) \subseteq W$. If $\dim W = d$, then the line $\Lambda^d W$ in $\Lambda^d(T^{a,b}_\varphi)$ consists of Hodge classes and therefore, by assumption, is left fixed by M'_φ. It then follows that $M'_\varphi(W) \subseteq W$. $\square$

Other properties of M_φ are:

(i) M_φ is a connected, reductive $\mathbb{Q}$-algebraic group.

We note that $M_\varphi(\mathbb{R})$ may not be connected as a real Lie group.

(ii) If $\rho : \mathrm{Aut}(V,Q) \to \mathrm{Aut}(V_\rho, Q_\rho)$ is a representation of $G = \mathrm{Aut}(V,Q)$ and φ is a Hodge structure for V, then $\rho(\varphi)$ is a Hodge structure for V_ρ and

$$M_{\rho(\varphi)} = \rho(M_\varphi).$$

(iii) When M_φ is a simple $\mathbb{Q}$-algebraic group, $M_\varphi(\mathbb{R})$ may not be a simple real Lie group. More generally, the almost product decomposition of $M_\varphi(\mathbb{R})$ into simple factors and an abelian part may be finer than the $\mathbb{Q}$-almost product decomposition of M_φ.

Using (I.C.1) we may give the following

Definition. For $F^\bullet \in \check{D}$, we define the *Mumford-Tate group* $M_{F^\bullet}$ to be the subgroup of G that pointwise fixes the algebra $\mathrm{Hg}^{\bullet,\bullet}_{F^\bullet}$ of Hodge tensors.

Of particular interest are the Mumford-Tate groups $M_{F^\bullet}$ that arise when $F^\bullet \in \check{D}$ is the Hodge filtration arising from a mixed Hodge structure, especially in the case of limiting mixed Hodge structures. The relation of these, together with that of Mumford-Tate domains as defined in the next section, to the Kato-Usui spaces ([KU]) remains to be carried out and seems to us a project of fundamental interest.

Example. The classical example is when the weight $n = 1$ and the Hodge number $h^{1,0} = 1$. We take $V = \mathbb{Q}^2$ written as column vectors,

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$Q(u,v) = {}^t v\, Q u.$$

Then $\check{D} = \mathbb{P}^1(\mathbb{C})$ and every point $\varphi \in D$ has a unique homogeneous coordinate representative

$$\varphi \leftrightarrow \begin{bmatrix} \tau \\ 1 \end{bmatrix}, \qquad \operatorname{Im}\tau > 0.$$

The group $G(\mathbb{R})$ is equal to $\mathrm{SL}_2(\mathbb{R})$ with elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, acting as usual by linear fractional transformations. Taking our reference point φ_0 to be given by $\tau = i$ and letting $\varphi \in D$ correspond to the conjugacy class $g^{-1} H_{\varphi_0} g$, for $t \in \mathbb{S}(\mathbb{R})$ represented as above by (v,u) with $u^2 + v^2 = 1$, we have

$$\varphi(t) = \begin{pmatrix} u + v(ab+cd) & v(b^2+d^2) \\ -v(a^2+c^2) & u - v(ab+cd) \end{pmatrix}.$$

The only proper reductive subgroups of $\mathrm{SL}_2(\mathbb{Q})$ defined over $\mathbb{Q}$ are algebraic 1-tori, such as

$$H_{\varphi_0} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a^2 + b^2 = 1 \right\}.$$

So M_φ is either SL_2 or a 1-torus, and the latter happens precisely when $\tau = g \cdot i$ is a quadratic irrationality.

I.D. **Mumford-Tate domains.** Let $\varphi \in D$ be a Hodge structure with Mumford-Tate group M_φ.

Definition. A *Mumford-Tate domain* $D_{M_\varphi} \subset D$ is the $M_\varphi(\mathbb{R})$-orbit of φ.

We shall sometimes omit reference to φ and shall simply speak of a Mumford-Tate domain $D_M \subset D$.

A fundamental property is

(I.D.1) *D_{M_φ} is a homogeneous complex submanifold of D.*

Proof. The basic observation is that

(I.D.2) *$\mathfrak{m}_\varphi$ is a sub-Hodge structure of $\mathfrak{g}_\varphi$.*

This is simply because

$$\mathrm{Ad}_\varphi : \mathbb{S}(\mathbb{R}) \to \mathfrak{m}_\varphi(\mathbb{R}).$$

We thus have

$$\begin{aligned}\mathfrak{m}_{\varphi,\mathbb{C}} &= \underset{i}{\oplus}\,\mathfrak{m}_\varphi^{-i,i}\\ &= \mathfrak{m}_\varphi^- \oplus \overline{\mathfrak{m}_\varphi^-} \oplus \mathfrak{m}_\varphi^{0,0}\end{aligned}$$

where $\mathfrak{m}_\varphi^- = \underset{i>0}{\oplus}\,\mathfrak{m}_\varphi^{-i,i}$ and $\mathfrak{m}_\varphi^{0,0}$ is the complexified Lie algebra of the isotropy group $M_\varphi(\mathbb{R}) \cap H_\varphi$ at φ of the action of $M_\varphi(\mathbb{R})$ on D_{M_φ}. As in the discussion in I.B above,$\mathfrak{m}_\varphi^-$ gives the $(1,0)$ tangent space at $\varphi \in D_{M_\varphi}$ to an $M_\varphi(\mathbb{R})$-invariant integrable almost complex structure. □

Let

$$\mathfrak{m}_{\varphi,\mathbb{R}} = \overset{k}{\underset{\alpha=1}{\oplus}}\,\mathfrak{m}_\alpha \oplus \mathfrak{a}$$

be the direct sum decomposition of $\mathfrak{m}_{\varphi,\mathbb{R}}$ into $\mathbb{R}$-simple factors $\mathfrak{m}_\alpha$ and an abelian part $\mathfrak{a}$. We observe that $\operatorname{Ad}\varphi(\mathbb{S}(\mathbb{R}))$ preserves this decomposition. Moreover, since $\mathfrak{h}_{\varphi,\mathbb{R}}$ is the centralizer in $\mathfrak{g}_\mathbb{R}$ of $\operatorname{Ad}\varphi(\mathbb{S}(\mathbb{R}))$ it follows that $\mathfrak{a} \subset \mathfrak{h}_{\varphi,\mathbb{R}}$. Thus the Mumford-Tate domain

$$D_{M_\varphi} = D_1 \times \cdots \times D_k$$

is a product of homogeneous, complex submanifolds where D_α is the exp M_α-orbit of φ, where M_φ is the almost direct product $\prod_\alpha M_\alpha \times T$.

In practice we will be interested in the quotient $\Gamma \setminus D_{M_\varphi}$ by an arithmetic subgroup of M_φ. The group Γ will not in general split so that $\Gamma \setminus D_{M_\varphi}$ may be irreducible whereas D_{M_φ} is not. An example of this is in Mumford's original paper [Mu] where D is the Siegel upper-half space $\mathcal{H}_4$ and M_φ is simple whereas $\mathfrak{m}_{\varphi,\mathbb{R}} \cong \mathrm{sl}_2(\mathbb{R}) \oplus \mathrm{sl}_2(\mathbb{R}) \oplus \mathrm{su}(2)$. Higher weight examples are given in section III of [GGK].

A fundamental difference between the classical and general cases is the following:

(I.D.3) *In the weight $n = 1$ case, the orbit of any compact factor of $M_\varphi(\mathbb{R})$ is a point. This is not true when $n \geqq 2$.*

In the classical case, the orbit of any connected compact factor of $M_\varphi(\mathbb{R})$ is a compact, complex submanifold of a bounded Hermitian symmetric domain and therefore is a point. In the general case the fibres of

$$G(\mathbb{R})/H_\varphi \to G(\mathbb{R})/K_\varphi,$$

where K_φ is the unique maximal compact subgroup of $G(\mathbb{R})$ that contains H_φ, are compact, complex submanifolds of D. One may show by examples that there are points $\varphi \in D$ where $H_\varphi \subsetneq M_\varphi \subseteq K_\varphi$.

Example. For a weight $n = 2$ Hodge structure with decomposition over $\mathbb{R}$

$$V_{\mathbb{R}} = (V^{2,0} \oplus \overline{V}^{2,0})_{\mathbb{R}} \oplus V^{1,1},$$

assume that this decomposition is actually defined over $\mathbb{Q}$. Then for a *general* Hodge structure of this type with $W = (V^{2,0} \oplus \overline{V}^{2,0})_{\mathbb{Q}}$ and $Q_W = Q\,|_W$, it may be shown that

$$M_\varphi = \mathrm{Aut}(W, Q_W) \subset G.$$

Since Q_W is positive definite, it follows that $M_\varphi(\mathbb{R})$ is compact and that D_{M_φ} is the fibre through φ of the map $G(\mathbb{R})/H_\varphi \to G(\mathbb{R})/K_\varphi$.

II. The structure theorem

II.A. **Variations of Hodge structure.** The main difference between the classical case when the weight $n = 1$ and D is a bounded Hermitian symmetric domain and the higher weight case is that in the latter case the maps to D arising from algebraic geometry satisfy a differential constraint. To explain this, we recall the natural identification

$$T_E \,\mathrm{Grass}(d, V_{\mathbb{C}}) \cong \mathrm{Hom}(E, V_{\mathbb{C}}/E)$$

for the tangent space to the Grassmannian of d-planes in $V_{\mathbb{C}}$ at a point $E \in \mathrm{Grass}(d, V_{\mathbb{C}})$. It follows that there is a natural inclusion

$$T_{F^\bullet}\check{D} \subset \underset{p}{\oplus} \mathrm{Hom}(F^p, V_{\mathbb{C}}/F^p).$$

Definition. The canonical sub-bundle $W \subset T\check{D}$ given by the *infinitesimal period relation* is defined by

$$W_{F^\bullet} = T_{F^\bullet}\check{D} \cap \left(\underset{p}{\oplus} \mathrm{Hom}(F^p, F^{p-1}/F^p) \right). \tag{II.A.1}$$

We will continue to denote by W the restriction to D of the infinitesimal period relation. The bundle $W \to \check{D}$ is acted on by $G(\mathbb{C})$ and the action of $G(\mathbb{R})$ on $W \to D$ leaves invariant the metric given by the Cartan-Killing form at each point. With the identification of the holomorphic tangent space

$$T_\varphi D \cong \underset{i>0}{\oplus} \mathfrak{g}_\varphi^{-i,i}$$

we have that

$$W_\varphi = \mathfrak{g}_\varphi^{-1,1}. \tag{II.A.2}$$

Although in this paper we shall not get into a detailed discussion, we note that the negative curvature properties of period domains that hold in the sub-bundle $W \subset TD$ — but except in the classical case not in the whole tangent bundle — carry over to Mumford-Tate domains and allow the application of hyperbolic complex analysis to variations of Hodge structure in Mumford-Tate domains as in the structure theorem (II.B.6) below.

We will define variations of Hodge structure in the setting most appropriate for this paper. For this we assume given a lattice

$$V_{\mathbb{Z}} \subset V$$

with $V_{\mathbb{Z}} \otimes \mathbb{Q} = V$. We then have the subgroup

$$G(\mathbb{Z}) = \mathrm{GL}(V_{\mathbb{Z}}) \cap G$$

of G.

Definition. A *variation of Hodge structure* is given by

$$\Phi : S \to \Gamma \setminus D \tag{II.A.3}$$

where

(i) S is a connected, smooth, quasi-projective complex algebraic variety;
(ii) Γ is a subgroup of $G(\mathbb{Z})$; and
(iii) Φ is a locally liftable holomorphic mapping whose local lifts are integral manifolds of the canonical distribution given by W.

Condition (iii) means the following: Around each point $s \in S$ there is a neighborhood $\mathcal{U}$ and a local lifting

$$\begin{tikzcd} & D \arrow[d] \\ \mathcal{U} \arrow[ur, "\tilde{\Phi}"] \arrow[r, "\Phi"] & \Gamma \setminus D \end{tikzcd}$$

where $\tilde{\Phi}$ is holomorphic and

$$\tilde{\Phi}_* : T\mathcal{U} \to W. \tag{II.A.4}$$

Although general period domains are very far from being Hermitian symmetric, a guiding principle in Hodge theory has been

Variations of Hodge structure have the same properties as they do when D is Hermitian symmetric.[1]

In fact, so far as we know, the *only* properties special to the classical case have to do with the presence of automorphic forms. As an example of this we have the result

[1]A similar statement holds with respect to variations of Hodge structure that arise from families of algebraic varieties. Again, so far as we are aware all properties that hold for these have been shown to hold for general variations of Hodge structure.

(II.A.5) *Let M be a Mumford-Tate group with $M(\mathbb{R})$ compact. Then the image of D_M in $\Gamma\backslash D$ meets the image $\Phi(S)$ of a variation of Hodge structure in points.*

Idea of the proof. There are two steps:

(i) Any Mumford-Tate group M_φ with $M_\varphi(\mathbb{R})$ compact is contained in the maximal compact subgroup K_φ.[2]

(ii) At any point $\varphi \in D$, the subspaces W_φ and $T_\varphi(K_\varphi \cdot \varphi)$ of $T_\varphi D$ intersect only in zero.

Here, $K_\varphi \cdot \varphi \subset D$ is the K_φ-orbit of φ. Except in the classical case, it is positive dimensional.

The proof of (ii) comes by observing that the complexified Lie aglebra of K_φ is

$$\mathfrak{k}_{\varphi,\mathbb{C}} = \bigoplus_{i\equiv 0(2)} \mathfrak{g}_\varphi^{-i,i}$$

whereas

$$W_\varphi = \mathfrak{g}_\varphi^{-1,1}.$$

For (i), since $\mathfrak{m} \subset \mathfrak{g}$ is a sub-Hodge structure, we have

$$\text{(II.A.6)} \qquad \mathfrak{m}_\mathbb{C} = \bigoplus_i \mathfrak{m}^{-i,i} \text{ where } \mathfrak{m}^{-i,i} = \mathfrak{g}^{-i,i} \cap \mathfrak{m}_\mathbb{C}.$$

The assumption that $M(\mathbb{R})$ is compact comes in by observing that, since the Cartan-Killing form is negative on the semi-simple part of $\mathfrak{m}_\mathbb{R}$, only terms $\mathfrak{m}^{-i,i}$ with $i \equiv 0(2)$ enter in (II.A.6).

II.B. **The structure theorem.** We begin by defining the Mumford-Tate group M_Φ for a variation of Hodge structure (II.A.3). For this we observe that for any $\gamma \in \Gamma$ and $\varphi \in D$, we have

$$\gamma(\mathrm{Hg}_\varphi^{\bullet,\bullet}) = \mathrm{Hg}_{\gamma(\varphi)}^{\bullet,\bullet}.$$

Thus the algebra of Hodge tensors is well-defined at a point of $\Gamma \setminus D$.

Definition. The *Mumford-Tate group M_Φ of a variation of Hodge structure* $\Phi : S \to \Gamma \setminus D$ is defined to be the Mumford-Tate group $M_{\Phi(\eta)}$ where η is a generic point of S.

More precisely, in the path-connected complement of a countable union Z of proper analytic subvarieties of S, the algebra of Hodge tensors will be locally constant. Generic means that $\eta \in S \setminus Z$.

Since Φ is locally liftable, choosing a base point $s_0 \in S$ there will be a *monodromy representation*

$$\text{(II.B.1)} \qquad \rho : \pi_1(S, s_0) \to \Gamma.$$

[2]More precisely, we have $M_\varphi(\mathbb{R}) \subset K_\varphi$.

In what follows, we will take Γ to be the image of $\pi_1(S, s_0)$.

We shall consider the variation of Hodge structure (II.B.3) *up to isogeny*. This means that if $\tilde{S} \xrightarrow{\pi} S$ is a finite cover, then Φ and

$$\tilde{\Phi} = \Phi \circ \pi : \tilde{S} \to \tilde{\Gamma} \setminus D$$

are considered equivalent. Here $\tilde{\Gamma}$ is the subgroup of finite index in Γ given by the $\tilde{\Phi}$ image of $\pi_1(\tilde{S}, \tilde{s}_0)$ where $\pi(\tilde{s}_0) = s_0$. The reason for doing this is the following: Assuming that the base point $s_0 \in S$ is generic in the above sense, the algebra of Hodge tensors at any point $\varphi(s_0) \in D$ lying over $\Phi(s_0)$ is invariant under the monodromy group Γ. Since the polarizing form is definite on each $\mathrm{Hg}^{a,b}_{\varphi(s_0)}$ and Γ acts by integral matrices, it follows that Γ acts on $\mathrm{Hg}^{a,b}_{\varphi(s_0)}$ as a finite group. Thus, by passing to a finite covering $\tilde{S} \to S$ we may assume that monodromy acts trivially on $\mathrm{Hg}^{\bullet,\bullet}_{\tilde{\varphi}(s_0)}$.

Assuming this has been done, so that Γ fixes $\mathrm{Hg}^{\bullet,\bullet}_{\varphi(s_0)}$, we observe first that

$$\Gamma \subset G(\mathbb{Z}) \cap M_\Phi. \tag{II.B.2}$$

Secondly, if $\varphi \in D$ is any point lying over $\Phi(\eta) \in \Gamma \setminus D$, then $M_\varphi = M_\Phi$ and we have that

$$\Phi : S \to \Gamma \setminus D_{M_\Phi}. \tag{II.B.3}$$

That is, by (II.B.1) the Mumford-Tate domain D_{M_Φ} is invariant under Γ so that $\Gamma \setminus D_{M_\Phi}$ is defined. Then, by the definition of genericity of η, the image $\Phi(S)$ lies in the subvariety $\Gamma \setminus D_{M_\Phi} \subset \Gamma \setminus D$.

We now let

$$M_\Phi = M_1 \times \cdots \times M_\ell \times T \tag{II.B.4}$$

be the almost product decomposition of M_Φ into $\mathbb{Q}$-simple factors M_i and an algebraic torus T. Denoting by $D_i \in D_\Phi$ the $M_i(\mathbb{R})$ orbit of $\varphi(s_0)$, and noting that T fixes $\varphi(s_0)$, we have a splitting of the Mumford-Tate domain

$$D_{M_\Phi} = D_1 \times \cdots \times D_\ell.$$

(II.B.5) **Structure theorem:** *The monodromy group Γ splits as an almost direct product*

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_k, \qquad k \leqq \ell$$

where $\Gamma_i(\mathbb{Q}) = M_i$.

The notation means that $\Gamma_i \subset M_i \cap G(\mathbb{Z})$ and that the $\mathbb{Q}$-Zariski closure $\Gamma_i(\mathbb{Q})$ of Γ_i in M_i is equal to M_i. Letting $D' = D_{k+1} \times \cdots \times D_\ell$ be the factors where the monodromy is trivial, the structure theorem and its proof have the implication:

(II.B.6) *The variation of Hodge structure is given by*

$$\Phi : S \to \Gamma_1 \setminus D_1 \times \cdots \times \Gamma_k \setminus D_k \times D' \subset \Gamma \setminus D$$

where $\Gamma_i(\mathbb{Q}) = M_i$. *Moreover,* Γ_i *and* $M_i(\mathbb{Z}) = M_i \cap G(\mathbb{Z})$ *have the same algebra of tensor invariants in* $T^{\bullet,\bullet}$.

Thus, since it seems not to be known — one way or the other — whether Γ_i is an arithmetic group; i.e., a subgroup of finite index in $M_i(\mathbb{Z})$, we know at least that as far as their tensor invariants go Γ_i and $M_i(\mathbb{Z})$ are indistinguishable.

The structure theorem is a consequence of the *theorem on the fixed part* (cf. [Sc] and the argument given by André [A]).

Discussion. It is known that a variation of Hodge structure (II.A.3) is *algebro-geometric* in the following sense: The image $\Phi(S) \subset \Gamma \setminus D$ is a quasi-projective algebraic subvariety of the analytic variety $\Gamma \setminus D$. It is also known that $\Phi(S)$ has finite volume relative to the volume form induced by the natural $G(\mathbb{R})$-invariant metric on D. One may then infer that the same properties hold for the non-trivial irreducible factors

$$\Phi_i : S \to \Gamma_i \setminus D_i \tag{II.B.7}$$

in the structure theorem. If $\Gamma_i' \subset M_i(\mathbb{Z})$ is any subgroup with $\Gamma_i \subset \Gamma_i'$ and such that Γ_i' leaves invariant the inverse image of $\Phi_i(S)$ in D_i, it follows that Γ_i *is of finite index in* Γ_i'.

Suppose now that the variation of Hodge structure (II.A.3) is one of the irreducible factors in (II.B.6) and set $D = D_M$ where M is the Mumford-Tate group. We say that $\Phi : S \to \Gamma \setminus D_M$ is *maximal* if, as always up to isogeny, S is a Zariski open set in any variation of Hodge structure

$$\Phi' : S' \to \Gamma' \setminus D_M$$

where $S \subset S'$, $\Gamma \subset \Gamma'$ and $\Phi' = \Phi \mid_S$. In the classical case it is known that

(i) any variation of Hodge structure is contained in a maximal one; and
(ii) for any maximal variation of Hodge structure the monodromy group is arithmetic.

Question. *Are (i) and (ii) true in the general case?*

Remark. Once one leaves the classical case, certain phenomena that might be described as *anomalous* enter. The use of Mumford-Tate groups and the structure theorem help to siphon out anomalous phenomena. Here is an example.

Definition. A set of Hodge numbers $\{h^{p,q}\}_{p+q=n}$ is *connected* if h^{p_0,q_0}, $h^{p_1,q_1} \neq 0 \Rightarrow h^{p,q} \neq 0$ for all integers $p \in [p_0, p_1]$. This property is equivalent to any two points of D being connected by an integral curve of the distribution $W \subset TD$. If the set of Hodge numbers is not connected then anomalous phenomena arise. For example, suppose $n = 5$ and $h^{4,1} = h^{1,4} = 0$ but all other $h^{p,q} \neq 0$. The Hodge structures are then

$$\text{(II.B.8)} \qquad \left\{ \begin{array}{c} V_{\mathbb{C}} = V^{5,0} \oplus V^{3,2} \oplus V^{2,3} \oplus V^{0,5} \\ V^{0,5} = \overline{V}^{5,0}, V^{2,3} = \overline{V}^{3,2}. \end{array} \right.$$

We note that for such a Hodge structure there is an associated abelian variety arising from

$$V' := V^{5,0} \oplus V^{3,2},$$

where we have

$$\left\{ \begin{array}{ll} Q(V', V') = 0 & \\ Q(u, C\bar{u}) > 0 & \qquad 0 \neq u \in V'. \end{array} \right.$$

For any variation of Hodge structure of type (II.B.8), the subspace $V^{5,0}$ is constant. Hence, if $V^f \subset V$ is the fixed part where monodromy is trivial, we have

$$(V^{5,0} \oplus \overline{V}^{5,0})_{\mathbb{R}} \subset V^f_{\mathbb{R}}.$$

Writing $V^\nu = (V^f)^\perp$ for the part on which monodromy has no trivial factors, the variation of Hodge structure is just that of the family of abelian varieties given by the Hodge decomposition of $V^\nu_{\mathbb{C}}$

$$(V^{3,2} \cap V^\nu_{\mathbb{C}}) \oplus (\overline{V^{3,2} \cap V^\nu_{\mathbb{C}}}).$$

This phenomenon is detected by splitting off the constant factors in the structure theorem.

III. Universal characteristic cohomology of Mumford-Tate domains

III.A. **Characteristic cohomology.** Let X be a complex manifold and $W \subset TX$ a holomorphic sub-bundle. Then

$$I := W^\perp \subset T^*X$$

is a holomorphic bundle and we let $\overline{I}$ be its conjugate in $(T^*_{\mathbb{R}}X) \otimes \mathbb{C}$. Denoting by $A^{\bullet,\bullet}(X) = \bigoplus_{p,q} A^{p,q}(X)$ the algebra of smooth differential forms on X we have the

Notation. $\mathcal{I}^{\bullet,\bullet} \subset A^{\bullet,\bullet}(X)$ is the *differential ideal* generated by the smooth sections of $I \oplus \overline{I}$.

Concretely, if locally I is generated over an open set U by holomorphic 1-forms θ^α, then $\mathcal{I}^{\bullet,\bullet}$ consists of global forms ψ that locally are expressed as

$$\psi = \psi_\alpha \wedge \theta^\alpha + \psi_{\overline{\alpha}} \overline{\theta}^\alpha + \Psi_\alpha \wedge d\theta^\alpha + \Psi_{\overline{\alpha}} \wedge \overline{d\theta^\alpha}.$$

An *integral manifold* of $\mathcal{I}^{\bullet,\bullet}$ is given by a manifold Y and a smooth mapping

$$f : Y \to X \tag{III.A.1}$$

such that $f^*(\mathcal{I}^{\bullet,\bullet}) = 0$. We may think of integral manifolds as solutions to the differential equations

$$\begin{cases} \theta^\alpha = \overline{\theta}^\alpha = 0 \\ d\theta^\alpha = \overline{d\theta^\alpha} = 0. \end{cases} \tag{III.A.2}$$

In case Y is a complex manifold and f is holomorphic – the case we will be considering in this paper — integral manifolds satisfy

$$f_* : TY \to W \subset TX.$$

The quotient space

$$Q^{\bullet,\bullet} = A^{\bullet,\bullet}(X)/\mathcal{I}^{\bullet,\bullet}$$

has an induced exterior derivative d_Q.

Definition. The *characteristic cohomology* $H^*_{\mathcal{I}}(X)$ is the cohomology of the complex $(Q^\bullet, d_Q)$ where $Q^n = \bigoplus_{p+q=n} Q^{p,q}$.

For an integral manifold, we have a map

$$f^* : H^n_{\mathcal{I}}(X) \to H^n(Y) \tag{III.A.3}$$

from the characteristic cohomology asssociated to $(X, \mathcal{I}^{\bullet,\bullet})$ to the usual cohomology of Y. We may think of characteristic cohomology as *that cohomology that induces ordinary cohomology on integral manifolds by virtue of their being solutions to the PDE system* (III.A.2).

In this paper we will be interested in the case where X is a Mumford-Tate domain and W is the infinitesimal period relation. In preparation for this we assume that X is a homogeneous complex manifold of the form

$$X = A/B$$

where A is a real, semi-simple Lie group and B is a compact, connected subgroup. For the complexified Lie algebras, we have the $\mathrm{Ad}\,B$-invariant splitting

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t} \oplus \mathfrak{b}_{\mathbb{C}}$$

where

$$\begin{cases} \mathfrak{t} = \mathfrak{t}^{1,0} \oplus \mathfrak{t}^{0,1} \\ \mathfrak{t}^{0,1} = \bar{\mathfrak{t}}^{1,0} \end{cases}$$

defines the complex structure on X. We also assume that the Cartan-Killing form defines an A-invariant Hermitian metric h on X. We denote by $\mathfrak{w} \subset \mathfrak{t}^{1,0}$ the subspace given by the fibre W at the identity coset, and we have the $\mathrm{Ad}\,B$-invariant decomposition

$$\mathfrak{t}^{1,0} = \mathfrak{w} \oplus \mathfrak{v}.$$

Using the metric, we identify $\mathfrak{t}$ with its dual $\check{\mathfrak{t}}$ and $\mathfrak{v}$ with its dual $\check{\mathfrak{v}}$. Finally we denote by $[\mathfrak{w}, \mathfrak{w}]_{\mathfrak{v}}$ the image in $\mathfrak{v}$ of the brackets of elements in $\mathfrak{w}$. Since $[\mathfrak{b}_{\mathbb{C}}, \mathfrak{w}] \subseteq \mathfrak{w}$, this is well-defined and we set

$$\begin{cases} \mathfrak{k} = \ker \{[\ ,\ ]_{\mathfrak{v}} : \wedge^2\mathfrak{w} \to \mathfrak{v}\} \\ \wedge^2\mathfrak{w} = \mathfrak{l} \oplus \mathfrak{k} \end{cases}$$

Notation. We set

$$\begin{cases} \Lambda^{1,0} = \mathfrak{w}, \Lambda^{0,1} = \overline{\mathfrak{w}} \\ \Lambda^{p,q} = (\mathfrak{l} \wedge \Lambda^{p-1}\mathfrak{w}) \otimes (\bar{\mathfrak{l}} \wedge \Lambda^{q-1}\overline{\mathfrak{w}}). \end{cases}$$

Basic observation. $\Lambda^{\bullet,\bullet}$ *is isomorphic to the complex of A-invariant forms in $\mathcal{I}^{\bullet,\bullet}$.*

The points are: (i) We identify the various vector spaces with their duals using the invariant metric; (ii) In degree one, $\check{\mathfrak{w}} \oplus \check{\overline{\mathfrak{w}}}$ gives the fibres of $\check{T}X/I \oplus \overline{\check{T}X/I}$; and (iii) Using the Maurer-Cartan equation, $\check{\mathfrak{k}} \subset \Lambda^2\check{\mathfrak{w}}$ gives the values at the identity coset of the forms $d\theta$ where θ is a section of I — this is the main point.

The induced exterior derivation d_Q induces a differential

$$\delta : \Lambda^n \to \Lambda^{n+1}$$

where $\Lambda^n = \bigoplus_{p+q=n} \Lambda^{p,q}$ and we have the result

(III.A.4) *The cohomology of the A-invariant forms in $\mathcal{I}^{\bullet,\bullet}$ is isomorphic to $H^*(\Lambda^\bullet, \delta)$.*

Definition. We shall call $H^*(\Lambda^\bullet, \delta)$ the *invariant characteristic cohomology.*

It is this that in the next section we shall be interested in for Mumford-Tate domains. To conclude this section we mention what we feel is a very interesting question.

To explain this we assume that X is compact but do not assume that it is homogeneous. We also assume that the sections of

$$W + [W, W]$$

generate a sub-bundle of TX. Using an Hermitian metric h on X, we may define $Q^{\bullet,\bullet}$ to be a subspace of $A^{\bullet,\bullet}(X)$ by taking the orthogonal complements to the values of the forms in $\mathcal{I}^{\bullet,\bullet}$ at each point of X. This enables us to define a global inner product on $Q^{\bullet,\bullet}$, an adjoint d_Q^* to d_Q, and a Laplacian

$$\Delta_Q = d_Q d_Q^* + d_Q^* d_Q.$$

Then as usual we may define a "harmonic" space

$$\mathcal{H}^n_{\mathcal{I}}(X, h) = \ker\{\Delta_Q : Q^n \to Q^n\}.$$

In order for this to have relevance to $H^n_{\mathcal{I}}(X)$ we make the following

Assumption. W is *bracket-generating*.

This means that

$$W + [W, W] + [W, [W, W]] + \cdots = TX.$$

It may be checked that this assumption is equivalent to the operator Δ_Q being hypoelliptic, and in this case we have the following Hodge-type theorem communicated to us by Michael Taylor:

> *The natural map*
>
> $$\mathcal{H}^n_{\mathcal{I}}(X) \xrightarrow{\sim} H^n_{\mathcal{I}}(X)$$
>
> *is an isomorphism.*

For applications one would like to have a Hodge structure on $H^n_{\mathcal{I}}(X)$. The natural way this could arise is if one has the commutation relation

$$[\Delta_Q, \Pi^{p,q}] = 0 \qquad \text{(III.A.5)}$$

where

$$\Pi^{p,q} : Q^\bullet \to Q^{p,q}$$

is the projection onto forms of type (p, q). For Hermitian manifolds when $W = TX$, Chern [C] proved that (III.A.5) is equivalent to the metric h being Kähler. This leads to the

Question. *What are the necessary and sufficient conditions, expressed in terms of the metric h and distribution W, that* (III.A.5) *holds?*

III.B. **Invariant characteristic cohomology for Mumford-Tate domains.** We consider the situation of section III.A when $X = D_M \subset D$ is a Mumford-Tate domain and $W_M \subset TD_M$ is the infinitesimal period relation. We shall denote by $\Lambda_M^{\bullet,\bullet}$ the complex of $G(\mathbb{R})$-invariant forms in $A^{\bullet,\bullet}(D_M)/\mathfrak{I}^{\bullet,\bullet}$, with the operator $\delta_M : \Lambda_M^\bullet \to \Lambda_M^{\bullet+1}$ being induced by the exterior derivative.

Definition. In this situation we shall refer to $H^*(\Lambda_M^\bullet, \delta_M)$ as the *universal characteristic cohomology.*

The reason for the term is that the image of $H^*(\Lambda_M^\bullet, \delta_M)$ in $H_{\mathfrak{I}}^*(D_M)$ is $G(\mathbb{R})$ *invariant.* Hence, for any variation of Hodge structure (II.A.3) there is an induced map

$$\Phi^* : H^*(\Lambda_M^\bullet, \delta_M) \to H^*(S)$$

independent of the monodromy group Γ.

(III.B.1) **Proposition:** *For the universal characteristic cohomology we have* $H^{2p+1}(\Lambda_M^\bullet, \delta_M) = 0$ *and*

$$H^{2p}(\Lambda_M^\bullet, \delta_M) \cong (\Lambda^{p,p})^{\mathfrak{m}^{0,0}}.$$

That is, the universal characteristic cohomology vanishes in odd degree, and in even degrees it is all of Hodge type (p,p) and given by the $G(\mathbb{R})$-invariant (p,p) forms in the complex $Q^{\bullet,\bullet}$.

Proof. Using the Hodge structure on $\mathfrak{m}$ we have

$$\mathfrak{m}_{\mathbb{C}} = \oplus \mathfrak{m}^{-i,i}$$

and the notational correspondence with that in the previous section is

$$\begin{cases} \mathfrak{k} & \leftrightarrow \bigoplus\limits_{i>0} \mathfrak{m}^{-i,i} \\ \mathfrak{w} & \leftrightarrow \mathfrak{m}^{-1,1} \\ \mathfrak{b}_{\mathbb{C}} & \leftrightarrow \mathfrak{m}^{0,0}. \end{cases}$$

The proposition will follow from

$$\text{(III.B.2)} \qquad \left(\Lambda^p \mathfrak{m}^{-1,1} \otimes \Lambda^q \mathfrak{m}^{1,-1}\right)^{\mathfrak{m}^{0,0}} = 0 \ \text{ if } \ p \neq q.$$

The reason for this is that the circle S^1 acts by $\operatorname{Ad}\varphi(\mathbb{S}(\mathbb{R}))$ on $\mathfrak{m}$ with eigenspaces $\mathfrak{m}^{-i,i}$, and for $t \in S^1$ we have

$$\begin{cases} t(u) = t^{-2}u, & u \in \mathfrak{m}^{-1,1} \\ t(v) = t^2 v, & v \in \mathfrak{m}^{1,-1}. \end{cases}$$

Since $\varphi(\mathbb{S}(\mathbb{R})) \subset H$, where $H = M(\mathbb{R}) \cap H_\varphi$ is the isotropy group, the result follows. $\square$

Remark. We note that, consistent with (II.A.5), when $M(\mathbb{R})$ is compact it follows that $\Lambda_M^{\bullet,\bullet} = (0)$.

Discussion. The proposition should be viewed as a very partial result. The desired result would be to explicitly identify the $\mathrm{Ad}\,H$-invariants in $\Lambda_M^{p,p}$. We note that the *integrability condition* arising from

$$[\ ,\ ] : \Lambda^2 \mathfrak{m}^{-1,1} \to \mathfrak{m}^{-2,2}$$

did not enter into the proof, and those will need to play a crucial role in determining $(\Lambda_M^{p,p})^{\mathfrak{m}^{0,0}}$.

For any period domain D there are defined the *Hodge bundles*

$$\mathbb{V}^{p,q} \to D$$

with fibre $V_\varphi^{p,q}$ over $\varphi \in D$. In a separate work it will be proved that

(III.B.3) *In the case when $D_M = D$; i.e., $M = G$, the universal characteristic cohomology is generated by the Chern forms of the Hodge bundles.*

The proof is based on a detailed analysis of the representation theory of the isotropy group H_φ. An interesting point is that *except in the classical case, where the relations among the Chern forms are universal and well-known, there are additional relations*

$$c_i(\mathbb{F}^p)c_j(\mathbb{F}^{n-p}) = 0 \ \textit{ if } i + j > h^{p,n-p} \tag{III.B.4}$$

imposed by the integrability conditions.

For a Mumford-Tate domain $D_M \subset D$, the Hodge bundles on D restrict to Hodge bundles on D_M. We ask the question

(III.B.5) *In general, do the Chern forms generate the universal characteristic cohomology?*

Since the conditions that a reductive $\mathbb{Q}$-algebraic group be a Mumford-Tate group are not known,[3] one may ask more specifically the following

(III.B.6) *Let M be a Mumford-Tate group such that $M(\mathbb{R})$ is a simple Lie group and the representation $M(\mathbb{R}) \to \mathrm{GL}(V_\mathbb{R})$ is irreducible. What are the conditions in the highest weight that the Chern forms generate the characteristic cohomology?*

In case this holds, what are the relations on the generators?

This seems to us a particularly interesting question because the interaction between representation theory and the integrability conditions will have to enter.

[3]We believe that this is an interesting and feasible question.

Remark. At the end of section II.B we have remarked on "anomalous phenomenon" that arise in non-classical Hodge-theoretic considerations. Another type of anamolous phenomenon concerns the old

(III.B.7) **Question:** *Can one give sufficient conditions that a variation of Hodge structure arise from algebraic geometry?*

As previously noted, all to us known properties of a variation of Hodge structure arising from algebraic geometry[4] hold for general variations of Hodge structure. However, a guess such as "a variation of Hodge structure given by

$$\Phi : S \to \Gamma_1 \setminus D_1 \times \cdots \times \Gamma_k \setminus D_k \tag{III.B.8}$$

in (II.B.6) is motivic" is false, because one may do something stupid like taking a motivic variation of Hodge structure and tensoring it with a general constant, non-motivic Hodge structure. What has been missing is some good notion of *primitive*, which would mean that the variation of Hodge structure is not composed as above from lower weight variations of Hodge structure. It seems possible that universal characteristic cohomology might be relevant to this issue.

IV. Noether-Lefschetz loci

IV.A. **Mumford-Tate domains and Noether-Lefschetz loci.** Let $\varphi \in D$ be a Hodge structure with algebra $\mathrm{Hg}_\varphi^{\bullet,\bullet} \subset T^{\bullet,\bullet}$ of Hodge tensors. Because we will be interested in the algebra $\mathrm{Hg}_\varphi^{\bullet,\bullet}$ and not the particular φ, we denote $\mathrm{Hg}_\varphi^{\bullet,\bullet}$ by $\mathbb{H}^{\bullet,\bullet}$ and by M the Mumford-Tate group M_φ. We think of M as a subgroup of G and not as an abstract group.

Definition. The *Noether-Lefschetz locus* $\mathrm{NL}_M \subset D$ is defined by

$$\mathrm{NL}_M = \left\{ \varphi \in D : \mathbb{H}^{\bullet,\bullet} \subseteq \mathrm{Hg}_\varphi^{\bullet,\bullet} \right\}.$$

That is, NL_M consists of all Hodge structures whose algebra of Hodge tensors is at least as big as the algebra $\mathbb{H}^{\bullet,\bullet}$.

We may give the same definition of the Noether-Lefschetz locus $\check{\mathrm{NL}}_M$ associated to a point $F^\bullet \in \check{D}$. When $F^\bullet = F_\varphi^\bullet$ for some $\varphi \in D$, it is clear that

$$\mathrm{NL}_M = \check{\mathrm{NL}}_M \cap D.$$

It is also clear that

(IV.A.1) $\check{\mathrm{NL}}_M \subset \check{D}$ *is an algebraic subvariety defined over* $\mathbb{Q}$.

[4] We shall refer to these as *motivic variations of Hodge structure.*

As will be discussed below, $\breve{\mathrm{NL}}_M$ will not in general be irreducible and NL_M will be smooth but generally will not be connected. For the individual components we have the

(IV.A.2) **Theorem:** *When $\mathbb{H}^{\bullet,\bullet} = \mathrm{Hg}_\varphi^{\bullet,\bullet}$ for a point $\varphi \in D$, the component of NL_M passing through φ is equal to the component $D^\circ_{M_\varphi}$ of the Mumford-Tate domain passing through φ.*

This result is plausible because both NL_M and M_φ are defined over $\mathbb{Q}$ in terms of the algebra of Hodge tensors associated to the Hodge structure φ. However, since D_{M_φ} is defined using $M_\varphi(\mathbb{R})$ it does not seem to be entirely obvious.

One consequence of the theorem is that the components of NL_M are smooth submanifolds of D. Another is the dimension count

$$\operatorname{codim}_D \mathrm{NL}_M = \operatorname{codim}_{\mathfrak{g}^-}(\mathfrak{m}^-_\varphi). \tag{IV.A.3}$$

This result is also not obvious. For example, if the weight $n = 2m$ is even and $\zeta \in V$, then the locus

$$\mathrm{NL}_\zeta \subset D$$

where $\zeta \in \mathrm{Hg}^{2m}_\varphi = V \cap V^{m,m}_\varphi$ is easily seen to be smooth and of codimension

$$h^{2m,0} + \cdots + h^{m+1,m-1} \tag{IV.A.4}$$

in D. However, since to our knowledge there is as of yet no information on the *effective* generators of the algebra $\mathrm{Hg}_\varphi^{\bullet,\bullet}$,[5] there does not seem to be a direct way to iterate this procedure to estimate the codimension of NL_M.

For a variation of Hodge structure (II.A.3), the pullback under Φ of the image in $\Gamma \backslash D$ of a Noether-Lefschetz locus is an object of classical and continuing algebro-geometric interest. The quantity on both sides of (IV.A.3) gives an upper bound which, except in the classical case, is way off. A first correction comes from the 1st order information in the differential constraints given by the infinitesimal period relation. For example, in case $\mathbb{H}^{\bullet,\bullet}$ is effectively generated by the polarization and a single Hodge class $\zeta \in V^\bullet$, the quantity (IV.A.4) is replaced by $h^{m+1,m-1}$. When the second order, or integrability conditions, are taken into account a further decrease occurs. This is explained in section

[5] *Effective generator* means a generator in $\mathrm{Hg}^{a,b}_\varphi$ of the algebra such that adding it to the algebra generated by the tensors in $\mathrm{Hg}^{a',b'}_\varphi$ for all a', b' with $a'+b' < a+b$ decreases the corresponding locus, either by decreasing the dimension or by eliminating one or more components.

(II.F) of [GGK] (cf. Theorem (II.F.5)), where the estimate given there is shown by example to in general be sharp.

IV.B. **Arithmetic properties of Noether-Lefschetz loci.** For simplicity of exposition, in this section we will assume that Hodge structures V_φ are *simple*, i.e. have no non-trivial sub-Hodge structures.[6] Then the endomorphism algebra

$$E_\varphi = \left\{ \begin{array}{c} \alpha : V \to V \text{ with} \\ \alpha(V_\varphi^{p,q}) \subseteq V_\varphi^{p,q} \end{array} \right\}$$

is a division ring over $\mathbb{Q}$.

One extreme of Mumford-Tate domains occurs when $D_{M_\varphi} = D$; i.e. $M_\varphi = G$. The other extreme is when D_{M_φ} is a point. We recall that a *CM-field* is a totally imaginary extension L of $\mathbb{Q}$ having a totally real subfield K with $[L : K] = 2$.

(IV.B.1) **Proposition:** *The following are equivalent:*

(i) D_{M_φ} *is a point;*
(ii) M_φ *is an algebraic torus* T_φ*;*
(iii) E_φ *is a CM-field of degree* $\dim(V)$ *over* $\mathbb{Q}$*.*

Sketch of the proof. (i) $\implies$ (ii) If $D_{M_\varphi} = M_\varphi(\mathbb{R}).\varphi$ is just the point $\{\varphi\}$, then M_φ is contained in the isotropy group[7] H_φ. Now $H_\varphi(\mathbb{Q})$ is equal to $\mathrm{Aut}(V, Q_\varphi) = E_\varphi \cap G(\mathbb{Q})$ and thus

$$M_\varphi(\mathbb{Q}) \subseteq E_\varphi.$$

But M_φ always commutes with E_φ, so here it commutes with $M_\varphi(\mathbb{Q})$. Since M_φ is defined over $\mathbb{Q}$, $M_\varphi(\mathbb{Q})$ is $\mathbb{Q}$-Zariski dense in M_φ and thus M_φ is abelian and is an algebraic torus.

(ii) $\implies$ (iii) This is a standard argument in the literature (cf. [Mo] and [GGK]).

(iii) $\implies$ (i) The assumption that E_φ is a field of degree[8] $\dim(V)$ implies that E_φ^* are the $\mathbb{Q}$-points of a maximal torus $\widetilde{T}$ (defined over $\mathbb{Q}$) of $\mathrm{GL}(V)$ which diagonalizes with respect to some Hodge basis. Therefore $\widetilde{T}(\mathbb{R}) \supset \varphi(\mathbb{S}(\mathbb{R}))$, which implies $M \subset \widetilde{T} \cap G$. Moreover, since $\widetilde{T} \cap G \subset H_\varphi$, $M_\varphi \subset H_\varphi$ so that $M_\varphi(\mathbb{R}).\varphi = \{\varphi\}$. $\square$

[6] The general case is treated in detail in sections III and IV of [GGK].

[7] In this proof, H_φ is treated (notationally) as an algebraic group rather than as a real Lie group.

[8] Note that we do not need to use the fact that E_φ is a CM-field.

In section III of [GGK] there is a detailed analysis and numerous examples of CM-Hodge structures. There it is also shown that the CM-Hodge structures are topologically dense in NL_M. There also the following result of Abdulali [Ab] is discussed:

A CM-Hodge structure is motivic.

This is the only result we know, aside from the classical case, where sufficient conditions are given to ensure that an abstract Hodge-theoretic object is motivic. In this regard, we know no example of an explicitly given Hodge structure that is not motivic.

As mentioned above, $\breve{\mathrm{NL}}_M$ is in general not irreducible and $\mathrm{NL}_M = \breve{\mathrm{NL}}_M \cap D$ is not connected. For example, there may be several CM-Hodge structures corresponding to a fixed CM-field E. It is interesting to know the answer to questions such as:

(i) What is the largest subgroup of $G(\mathbb{R})$ stabilizing NL_M?
(ii) Does this group transitively permute the components of NL_M?
(iii) What is the field of definition of the irreducible components of $\breve{\mathrm{NL}}_M$?

Regarding (i) one has the

(IV.B.2) **Theorem:** *The largest subgroup of $G(\mathbb{R})$ stabilizing* NL_M *is the normalizer $N_G(M,\mathbb{R})$ of $M(\mathbb{R})$ in $G(\mathbb{R})$. A similar result holds regarding* $\breve{\mathrm{NL}}_M$ *and $N_G(M,\mathbb{C})$.*

Regarding (ii), we first note that in the classical case, NL_M is a single $M(\mathbb{R})$-orbit, whereas this is definitely *not* so in the non-classical case.

(IV.B.3) **Proposition:** *Let $T \subset G$ be a maximal anisotropic real torus which is defined over $\mathbb{Q}$. Then the Weyl group $W_G(T,\mathbb{R})$ of $T(\mathbb{R})$ in $G(\mathbb{R})$ acts transitively on the components of* NL_T. *There is a similar statement for $W_G(T,\mathbb{C})$ and the components of* $\breve{\mathrm{NL}}_T$.

Turning to (iii), we say that a Hodge structure φ is *non-degenerate* if $\mathrm{Hg}_\varphi^{\bullet,\bullet}$ is effectively generated by $\mathrm{Hg}_\varphi^{1,0}$ and $\mathrm{Hg}_\varphi^{1,1}$. For example, in the situation where the equivalent conditions of (IV.B.1) hold, V_φ is nondegenerate if and only if T_φ is a maximal torus in G.

Let φ_0 be a non-degenerate CM-Hodge structure of odd weight and $L \subset \mathrm{End}(V_{\varphi_0})$ its CM-field (assumed Galois). Using the non-degeneracy assumption, it follows that the Mumford-Tate group T_{φ_0} is just the commutator of L in G, written $T = G^L$. Letting $K \subseteq L$ be a *normal* subfield we have the

(IV.B.4) **Theorem:** *For $M = G^K$, the permutation action of*

$$W_G(M) := \frac{N_G(M) \cap N_G(T)}{N_M(T)}$$

on the components of $\breve{\mathrm{NL}}_M$ *reproduces that of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. *In particular, the orbits of $W_G(M)$ acting on components are defined over $\mathbb{Q}$, and individual components of* $\breve{\mathrm{NL}}_M$ *are defined over K.*

References

[Ab] S. Abdulali, Hodge Structures of CM type, *J. Ramanujan Math. Soc.* **20** (2005), no. 2, 155–162.

[A] Y. André, Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part, *Compositio Math.* **82** (1992), 1–24.

[Bo] A. Borel, *Linear Algebraic Groups*, *Grad. Texts in Math.* **126**, Springer-Verlag, New York, 1991.

[C] S. S. Chern, On a generalization of Kähler geometry, in Algebraic Geometry and Topology. A symposium in honor of S. Lefschetz, pp. 103–121, Princeton Univ. Press, Princeton, NJ, 1957.

[C-MS-P] J. Carlson, S. Mueller-Stach, and C. Peters, *Period Mappings and Period Domains*, Cambridge Univ. Press, 2003, 559pp.

[CGG] J. Carlson, M. Green, and P. Griffiths, Variations of Hodge structures considered as an exterior differential system: Old and new results, *SIGMA* **5**, 087, 40 pp. http://www.emis.de/journals/SIGMA/2009/087/

[DMOS] P. Deligne, Hodge cycles on abelian varieties (Notes by J. S. Milne), in *Hodge Cycles, Motives, and Shimura Varieties*, *Lect. Notes in Math.* **900**, Springer-Verlag, New York, 1982, 9–100.

[GGK] M. Green, P. Griffiths, and M. Kerr, Mumford-Tate and the geometry and arithmetic of period domains, to appear.

[KU] K. Kato and S. Usui, *Classifying Spaces of Degenerating Polarized Hodge Structures*, *Ann. of Math. Studies* **169**, Princeton Univ. Press, Princeton, NJ, 2009.

[Mo] B. Moonen, Notes on Mumford-Tate groups, preprint, 1999.

[Mu] D. Mumford, A note of Shimura's paper "Discontinuous groups and abelian varieties, *Math. Ann.* **181** (1969), 345–351.

[PS] C. Peters and J. Steenbrink, *Mixed Hodge Structures*, *Ergeb. Math. Grenzgebiete* **52**, Springer-Verlag, New York, 2008, 470 pp.

[Sc] W. Schmid, Variation of Hodge structure: The singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.

Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095
E-mail address: mlg@ipam.ucla.edu

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540
E-mail address: pg@ias.edu

Department of Mathematical Sciences, University of Durham, Science Laboratories, South Rd., Durham DH1 3LE, United Kingdom
E-mail address: matthew.kerr@durham.ac.uk