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Mumford-Tate groups and the classification of Hodge structures†

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I. Hodge structures

II. Mumford-Tate groups

talk will illustrate

III. "Complex multiplication" Hodge structures

arithmetic aspects

IV. Variations of Hodge structure and Mumford-Tate domains

algebro-geometric aspects

V. Mumford-Tate Lie algebras

representation theoretic aspects

I. Hodge structures

(I.A) Definitions

Given $V = \mathbb{Q}$-vector space, $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$.

Let $\mathbb{U} \subset SL(2)$ be the algebraic subgroup defined by

$$\mathbb{U}(K) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \;\middle|\; \begin{matrix} \alpha^2+\beta^2=1 \\ \alpha,\beta \in K \end{matrix} \right\} \quad \left(K \subset \mathbb{C} \text{ any subfield} \right).$$

One has $\mathbb{U}(\mathbb{C}) \cong \mathbb{C}^* \supset$ "unit S^1" $\cong \mathbb{U}(\mathbb{R})$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mapsto \alpha + i\beta =: z$$

A Hodge structure (HS) of weight n on V is:

(i) a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ where $\overline{V^{p,q}} = V^{q,p}$

or

(ii) a decreasing filtration $\cdots \supset F^p V_{\mathbb{C}} \supset F^{p+1} V_{\mathbb{C}} \supset \cdots$

satisfying $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{n-p+1} V_{\mathbb{C}}}$

or

(iii) a real representation $\varphi : \mathbb{U} \to SL(V)$

satisfying $\varphi(-1) = (-1)^n$.

(i) ↔ (ii): $V^{p,q} = F^p \cap \overline{F^q}$, $F^p = \bigoplus_{p' \geq p} V^{p',q'}$

(i) ↔ (iii): $\varphi(z)$ = multiplication by z^{p-q} on $V^{p,q}$

Write $h^{p,q} := \dim_{\mathbb{C}} V^{p,q}$ for the Hodge numbers.

(I.B) Polarizations

Given (V, φ) HS of weight n.

Let $Q : V \otimes V \to \mathbb{Q}$ be a nondegenerate bilinear form :

- symmetric if n even,
- alternating if n odd.

Q is a <u>polarization</u> of (V, φ) $\overset{\text{def.}}{\Longleftrightarrow}$ the Hodge-Rieman bilinear relations

(I) $Q(V^{p,q}, V^{p',q'}) = 0$ unless $(q,p) = (p',q')$

(II) $\sqrt{-1}^{\,p-q} Q(v, \bar{v}) > 0$ for any nonzero $v \in V^{p,q}$

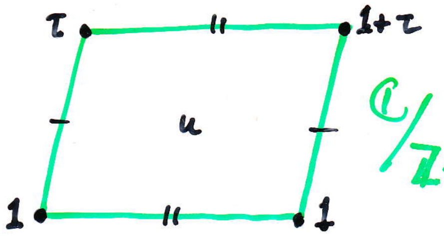
hold.

We say (V, φ, Q) is a <u>polarized Hodge structure</u> (PHS) of weight n.

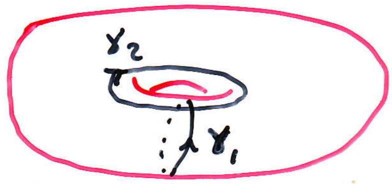
The category of PHS is semisimple.

(I.C) link to Algebraic Geometry

Ex // $\begin{pmatrix} \text{compact complex} \\ \text{1 torus} \end{pmatrix} \xrightarrow[\wp\text{-function}]{\text{Weierstraß}} \begin{pmatrix} \text{genus 1 [elliptic]} \\ \text{algebraic curve} \end{pmatrix}$



$$\mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle \xrightarrow{(\wp,\wp')} E_\tau$$



$$V := H^1(E_\tau, \mathbb{Q}) = \mathbb{Q}\langle \gamma_1^*, \gamma_2^* \rangle$$

$$V_{\mathbb{C}} = H^1(E_\tau, \mathbb{C}) = \underbrace{\Omega^1(E_\tau)}_{V^{1,0}} \oplus \underbrace{\overline{\Omega^1(E_\tau)}}_{V^{0,1}} = \mathbb{C}\langle \overset{du}{\overset{\shortparallel}{\omega}} \rangle \oplus \mathbb{C}\langle \bar{\omega} \rangle$$

In $V_{\mathbb{C}}$, $\quad \omega = \gamma_1^* + \tau \gamma_2^*$.

$Q(\gamma_i^*, \gamma_j^*) := (-1)^j \delta_{ij}$ gives a polarization $\iff \tau \in \mathfrak{h}$. //

This yields the first of the 1-1 correspondences:

(i) weight 1 rank 2 PHS $\longleftrightarrow$ elliptic curves
(ii) weight 1 HS $\longleftrightarrow$ compact complex tori
(iii) weight 1 PHS $\longleftrightarrow$ abelian varieties

} up to isogeny

<u>Regarding (ii) vs. (iii)</u>: a polarization yields an ample line bundle on the torus, hence (by Kodaira / theta functions) a projective embedding. For rank > 2, there do exist <u>nonpolarizable</u> weight 1 HS $\longleftrightarrow$ <u>nonalgebraic</u> compact complex tori.

More generally, the n^{th} cohomology groups of a smooth projective algebraic D-fold $X/\mathbb{C}$ yields a HS of weight n. To see this, let

$$\left.\begin{array}{l} A^m(X) := \text{smooth } \mathbb{C}\text{-valued differential } m\text{-forms} \\ C_m(X) := \text{piecewise } C^\infty \text{ chains of real dimension } m \end{array}\right\} \text{on } X^{an}$$

Define $H_n(X,\mathbb{Z}) := H_n(C_\bullet(X), \underset{\text{boundary operator}}{\partial})$,

$H^n(X,\mathbb{A}) := \operatorname{Hom}(H_n(X,\mathbb{Z}), \underset{\text{subfield of } \mathbb{C}}{\mathbb{A}})$;

then $H^n(A^\bullet(X), d) \xrightarrow{\cong} H^n(X,\mathbb{C})$. [de Rham theorem]

$$\omega \longmapsto \int_{(\cdot)} \omega$$

Next put $A^{p,q}(X) :=$ C^∞ forms of degree $(p+q)$, which in local holomorphic coordinates $(z_1,\dots,z_D)$ take the form

$$\sum_{\substack{I,J\subset\{1,\dots,N\} \\ |I|=p,\ |J|=q}} \underbrace{f_{IJ}(z)}_{(C^\infty \text{ function})}\, dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q}$$

Denote by $H^{p,q}(X)$ the image of

$$\ker(d)\subset A^{p,q}(X)$$

in $H^{p+q}(X,\mathbb{C})$. Clearly $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Hodge Theorem: $H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$

valid more generally for compact Kähler manifolds

We can polarize the resulting HS on $H^n(X)$ using the Hard Lefschetz Theorem in general. In case $n \leq D$ and $H^n(X)$ is primitive, then for $[\omega], [\eta] \in H^n(X,\mathbb{C})$

$$Q([\omega],[\eta]) := \int_Y \omega \wedge \eta$$

where Y is the intersection of X with $(D-n)$ generic hypersurfaces.

The algebraic cycles of codimension p on X are elements of the free abelian group on irreducible subvarieties of X of (complex) codimension p:

$$Z = \sum q_i Z_i \in Z^p(X), \quad q_i \in \mathbb{Q}.$$

The fundamental class map

$$cl : Z^p(X) \longrightarrow Hg^p(X) := H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X,\mathbb{C})$$

is defined by Poincaré duality, and the $$$$ Hodge Conjecture (HC) asserts that cl is surjective: i.e., that

all rational (p,p) classes are algebraic.

(I.D) link to Representation Theory

If a HS (V, φ) is polarized by Q, then

$$\varphi : \mathbb{U} \to \mathrm{Aut}(V, Q) \cong \begin{cases} SO\left(\sum_{p \text{ even}} h^{p,q}, \sum_{p \text{ odd}} h^{p,q}\right) & , n \text{ even} \\ Sp\left(\sum h^{p,q} =: r\right) & , n \text{ odd} \end{cases}$$

An algebraic group M over a field $k (\subset \mathbb{C})$ is an algebraic variety $/k$ together with k-morphisms of varieties inducing group structure on the K-points $M(K)$ for any field extension K/k.

Let M be a $\mathbb{Q}$-algebraic group.

A <u>Hodge representation</u> of M is

- a $\mathbb{Q}$-vector space V,
- a nondegenerate symmetric or alternating bilinear form Q on V,
- a faithful representation $\rho : M \to \mathrm{Aut}(V, Q)$ (defined $/\mathbb{Q}$), and
- a cocharacter $\tilde{\varphi} : \mathbb{U} \to M$ (defined $/\mathbb{R}$),

such that $(V, Q, \rho \circ \tilde{\varphi})$ is a PHS.

This will become the basis for connections between HS and Lie theory.

(I.E) link to Arithmetic

A CM field is a totally imaginary quadratic extension of a totally real number field.

Ex// $\mathbb{Q}(\sqrt{-d})$; $\mathbb{Q}(\sqrt{-2},\sqrt{-3})$; $\mathbb{Q}(\zeta_p := e^{\frac{2\pi i}{p}})$ //

Let (V,φ) be a polarizable HS. The (semisimple) endomorphism algebra $\mathcal{E}_\varphi := \mathrm{End}(V,\varphi)$ consists of $\mathbb{Q}$-linear transformations intertwining φ (or "respecting $V^{p,q}$'s").

We say (V,φ) has a K-multiplication if there is an embedding of a # field $K \underset{\imath}{\hookrightarrow} \mathcal{E}_\varphi$. Necessarily $[K:\mathbb{Q}] \leq \dim_{\mathbb{Q}} V$; if equality holds, K must be a CM field.

If $(V,\varphi) = (V_1^{\oplus m_1} \oplus \cdots \oplus V_\ell^{\oplus m_\ell}, \varphi_1^{\oplus m_1} \oplus \cdots \oplus \varphi_\ell^{\oplus m_\ell})$ is the decomposition into simple factors, then $\mathcal{E}_\varphi = \mathrm{Mat}_{m_1}(\mathcal{E}_{\varphi_1}) \times \cdots \times \mathrm{Mat}_{m_\ell}(\mathcal{E}_{\varphi_\ell})$ where $\mathcal{E}_{\varphi_i}$ are division algebras of the following types:

"Albert classification"

(I) totally real # field

(II) / (III) $\dfrac{\text{quaternion algebra}}{\text{totally real \# field}}$ which $\left\{\begin{array}{l} \text{splits} \ (\cong M_2(\mathbb{R})) \\ \text{doesn't split} \ (\cong \mathbb{H}) \end{array}\right\}$ under real embeddings

(IV) CM field or division algebra over one

II. Mumford-Tate groups

(II.A) Motivation

MT groups are the natural ($\mathbb{Q}$-algebraic) symmetry groups of HS, in the sense of stabilizing Hodge substructures, commuting with endomorphisms, etc. — hence <u>shrinking</u> as a HS becomes more special.

They were introduced for abelian varieties by Mumford (1966) to

AG • give a Hodge-theoretic characterization of <u>families of abelian varieties</u> (over Hermitian symmetric domains) studied by Kuga and Shimura, particularly those <u>not</u> characterized by their algebra of (holomorphic) endomorphisms.

NT • "generalize" the <u>main theorem of Complex Multiplication</u> (Shimura-Taniyama) to more general abelian varieties

RT • [in retrospect] reduce the study of the HC for abelian varieties to invariant theory of classical Lie groups

Deligne (1972) generalized the definition to Hodge structures.

(II.B) Definitions

Let (V, φ, Q) be a PHS of weight n, and put $G := \mathrm{Aut}(V, Q)$ ($\mathbb{Q}$-algebraic group).

Then φ induces on the tensor spaces

$$T^{a,b}V := V^{\otimes a} \otimes (V^\vee)^{\otimes b}$$

a HS of weight $(a-b)n$. The Hodge tensor spaces are:

$$Hg^{a,b}V := \begin{cases} T^{a,b}V \cap (T^{a,b}V_{\mathbb{C}})^{m,m} & , (a-b)n = 2m \\ 0 & , (a-b)n \text{ odd} \end{cases}$$

$$\boxed{\begin{array}{l} \text{The MT group } M_\varphi \text{ of } (V,\varphi) \text{ is given by} \\ \text{(i) the (largest) algebraic subgroup of } G \text{ fixing all } Hg^{a,b}V \text{ pointwise} \\ \text{or} \\ \text{(ii) the smallest } \mathbb{Q}\text{-algebraic subgroup of } G \text{ whose} \\ \text{real points contain } \varphi(S^1) \text{ (or whose complex points contain } \varphi(\mathbb{C}^*)) \end{array}}$$

M_φ is $\begin{cases} \underline{\text{Connected}}: M(\mathbb{C}) \text{ is a connected Lie group} \\ \underline{\text{reductive}}: \text{product of simple algebraic groups and algebraic tori (complex pts.} \cong (\mathbb{C}^*)^s) \end{cases}$

$\Longrightarrow$ linear representations completely reducible.

(III.C) Example

Let (V, φ, Q) be the PHS on $H^1(E_\tau)$ $(\tau \in \mathfrak{h})$. (cf. §I.C)

$M_\varphi \underset{(ii)}{:=}$ $\mathbb{Q}$-closure of $\varphi(\mathbb{C}^*)$ = subgroup of G generated by $\left\{ \bigcup_{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})} {}^{\sigma}(\varphi(\mathbb{C}^*)) \right\}$

To get a feel for this, consider the basis $\Omega = \{\omega, \bar{\omega}\}$ of $V_{\mathbb{C}}$.

If we rig σ to fix generic $z \in \mathbb{C}^*$ and entries of $[\sigma]_\Omega$, then

$$[{}^{\sigma}\varphi(z)]_\Omega = [\sigma]_\Omega [\varphi(z)]_\Omega [\sigma^{-1}]_\Omega \quad \text{where } [\varphi(z)]_\Omega = \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}.$$

$\boxed{\text{quadratic } \tau:}$ $\tau = \sqrt{-d}$ $(d \in \mathbb{N})$, $\sigma : \sqrt{-d} \leftrightarrow -\sqrt{-d}$

$$\Rightarrow [{}^{\sigma}\varphi(z)] = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} z^{-1} & \\ & z \end{pmatrix} = [\varphi(z^{-1})]_\Omega$$

$\boxed{\text{Cubic } \tau:}$ $\tau = \sqrt[3]{2}\,\zeta_3$, $\sigma : \sqrt[3]{2} \to \sqrt[3]{2}\,\zeta_3 \to \sqrt[3]{2}\,\zeta_3^2 \to \sqrt[3]{2}$, fixes ζ_3

$$\Rightarrow [{}^{\sigma}\varphi(z)] = \begin{pmatrix} 0 & -\zeta \\ 1 & -\zeta^2 \end{pmatrix}\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}\begin{pmatrix} -\zeta & 1 \\ -\zeta^2 & 0 \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ \zeta(z - z^{-1}) & z \end{pmatrix} \notin \mathrm{im}(\varphi)$$

More generally: • $[\mathbb{Q}(\tau):\mathbb{Q}] \neq 2 \Rightarrow M_\varphi = G = SL(2)$ and $\mathcal{E}_\varphi \cong \mathbb{Q}$

• $[\mathbb{Q}(\tau):\mathbb{Q}] = 2 \Rightarrow M_\varphi = \mathrm{im}(\varphi)$ = algebraic 1-torus, with $M_\varphi(\mathbb{Q}) \cong \mathbb{Q}(\tau)^*$ and $\mathcal{E}_\varphi \cong \mathbb{Q}(\tau)$.

We say E_τ has <u>Complex Multiplication</u> (CM) by $\mathbb{Q}(\tau)$.

(II.D) Hodge Conjecture (application #1)

For any PHS, $\mathcal{E}_\varphi \cong Hg^{1,1}V$ and $Q \in Hg^{0,2}V$.

That M_φ "fixes" $Hg^{1,1}V$ translates to commuting with endomorphisms:

$$M_\varphi \subseteq G^{\mathcal{E}_\varphi}.$$

When this is an equality, M_φ (and the PHS) is said to be <u>nondegenerate</u>.

Now let (V, φ, Q) be the PHS on H^1 of an abelian variety A. The cohomology groups of self-products of A are built out of the $T^{a,b}V$, and the Hodge classes out of the $Hg^{a,b}V$. Moreover, by the Lefschetz (1,1) theorem, $\mathcal{E}_\varphi$ and Q correspond to algebraic cycles. If the latter <u>generate</u> $\bigoplus_{a,b} Hg^{a,b}V =: Hg^{\bullet,\bullet}V$, then $Hg^*(A \times \cdots \times A)$ are also algebraic. Since M_φ controls $Hg^{\bullet,\bullet}V$, nondegeneracy is part of making this happen:

<u>Theorem</u> [Murty/Hazama 1984]: If $H^1(A)$ is nondegenerate <u>and</u> no simple factor of $\mathcal{E}_\varphi$ ($= \mathrm{End}(A) \otimes \mathbb{Q}$) is of type <u>III</u>, then <u>the HC holds for all powers of A</u>.

Ex//
- A simple of prime dimension
- A arbitrary product of elliptic curves

(II.E) Absolute Hodge Conjecture

(application #2)

Let X be an algebraic variety $/\mathbb{C}$. The action of $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on coefficients of its defining equations yields ${}^{\sigma}X$, and its action on algebraic differential forms yields

$$\sigma_* : H^{m,m}(X,\mathbb{C}) \to H^{m,m}({}^{\sigma}X,\mathbb{C}).$$

Recalling $Hg^m(X) := H^{m,m}(X,\mathbb{C}) \cap H^{2m}(X,\mathbb{Q})$, define the space of <u>absolute Hodge classes</u>

$$AHg^m(X) := \left\{ \xi \in Hg^m(X) \;\middle|\; \begin{array}{c} \sigma_*(\xi) \in Hg^m({}^{\sigma}X) \\ \forall\, \sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \end{array} \right\}.$$

We have

$$\overbrace{cl(Z^m(X)) \subset AHg^m(X) \underbrace{\subset}_{\text{"AHC"}} Hg^m(X)}^{\text{HC}}.$$

Theorem [Deligne, 1982]: AHC is true for all abelian varieties.

The (long and deep) proof makes extensive use of MT groups, the theory of CM (for abelian varieties), and the theory of Shimura varieties underlying families of abelian varieties.

There are many other applications of MT groups, e.g. C. Schoen's criterion for a variety not to be dominated by products of curves.

III. Complex Multiplication (CM) Hodge structures

(III.A) Definition

[working modulo trivial HS in even weight]

A HS (V, φ) is said to be "CM" if

M_φ is abelian,

i.e. an algebraic torus. CM-HS have "lots" of endomorphisms: for one thing $\mathcal{E}_\varphi \supset M_\varphi(\mathbb{Q})$; and if (V, φ) is simple and polarized, it has multiplication by a CM field L of degree $\dim V$. (as in §I.E)

Moreover, CM-HS turn out to be the natural base points for moduli of PHS.

An abelian variety A is said to be CM if $H^1(A)$ is. The eigenvalues of the action of $L \subset \mathcal{E}_\varphi = \mathrm{End}(A) \otimes \mathbb{Q}$ on T_0A yield a choice $\Theta = \{\theta_1, \ldots, \theta_g\}$ of $g = \dim A$ pairwise non-conjugate complex embeddings of L, called a CM type. The main theorem of CM sets up a beautiful correspondence between ideal class theory of L and Galois action on torsion points of A, with origins in Kronecker's "Jugendtraum" on abelian extensions of imaginary quadratic fields.

(III. B) Construction

Let L be a CM field of degree $2g$, with embeddings

$$\mathrm{Hom}(L,\mathbb{C}) = \{\theta_1,\dots,\theta_g\,;\,\bar{\theta}_1,\dots,\bar{\theta}_g\}.$$

An <u>n-orientation</u> of L is a partition

$$\mathrm{Hom}(L,\mathbb{C}) = \coprod_{p+q=n} \pi^{p,q} \quad , \quad \overline{\pi^{p,q}} = \pi^{q,p}.$$

Associated to an n-oriented CM field is a CM-HS (L, φ^n_π) of weight n:

$\left(\begin{array}{c}2g\text{-dim'l}\\ \mathbb{Q}\text{-vector space}\end{array}\right)$ $V := L$ ↺ action of mult. by $\ell \in L$

$$V_{\mathbb{C}} \cong L \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\theta\in\mathrm{Hom}(L,\mathbb{C})} E_\theta(V_{\mathbb{C}})$$

eigenspace on which ℓ acts as mult. by $\theta(\ell)$

$$V^{p,q} := \bigoplus_{\theta\in\pi^{p,q}} E_\theta(V_{\mathbb{C}})$$

This turns out to be polarizable. Moreover, any polarizable CM-HS decomposes as a direct sum of (L, φ^n_π)'s.

Ex// $(n=1)$ Given a CM type $\Theta = \{\theta_1,\dots,\theta_g\}$, take $\pi^{1,0} = \Theta$, $\pi^{0,1} = \overline{\Theta}$ and denote the resulting weight-1 CM-HS by (L, φ^1_Θ). //

(III. C) Example

Here are two ways to get the weight-1 CM-HS

$$(*) \qquad \left(\mathbb{Q}(\zeta_5),\ \varphi^1_{\{\theta_1,\theta_2\}}\right)$$

from algebraic geometry.

[NOTATION: $\theta_j : \mathbb{Q}(\zeta_5) \hookrightarrow \mathbb{C}$, $j=1,2,3,4$, sends $\zeta_5 \mapsto \zeta_5^j$.]

① Associated to a CM type $(L, \Theta = \{\theta_1, \ldots, \theta_g\})$ is an algebraic complex torus

$$A_{(L,\Theta)} := \mathbb{C}^g \Big/ \mathbb{Z}\left\langle \begin{pmatrix} \theta_1(\gamma_1) \\ \vdots \\ \theta_g(\gamma_1) \end{pmatrix}, \ldots, \begin{pmatrix} \theta_1(\gamma_{2g}) \\ \vdots \\ \theta_g(\gamma_{2g}) \end{pmatrix} \right\rangle \qquad (\gamma_1, \ldots, \gamma_{2g} \text{ basis of } \mathcal{O}_L)$$

with $H^1(A) = (L, \varphi^1_\Theta)$. To get $(*)$, take the abelian surface $A = \mathbb{C}^2 / \mathbb{Z}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \zeta_5 \\ \zeta_5^2 \end{pmatrix}, \begin{pmatrix} \zeta_5^2 \\ \zeta_5^4 \end{pmatrix}, \begin{pmatrix} \zeta_5^3 \\ \zeta_5 \end{pmatrix} \right\rangle$.

② The curve $C = \{y^5 = x(1-x)\} \subset \mathbb{C}^2$ has an automorphism $(x,y) \longmapsto (x, \zeta_5 y)$ which lifts to the desingularization $\tilde{C}$ of its closure in $\mathbb{P}^2$. Its action on $\Omega^1(\tilde{C}) = \mathbb{C}\left\langle \frac{dx}{y^4}, \frac{dx}{y^3} \right\rangle = H^{1,0}(\tilde{C})$ has eigenvalues $\theta_1(\zeta_5), \theta_2(\zeta_5)$, indicating that the HS on $H^1(\tilde{C})$ is $(*)$.

In fact, $A = J^1(\tilde{C})$ up to isogeny.

(III.D) CM-HS are motivic

Theorem [Abdulali, 2006]: Any polarizable CM-HS is of algebro-geometric origin.

Explanation: let (L, Π) be an n-oriented CM field,

$\Theta(\Pi) :=$ set of CM types refining Π,

$$A := \bigtimes_{\Theta \in \Theta(\Pi)} A_{(L,\Theta)}^{\times m_\Theta} \quad (= \text{abelian variety})$$

Abdulali's result is that $(L, \varphi^n_\Pi) \underset{\text{sub-HS}}{\subset} H^n(A)$.

Ex// $\left.\begin{array}{l} L = \mathbb{Q}(\zeta_5) \\ \Pi^{3,0} = \{\theta_1\} \\ \Pi^{2,1} = \{\theta_2\} \\ \Pi^{1,2} = \{\theta_3\} \\ \Pi^{0,3} = \{\theta_4\} \end{array}\right\} \longrightarrow (\mathbb{Q}(\zeta_5), \varphi^3_\Pi)$ weight 3, Hodge #'s $(1,1,1,1)$

$\bigcap$ sub-HS

$$H^3\Big(\underbrace{\big(A_{(L,\varphi^1_{\{\theta_1,\theta_2\}})}\big)^{\times 2}}_{\text{Griffiths intermediate Jacobian}} \times \underbrace{A_{(L,\varphi^1_{\{\theta_1,\theta_3\}})}}_{\text{Weil intermediate Jacobian}}\Big)$$ //

Problem: The HC holds for nondegenerate CM abelian varieties. If (L, φ^n_Π) is nondegenerate, are its Hodge tensors in $H^*(A)$ algebraic? (Is A nondegenerate?)

(III.E) Rank of MT

Nondegeneracy of a CM-HS means MT is "as big as possible".

If (L, φ^n_Π) is irreducible, one wants $\dim(M_{\varphi^n_\Pi}) = \frac{1}{2}[L:\mathbb{Q}]$.

So to investigate the "Problem" one needs a formula for dim(MT).

Assume $L/\mathbb{Q}$ is Galois and pick an embedding $\theta : L \hookrightarrow \mathbb{C}$,

so

$$\mathcal{G} := \mathrm{Gal}(L/\mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}(L, \mathbb{C}) = \coprod_{p+q=n} \Pi^{p,q}$$

$$\sigma \longmapsto \theta \circ \sigma$$

defines a function $p : \mathcal{G} \to \mathbb{Z}$. Denote "rank -1" of

$$\mathbb{Z}[\mathcal{G}] \longrightarrow \mathbb{Z}[\mathcal{G}]$$

$$[g] \longmapsto \sum_{\sigma \in \mathcal{G}} p(\sigma)[\sigma g]$$

by $R(L, \Pi)$. Generalizing work of Kubota (1965) and Ribet (1980) on the abelian variety case, we have the

$$\boxed{\textbf{Theorem } [GGK, 2010]: \ \dim(M_{\varphi^n_\Pi}) = R(L, \Pi)}$$

Ex// With $L = \mathbb{Q}(\zeta_{32})$, there is a 3-orientation Π and CM type $\Theta \in \Theta(\Pi)$ such that $R(L, \Pi) = 8$ while $R(L, \Theta) = 6$. So Abdulali's abelian variety need not be nondegenerate when the CM-HS is.

IV. Variations of HS and MT domains

(IV.A) Period domains

These are the classifying spaces for PHS of fixed weight n and Hodge #'s $\underline{h} = \{h^{p,q}\}_{p+q=n}$.

assume at least 2 nonzero $h^{p,q}$'s.

Put $r := \sum h^{p,q}$, $f^p := \sum_{p' \geq p} h^{p',q'}$

V = $\mathbb{Q}$-vector space of dimension r

Q = nondegenerate $(-1)^n$-symmetric bilinear form on V

$G = \mathrm{Aut}(V,Q)$ ($\mathbb{Q}$-algebraic group)

$$D_{\underline{h}} := \left\{ \begin{array}{c} \mathbb{R}\text{-cocharacters} \\ \varphi : \mathbb{U} \to G \end{array} \middle| \begin{array}{c} (V,\varphi,Q) \text{ is a PHS} \\ \text{w/ Hodge \#'s } \underline{h} \end{array} \right\}$$

is the period domain ("HS of type $\underline{h}$")

open subset $\cap$

$$\check{D}_{\underline{h}} := \left\{ \begin{array}{c} \text{descending} \\ \text{filtrations} \\ F^\bullet \text{ on } V_{\mathbb{C}} \end{array} \middle| \begin{array}{c} \dim(F^p) = f^p \\ Q(F^p, F^{n-p+1}) = 0 \end{array} (\forall p) \right\}$$

is the compact dual

projective subvariety $\cap$

$$\Pi_p \,\mathrm{Grass}(f^p, V_{\mathbb{C}})$$

A choice of reference PHS φ allows us to express both $D_{\underline{h}}$ and $\check{D}_{\underline{h}}$ as homogeneous spaces:

$$D_{\underline{h}} \xleftarrow{\cong} G(\mathbb{R})/H_\varphi \qquad \check{D}_{\underline{h}} \xleftarrow{\cong} G(\mathbb{C})/P_\varphi$$

$$g\varphi g^{-1} \longleftarrow\!\shortmid\, g \qquad gF^\bullet_\varphi \longleftarrow\!\shortmid\, g$$

compact "isotropy group" of φ

parabolic subgroup with $P_\varphi \cap G(\mathbb{R}) = H_\varphi$

(IV.B) Variations of Hodge structure (VHS)

Let $\{X_s\}_{s\in\mathcal{S}}$ be a smooth algebraic family of smooth projective varieties. We may identify $H^n(X_s^{an},\mathbb{Q}) \cong$ fixed V (up to action of monodromy), and consider the variation of the structures on $V_{\mathbb{C}}$ induced by the Hodge theorem:

- $V_s^{p,q}$ varies non holomorphically
- $F_s^p V_{\mathbb{C}}$ varies holomorphically, & satisfies $\boxed{dF_s^p \subset \Omega^1_{\mathcal{S}} \otimes F_s^{p-1}}$ infinitesimal period relation (IPR)

Abstractifying this,

a polarized VHS of type $\underline{h}$ over a $\mathbb{C}$-algebraic manifold $\mathcal{S}$ is (equivalently):

(i) a $\mathbb{Q}$-local system $\mathcal{H} \to \mathcal{S}$ with bilinear form Q and filtration $\mathcal{F}^\bullet$ of $\mathcal{H}_{\mathcal{O}} := \mathcal{H} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{S}}$ by holomorphic sub-bundles, such that

- the fibres are PHS of type $\underline{h}$, and
- the flat connection annihilating $\mathcal{H}$ satisfies the IPR

or

(ii) a locally liftable holomorphic mapping

$$\Phi : \mathcal{S} \longrightarrow \Gamma\backslash D_{\underline{h}} \qquad (\Gamma \leq G(\mathbb{R}) \text{ discrete sgp.})$$

with $\underbrace{\text{image an integral manifold}}_{\Phi^*(\mathcal{I})=0}$ of a G-invariant EDS $\mathcal{I} \subset \Omega^\bullet_{D_{\underline{h}}}$ related to the IPR

For $\underline{h} = (a,a)$ or $(1,b,1)$, $\mathcal{I} = 0$ ("classical case")

Otherwise, $\mathcal{I} \neq 0$.

(IV.C) Mumford-Tate domains (introduced in [GGK, Bollettino dell'UMI] (2010))

Let $\varphi \in D_{\underline{h}}$ be a PHS, $M \leq G$ a (connected/reductive) $\mathbb{Q}$-algebraic group. For each φ' in the orbit

$$M(\mathbb{R}).\varphi = \{(V, \mu\varphi\mu^{-1}; Q) \mid \mu \in M(\mathbb{R})\} \subseteq D_{\underline{h}},$$

the MT group $M_{\varphi'} \leq M$. If this is equality for some φ', then

$$M(\mathbb{R}).\varphi =: D_{M,\varphi}$$ (drop if φ understood/implied)

is called a MT domain. (Clearly $D_{M_\varphi,\varphi}$ is one.)

Key facts: 1 MT domains are homogeneous complex manifolds:

$$D_{M,\varphi} = M(\mathbb{R})/M(\mathbb{R}) \cap H_\varphi \underset{\text{open subset}}{\subset} M(\mathbb{C})/M(\mathbb{C}) \cap P_\varphi = \underbrace{\check{D}_{M,\varphi}}_{\text{proj. variety defined}/\#\text{ field}}$$

2 Every MT domain contains a CM-HS, making these the simplest "base points"

3 If we define the Noether-Lefschetz locus $NL_M \subset D_{\underline{h}}$ to be the set of PHS with MT group $\subseteq M$, then the connected components of NL_M and D_M through φ are equal.

4 Period domains are MT domains: in fact, $G\,(=Aut(V,Q))$ is the MT group of a "generic" $\varphi \in D_{\underline{h}}$.

A MT domain is called classical iff the pullback of the IPR $\mathcal{I} \subset \Omega^{\bullet}_{D_{\underline{h}}}$ to $\Omega^{\bullet}_{D_M}$ is zero. In this case,

- $M(\mathbb{R}) \cap H_\varphi \subset M(\mathbb{R})$ is a maximal compact subgroup
- D_M is a Hermitian symmetric domain
- if $\Gamma \subset M(\mathbb{R})$ is an arithmetic subgroup, $\Gamma \backslash D_M$ is an algebraic ("Shimura") variety (These typically parametrize families of algebraic varieties.)
- the nonabelian simple factors of $M(\mathbb{R})$ are of classical types A, B, C, D (i.e. SU, SO, Sp)

Apart from exotic cases, if D_M is nonclassical then the first 3 items are false. As we shall see the last point can also fail. Moreover, the motivic $\varphi \in D_M$ are a dense measure 0 subset (incl. all CM-HS).

Problem: Find a single explicit PHS not of algebro-geometric origin.

IV. D) Structure Theorem for VHS (UP TO FINITE DATA!)

Associated to a polarized VHS $\Phi : \mathcal{S} \to \Gamma \backslash D_{\underline{h}}$ is the <u>monodromy representation</u> $\rho : \pi_1(\mathcal{S}) \to \Gamma$; assume $\Gamma = \rho(\pi_1(\mathcal{S}))$ and preserves a lattice in V.

Let $M_\Gamma \subset G$ be the smallest $\mathbb{Q}$-algebraic group containing Γ.

Take a very general point $\eta \in \mathcal{S}$, and a lift $\varphi_\eta \in D_{\underline{h}}$ of $\Phi(\eta)$. Then

$$M_\Phi := M_{\varphi_\eta} \qquad (\text{"}\underline{\text{MT group of } \Phi}\text{"})$$

is independent of the lift, with decomposition into $\mathbb{Q}$-algebraic groups

$$M_\Phi = M_\Gamma \times M' = \overbrace{(M_1 \times \cdots \times M_k)}^{\mathbb{Q}\text{-simple}} \times M'.$$

Theorem [GGK, 2010]: The orbits

$$M_i(\mathbb{R}).\varphi =: D_i \quad , \quad M'(\mathbb{R}).\varphi =: D'$$

are homogeneous complex manifolds, and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$ where $\overline{\Gamma_i} = M_i$. The VHS factors

$$\mathcal{S} \longrightarrow \Gamma_1 \backslash D_1 \times \cdots \times \Gamma_k \backslash D_k \times D' \cong \Gamma \backslash D_{M_\Phi} \subset \Gamma \backslash D_{\underline{h}}$$

with D'-factor <u>constant</u>: i.e., "Γ is as large as possible".

The image $\Phi(\mathcal{S}) \subset \Gamma \backslash D_{M_\Phi}$ has canonically the structure of a <u>quasi-projective algebraic variety</u>, <u>even when</u> (as in the non-classical case) <u>$\Gamma \backslash D_{M_\Phi}$ does not</u>.

(IV. E) Examples: VHS from families of Calabi–Yau 3-folds — $\left\{\begin{array}{l}\text{smooth projective w./} \\ \dim H^{3,0}=1,\ \dim H^{2,0}=\dim H^{1,0}=0\end{array}\right.$

Ex// The VHS $\Phi : \mathbb{P}^1 \setminus \{0,1,\infty\} \to \Gamma \backslash D_{(1,1,1,1)}$ nonclassical, dimension $=4$

given by $\{H^3(X_t)\}_{t \in \mathbb{P}^1 \setminus \{0,1,\infty\}}$ of the mirror quintic family of CY's, has

- $M_\Phi = Sp(4) = G$
- Φ is maximal: i.e. $\Phi(\mathbb{P}^1 \setminus \{0,1,\infty\})$ is a maximal integral submanifold of $\mathcal{I}$
- $\Phi(0)$ is a CM point: $H^3(X_0) \cong (\mathbb{Q}(\zeta_5), \varphi^3_\pi)$ (cf. §III.D) //

In general, the image of even a maximal & motivic VHS Φ need not contain any CM points. For string-theorists in search of "mirror pairs" of CM CY's, the following generalization of the André–Oort Conjecture suggests looking at types of "CY-HS" parametrized by classical MT domains:

Expectation: The Zariski closure of the set of CM-HS in $\Phi(S)$ is a union of Shimura varieties.

Ex// Let $K = \mathbb{Q}(\sqrt{-d})$, and consider weight 3 PHS of type $(1,a,a,1)$ with a K-multiplication having $\left\{\begin{array}{l} V^{3,0} \oplus V^{2,1} \\ V^{1,2} \oplus V^{0,3}\end{array}\right.$ as eigenspaces.

- The MT group of a general such PHS is $M = SU(a,1)$
- The classifying space of such PHS is D_M (classical), a complex ball.
- In a maximal VHS of this type, the CM points are dense.

V. Mumford-Tate Lie algebras

(V.A) Two problems

① What are the possible $\underbrace{\text{MT groups for } \varphi \in D_{\underline{h}}}_{\text{OR: MT subdomains of } D_{\underline{h}}}$?

② When is a given $\mathbb{Q}$-algebraic group the MT group of some PHS ?

Both problems require us to look at Lie algebras and their representations. To see the context:

fix a period domain $D_{\underline{h}}$ and PHS (V, φ, Q) $\ni$ & put $G := Aut(V, Q)$.

Then on $\mathfrak{g} := Lie(G) \subset End(V)$,

$$\underline{Ad\,\varphi} : \mathbb{U} \to Aut(\mathfrak{g}, B)$$

$$z \mapsto \{X \mapsto \varphi(z) \cdot X \circ \varphi(z)^{-1}\}$$

Killing form

<u>induces a PHS of weight 0</u>, and

one identifies

$$T_\varphi D_{\underline{h}} = \mathfrak{g}^- := \bigoplus_{i<0} \mathfrak{g}^{(-i,i)}.$$

(V.B) Classification of MT subdomains (in $D_{\underline{h}}$)

One has isomorphisms of sets

$D_M := M(\mathbb{R}).\varphi \in$ **(i)** MT domains $D_M \, (\subset D_{\underline{h}})$ through φ

$\cong$

$M \in$ **(ii)** $\mathbb{Q}$-algebraic groups $M \supset M_\varphi$ such that $M = M_{\mu\varphi\mu^{-1}}$ for some $\mu \in M(\mathbb{R})$

$\cong$

$\mathfrak{m} := \mathrm{Lie}(M) \in$ **(iii)** sub-HS $\mathfrak{m} \subset \mathfrak{g}$, $\mathfrak{m} \supset \mathfrak{m}_\varphi$ which

(a) are also sub-Lie-algebras

&

(b) amongst such objects with given $\mathfrak{m}^-$, have smallest $\mathfrak{m}^{0,0}$

$T_\varphi D_M = \mathfrak{m}_{\mathbb{C}} \cap \mathfrak{g}^- =: \mathfrak{m}^-$

Theorem [GGK, 2010]:

How to apply this: Any D_M contains a CM-HS, and we have a classification of these (§ III.B). Using these as our base points φ:

- reduces verifying "$\mathfrak{m}/\mathbb{Q}$" to Galois theory of CM fields
- simplifies checking $\mathfrak{m} \supset \mathfrak{m}_\varphi$ by knowledge of $\dim(M_\varphi)$ (§ III.E)

(V.C) Example: weight 3, rank 4, $\underline{h} = (1,1,1,1)$ "mirror-quintic-type PHS"

It is easiest to describe the MT Noether-Lefschetz loci $\left(= \coprod_{\text{finite}} \overset{\text{connected}}{D_{M,\varphi_i}}\right)$ than the M's, D_M's, or $\mathfrak{m}$'s.

In the following classification, it is also understood that one gets countably infinite families in each case via translating by $G(\mathbb{Q})$ and/or changing the choice of CM field.

The MT-NL locus types with generic PHS irreducible are:

(I) $D_{\underline{h}}$

(II) PHS with multiplication by a real quadratic field

(III) PHS with multiplication by an $\underbrace{\text{imaginary quadratic field}}_{\text{IQF}}$

(IV) PHS with a certain Hodge 4-tensor

(V) CM-HS: $\mathrm{Gal}(L/\mathbb{Q}) \cong \begin{cases} \text{(i)} & \mathbb{Z}_4 \\ \text{(ii)} & \mathbb{Z}_2 \times \mathbb{Z}_2 \end{cases}$

MT-NL locus types with generic PHS $\underset{\text{HS}}{V} = \underset{(3,0)+(0,3)}{V'} \oplus \underset{(2,1)+(1,2)}{V''}$ reducible are:

(VI) V', V'' both general

(VII) (i) V' general, V'' CM (IQF); (ii) vice versa

(VIII) V', V'' CM by $\begin{cases} \text{(i) distinct} \\ \text{(ii) same} \end{cases}$ IQF's.

Amazing Fact: The existence of a type (IV) D_M through a type (VIII)(ii) CM point φ, is determined by a computation in the ideal class group of the IQF!!

Our approach gives fairly complete "generic" information about these types, e.g.:

MTNL locus type	# of D_M's in NL_M	$\dim(D_M)$	D_M classical?	$M(\mathbb{R}) \cong$?	$\mathcal{E}_\varphi \cong$?
(I)	1	4	no	$Sp(4)$	$\mathbb{Q}$
(II)	2	2	no	$SL(2)\times SL(2)$	$\mathbb{Q}(\sqrt{+d})$
(III) (i) / (ii)	2	1	yes / no	$U(1)\times SL(2)$	$\mathbb{Q}(\sqrt{-d})$
(IV)	1	1	yes	$SL(2)$	$\mathbb{Q}$
(V) (i) / (ii)	2	0	yes	$U(1)\times U(1)$	L (degree 4 CM field)
(VI)	2	2	no	$SL(2)\times SL(2)$	$\mathbb{Q}\times\mathbb{Q}$
(VII) (i) / (ii)	1 / 2	1	no / yes	$U(1)\times SL(2)$	$\mathbb{Q}\times\mathbb{Q}(\sqrt{-d})$
(VIII) (i) / (ii)	2 / 1	0	yes	$U(1)\times U(1)$ / $U(1)$	$\mathbb{Q}(\sqrt{-d'})\times\mathbb{Q}(\sqrt{-d''})$

While there are no nonclassical Lie groups here, the second column points out a striking difference to classical period domains, in which this column would consist of 1's.

Problems: (1) Use the above to classify the $\mathbb{Z}$-VHS in [Doran-Morgan, 2006].

(2) Carry out classification for higher rank CY-HS, and use the result to study moduli of K3-fibered CY's.

(V.D) The adjoint representation

Let M be a $\mathbb{Q}$-algebraic group.

A Hodge representation $\begin{cases} (V,Q) \\ \rho : M \to \mathrm{Aut}(V,Q) \\ \tilde{\varphi} : \mathbb{U} \to M \end{cases}$

is called a Mumford-Tate representation

if $M = M_{\rho\circ\tilde{\varphi}}$.

Proposition [GGK, 2010]: Assume M is $\mathbb{Q}$-simple.

Then

(a) the conjugate by a generic $\mu \in M(\mathbb{R})$ of a Hodge representation is a MT representation.

(b) M has a Hodge representation of the form

$$\begin{cases} \rho = \boxed{\mathrm{Ad} : M \to \mathrm{End}(\mathfrak{m})} \\ \tilde{\varphi} : \mathbb{U} \to M \end{cases}$$

$$\Updownarrow$$

$M(\mathbb{R})$ has a compact maximal torus T, and $\tilde{\varphi}$ is a cocharacter of T with

$$\langle \tilde{\varphi}, \text{compact roots} \rangle \underset{(4)}{\equiv} 0 \quad , \quad \langle \tilde{\varphi}, \text{noncompact roots} \rangle \underset{(4)}{\equiv} 2$$

(of T acting on $\mathfrak{m}$ by Ad)

This immediately rules out all $SL(n\geq 3)$ as MT groups.
HOWEVER, it also yields (with some work!)...

Theorem [GGK, 2010]: G_2 is a MT group!

More precisely, there is a $\mathbb{Q}$-algebraic group M with $M(\mathbb{C}) \cong G_{2,\mathbb{C}}$ and PHS φ on $\mathfrak{m}$ such that $M_\varphi = M$. The orbit $M(\mathbb{R}).\varphi$ yields a (necessarily nonclassical) MT domain D_M. There are infinitely many possibilities for the Hodge #'s $\underline{h}$.

Ex// ① $\underline{h} = (1,4,4,4,1)$
② $\underline{h} = (2,1,2,4,2,1,2)$
$\Big\}$ $\dim(D_M) = 5$ in both cases. //

For a general $D_M \ni \varphi$, the M-translates of $\mathfrak{m}^{(-1,1)} \subset T_\varphi D_M$ give a distribution $\mathbf{W} \subset TD_M$ with $W^\perp \subset T^* D_M$ generating $\mathcal{I} \subset \Omega^\bullet_{D_M}$. In ②, $\mathbf{W}$ is a contact 2-plane field first studied by Cartan a century ago.

Problems: **(a)** Give a Hodge-theoretic description of the duality (cf. [Bryant]) between integral curves of the EDS's in ① and ②.

(b) [Serre] Is G_2 the MT group of a <u>motivic</u> PHS?

(c) ["Lie bracket HC"] For any <u>motivic</u> adjoint MT representation, find an algebraic cycle corresponding to

$$[\,,\,] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \quad \text{(a Hodge 3-tensor!)}.$$