THE SHEAF OF NONVANISHING MEROMORPHIC FUNCTIONS IN THE PROJECTIVE ALGEBRAIC CASE IS NOT ACYCLIC

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ABSTRACT. Let $X/\mathbb{C}$ be a projective algebraic manifold, and $\mathcal{M}_X^*$ be the sheaf of nonvanishing meromorphic functions on X in the analytic topology. We prove a number of nonvanishing results for $H^\bullet(X, \mathcal{M}_X^*)$. In particular, $\mathcal{M}_X^*$ is acyclic iff $\dim X = 1$.

Titre français: Le faisceau des fonctions méromorphes non nulles sur une variété algébrique projective n'est pas acyclique.

Résumé. Sur une variété algébrique projective lisse $X/\mathbb{C}$, soit $\mathcal{M}_X^*$ le faisceau des germes de fonctions méromorphes non nulles pour la topologie analytique de X. Nous démontrons un certain nombre de résultats de non annulation pour la cohomologie $H^\bullet(X, \mathcal{M}_X^*)$. En particulier, le faisceau $\mathcal{M}_X^*$ est acyclique si et seulement si X est de dimension 1.

1. Introduction

For a compact complex manifold X, with sheaf of germs of nonvanishing meromorphic functions $\mathcal{M}_X^*$ on X, examples of X abound where $H^1(X, \mathcal{M}_X^*) \neq 0$, even among the class of Kähler manifolds. One of the motivations for studying the sheaf $\mathcal{M}_X^*$ has to do with the short exact sequence

$$0 \to \mathcal{O}_X^\times \to \mathcal{M}_X^* \to \mathcal{D}_X \to 0,$$

where $\mathcal{O}_X^\times$ is the sheaf of germs of nowhere vanishing holomorphic functions on X, and $\mathcal{D}_X$ is the sheaf of Cartier divisors on X. If $H^1(X, \mathcal{M}_X^*) = 0$, then every holomorphic line bundle on X is the line bundle of a Cartier divisor. However, if X is projective algebraic, it is well known that the latter property holds true. For example if X is a compact Riemann surface, one easily sees from the definition that $\mathcal{D}_X$ is a fine sheaf, which can be identified with the sheaf of 0-cycles on X. It follows that $H^i(X, \mathcal{D}_X) = 0$ for $i \geq 1$, thus $H^1(X, \mathcal{M}_X^*) = 0$. The exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ also implies $H^i(X, \mathcal{O}_X^\times) = 0$ for $i \geq 2$, hence $H^i(X, \mathcal{M}_X^*) = 0$ for $i \geq 2$ as

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well. The fact that any compact Riemann surface is also projective algebraic, together with the import of Serre's work (viz., GAGA and the vanishing of the corresponding cohomology, $H^1_{\text{Zar}}(X, \mathcal{M}^*_{X,\text{alg}}) = 0$, in the Zariski topology, which follows immediately from the fact that $\mathcal{M}^*_{X,\text{alg}}$ is a constant sheaf), may have led many others to speculate that $H^1(X, \mathcal{M}^*_X) = 0$ if X is projective algebraic. *A priori* this would be a reasonable expectation in light of the above discussion. Any known proof of the fact that in the projective algebraic arena, every holomorphic line bundle is the line bundle associated to a Cartier divisor, rests on showing that X projective algebraic implies that the morphism $H^1(X, \mathcal{O}^\times_X) \to H^1(X, \mathcal{M}^*_X)$ is zero (a true statement!), which is based on a polarization argument. We wish to make it clear that $\mathcal{M}_{X,\text{alg}} = \mathbb{C}(X)$, where $\mathbb{C}(X)$ the field of rational functions on a projective algebraic X, and $\mathcal{M}^*_{X,\text{alg}} = \mathbb{C}(X)^*$ (multiplicative group), whereas $\mathcal{M}_X$, $\mathcal{M}^*_X$ are the corresponding meromorphic sheaves.

It might have started off as folklore - indeed it seemed to be taken as evident at the time the third author was a graduate student, that $H^1(X, \mathcal{M}^*_X)$ vanishes for X projective algebraic. A case in point is the appearance of that very statement in [D](p. 130). Further, a cursory reading of (this statement in) [B-H](p. 334) may also suggest a similar issue, although in fairness to the authors in [B-H], they most likely expected the reader to interpret the statement (in a correct form) in the Zariski topology. Over the years, questions about the whereabouts of a proof of acyclicity of $\mathcal{M}^*_X$ for X projective algebraic have surfaced, and for good reason. The knowledge of a specific sheaf having acyclic properties indeed confers some important cohomological consequences.

The purpose of this note is to make it abundantly clear that in the projective algebraic arena, any general claims to the effect that $H^1(X, \mathcal{M}^*_X) = 0$ or $\mathcal{M}^*_X$ is acyclic, are *false.* This corrects that same erroneous assertion (viz., $H^1(X, \mathcal{M}^*_X) = 0$) in [Lew](p. 66). Quite generally we prove the following:

Theorem 1.1. *Let X be a projective algebraic manifold, with sheaf of germs of nonvanishing meromorphic functions $\mathcal{M}^*_X$ in the analytic topology. Then $H^i(X, \mathcal{M}^*_X) \neq 0$ if there is a smooth hypersurface $D \subset X$ such that the restriction map $H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$ is nontrivial.*

Corollary 1.2. *$\mathcal{M}^*_X$ is an acyclic sheaf iff* $\dim X = 1$.

Corollary 1.3. $H^i(X, \mathcal{M}^*_X) \neq 0$ *if* $\dim X \geq i + 1$ *and* $H^i(X, \mathbb{Z}) \neq 0$.

Note that by Corollary 1.3, and if $\dim X \geq 2$, then $H^1(X, \mathbb{Z}) \neq 0 \Rightarrow H^1(X, \mathcal{M}^*_X) \neq 0$. In other words, there is a topological obstruction to the triviality of $H^1(X, \mathcal{M}^*_X)$, (which may in particular be due to X not being simply-connected). The authors are unaware of any known and/or published proof of this result, despite the fact that it is a natural line of enquiry. Indeed given the aforementioned history, a published proof is probably nonexistent.

Having said this, hopefully our argument given here will be seen as a novel solution to this problem.

Finally we observe that if $\mathcal{M}^*_{X,\mathrm{alg}}$ is considered as a sheaf in the analytic topology, then our proof below exhibits nontrivial elements in $H^i(X, \mathcal{M}^*_{X,\mathrm{alg}})$; more specifically in the image $H^i(X, \mathcal{M}^*_{X,\mathrm{alg}}) \to H^i(X, \mathcal{M}^*_X)$. Using the fact that $\mathcal{M}^*_{X,\mathrm{alg}}$ is also the constant sheaf in the analytic topology, together with the universal coefficient theorem, we have a partial converse result:

Proposition 1.4. *Let $X/\mathbb{C}$ be a projective algebraic manifold. Then*

$$H^i(X,\mathbb{Z}) = 0 \text{ and } H_{i-1}(X,\mathbb{Z}) \text{ torsion free} \Rightarrow H^i(X, \mathcal{M}^*_{X,\mathrm{alg}}) = 0.$$

One wonders if the following is true:

Question 1.5. Is the sheaf $\mathcal{M}^*_X / \mathcal{M}^*_{X,\mathrm{alg}}$ acyclic?

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2. The proofs

Proof. (of Theorem 1.1) Let $X = \bigcup V_\alpha$ be a covering of X such that both $\{V_\alpha\}$ and $\{V_\alpha \cap D\}$ are acyclic coverings for the sheaf $\mathbb{Z}$, i.e.,

$$H^k(V_{\alpha_1} \cap V_{\alpha_2} \cap ... \cap V_{\alpha_l}, \mathbb{Z}) = H^k(V_{\alpha_1} \cap V_{\alpha_2} \cap ... \cap V_{\alpha_l} \cap D, \mathbb{Z}) = 0,$$

for all $k \geq 1$ and $\{V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_l}\} \subset \{V_\alpha\}$. So the Čech complexes associated to $\{V_\alpha\}$ and $\{V_\alpha \cap D\}$ calculate $H^k(X,\mathbb{Z})$ and $H^k(D,\mathbb{Z})$. Let $\{(V_{\alpha_0...\alpha_i}, m_{\alpha_0...\alpha_i})\}$ be a nontrivial cocycle in $H^i(X,\mathbb{Z})$, where $m_{\alpha_0...\alpha_i} \in \mathbb{Z}$, which by our assumptions, restricts to a nontrivial cocycle $\{(V_{\alpha_0...\alpha_i} \cap D, m_{\alpha_0...\alpha_i})\} \in H^i(D,\mathbb{Z})$. Let $f \in \mathbb{C}(X)^*$ be a rational function such that $(f) = D + A - B$ where A is a very ample divisor such that $\mathcal{O}_X(D + A)$ is very ample, and B is a member of the linear system $|D + A|$, both A and B being taken to be smooth (say) and intersecting D properly. Let us consider the i-cocycle $\{(V_{\alpha_0...\alpha_i}, g_{\alpha_0...\alpha_i})\}$ of $\mathcal{M}^*_X$ with $g_{\alpha_0...\alpha_i} = f^{m_{\alpha_0\cdots\alpha_i}}$. We claim that this is a nontrivial cocycle. Let us assume to the contrary. Then there exists $\{(V_{\alpha_0...\alpha_{i-1}}, h_{\alpha_0...\alpha_{i-1}})\}$ (after some refinement of $\{V_\alpha\}$) such that

$$g_{\alpha_0...\alpha_i} = \delta\big(h_{\alpha_0...\alpha_{i-1}}\big)_{\alpha_0...\alpha_i},$$

where δ is the Čech coboundary. Let $\mu_{\alpha_0...\alpha_{i-1}}$ be the multiplicity of $h_{\alpha_0...\alpha_{i-1}}$ along D. Then we have

$$m_{\alpha_0...\alpha_i} = \delta\big(\mu_{\alpha_0...\alpha_{i-1}}\big)_{\alpha_0...\alpha_i}.$$

when $V_{\alpha_0...\alpha_i} \cap D \neq \emptyset$. That is to say that $\{(V_{\alpha_0...\alpha_i} \cap D, m_{\alpha_0...\alpha_i})\}$ is the trivial cocycle in $H^i(D,\mathbb{Z})$, a contradiction. □

Proof. (of Corollary 1.2) This uses the aforementioned fact that for $\dim X = 1$, $H^i(X, \mathcal{M}_X^*) = 0$ for $i \geq 1$, and that $H^2(X, \mathbb{Z})$ contains a nontrivial Kähler class for $\dim X \geq 1$. In particular, $H_2(D, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ nontrivial implies that $H^2(X, \mathbb{Z}) \to H^2(D, \mathbb{Z})$ is nontrivial. Thus for $\dim X \geq 2$, $H^2(X, \mathcal{M}_X^*) \neq 0$. □

Proof. (of Corollary 1.3) Use the fact that by the Lefschetz hyperplane theorem, we have an injection $H^i(X, \mathbb{Z}) \hookrightarrow H^i(D, \mathbb{Z})$ for a smooth very ample divisor $D \subset X$. □

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