ALGEBRAIC AND ARITHMETIC PROPERTIES OF PERIOD MAPS

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ABSTRACT. We survey recent developments in Hodge theory which are closely tied to families of CY varieties, including Mumford-Tate groups and boundary components, as well as limits of normal functions and generalized Abel-Jacobi maps. While many of the techniques are representation-theoretic rather than motivic, emphasis is placed throughout on the (known and conjectural) arithmetic properties accruing to geometric variations.

1. INTRODUCTION

The last 40 years have seen the development of rich theories of *Hodge theory at the boundary* and *symmetries of Hodge structures* which have been strongly motivated by the study of non-classical, higher weight variations of Hodge structures such as those arising from families of Calabi-Yau 3-folds. Efforts to complete the moduli of families of toric hypersurfaces (e.g. via the secondary toric variety), and the centrality of the large complex structure limit in mirror symmetry, call for a compactification of period maps in higher weight such as that provded by Kato and Usui in [KU]. In the Greene-Plesser mirror constructions, the use of subfamilies of CY 3-folds with special symmetry suggests a systematic investigation of special loci in the target of the period map, and this is what Mumford-Tate subdomains provide [GGK2]. These more general perspectives on specific phenomena can have value, as seen in the generic global Torelli result of Usui [Us], or the Friedman-Laza classification of special families of "Calabi-Yau type" Hodge structures parametrized by Hermitian symmetric M-T domains [FL].

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Another major source of motivation for these developments is, of course, the Hodge conjecture. The invariant tensors of the M-T group of a "motivic" Hodge structure are exactly the classes that should come from algebraic cycles, if the conjecture holds. This was used by Hazama and Murty to establish the HC for a large subclass of abelian varieties [Ha, Mu], by Deligne to prove the (weaker) absoluteness of Hodge classes for all of them [De], and more recently by Arapura and Kumar to prove the Beilinson-Hodge conjecture in several cases [AK]. On the other hand, Lefschetz's original proof of his (1,1) theorem [Le], as well as the proof of the HC in special cases such as the cubic fourfold [Zu], relies on "filling in the singular fibers" of a 1-parameter Jacobian bundle, so as to extend normal functions to the "boundary". The more modern plan of attack on the HC suggested by Green and Griffiths [GG] employs singularities of normal functions in several variables. In this context, it seems natural to try to combine these asymptotic considerations with M-T groups: for instance, the influence of symmetries of the underlying VHS on normal functions seems largely unexplored.

Beyond the symmetries and asymptotics of Hodge structures, a third major theme of these notes is the *arithmetic of periods*, of which Euler's work on relations between multiple zeta values may be regarded as an early example. Now the period map is highly transcendental, but in "classical" cases (abelian varieties, K3 surfaces, and the like) strongly tied to modular forms. What about the "nonclassical" higher weight case? Whether we are concerned with maximal-dimensional images of period maps, extension classes of geometric limiting mixed Hodge structures, or limits of normal functions, Griffiths transversality exerts a rigidifying effect on periods, causing them to occur in countably many families which then have "arithmetic meaning". This is visible in the ubiquitous $\zeta(3)$ in the periods at the large complex structure limit in mirror symmetry. Another thread of the story has to do with vanishing of periods and the related appearance of new Hodge classes over so-called "generalized Noether-Lefschetz (or Hodge) loci", which include zero-loci of normal functions and preimages of M-T subdomains under the period map. For variations of Hodge structure (or normal functions) arising from algebraic geometry over a field k, these loci should be defined over $\bar{k}$, an expectation which is closely related to the HC as well as the conjectures of Beilinson and Bloch.

These notes are arranged in three sections: symmetries, arithmetic, and asymptotics. In the first, we introduce Mumford-Tate groups and domains and explain how they refine the period mapping. Then we look at what M-T groups are possible, and the construction of Hodge structures with given M-T group from representation theory [GGK2], followed by a result of Robles [Ro] on the maximal dimension of the image of a period map. The second section, on arithmetic, fleshes out the previous paragraph, including a result of Voisin on the field of definition of Hodge loci [Vo], and touches on CM points and transcendence of periods. In the final section, we explain how symmetries and asymptotics interact in the context of limit mixed Hodge structures, and how to extend normal functions arising from a family of cycles, and discuss arithmetic features of the limit in each case.

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2. Symmetries: Mumford-Tate groups and domains

The study of domains for Hodge structures with additional "symmetries" goes back to Picard's work on the moduli of curves of the form

$$C := \{y^3 = x(x-1)(x-\alpha)(x-\beta)\} \subset \mathbb{P}^2.$$

Writing $V := H^1(C, \mathbb{Q})$ with Hodge structure φ (see §2.1), the cubic automorphism $y \mapsto \zeta_3 y$ induces an embedding

$$\mathbb{F} := \mathbb{Q}(\zeta_3) \overset{\mu}{\hookrightarrow} End(V, \varphi).$$

The decomposition $V_{\mathbb{F}} = V_+ \oplus V_-$ into $\mu(\mathbb{F})$-eigenspaces is therefore compatible with the Hodge decomposition, with ranks

	V_+	V_-
(1,0)	2	1
(0,1)	1	2

.

Define an $\mathbb{F}$-Hermitian form on V_+ by

$$\langle \cdot, \cdot \rangle := \sqrt{-3}(\cdot, \bar{\cdot}),$$

where $(\cdot,\cdot)$ is cup-product. The generalized period domain (in this case, a Hermitian symmetric domain) parametrizing such Hodge structures is the 2-ball

$$D = U_{\langle \bullet, \bullet \rangle}(2,1).\varphi \; (\cong U(2,1)/U(2)\times U(1)) \; \cong \mathbb{B}_2.$$

Such domains are familiar in the theory of Shimura varieties, but they generalize quite naturally to higher weight and level (and the non-Hermitian setting), and this is the story we shall now flesh out.

2.1. **Mumford-Tate groups.** Begin with a $\mathbb{Q}$-vector space V (of finite dimension), and write $V_k := V \otimes_{\mathbb{Q}} k$. For us, a *HS (Hodge structure)* of weight $n \in \mathbb{Z}$ on V will be simply a homomorphism

$$\varphi : S^1 \to SL(V_{\mathbb{R}})$$

with $\varphi(-1) = (-1)^n \mathrm{id}_V$. Since φ is real,

$$V_\varphi^{p,n-p} := \{z^{2p-n}\text{-eigenspace of } \varphi(z)\} \subset V_{\mathbb{C}}$$

and $V_\varphi^{n-p,p}$ are conjugate. Write $F_\varphi^p := \oplus_{r \geq p} V_\varphi^{r,n-r}$.

Given a nondegenerate bilinear form $Q : V \times V \to \mathbb{Q}$, [anti]symmetric according to the parity of n, we say that φ is *polarized* by Q iff the two Hodge-Riemann relations hold:

(HRI) $\varphi(S^1) \subset Aut(V,Q)$ [or $Q(F^p, F^{p'}) = 0$ for $p + p' > n$];
(HRII) $Q(v, \varphi(i)\bar{v}) > 0 \;\forall v \in V_{\mathbb{C}} \backslash \{0\}$.

The tensor spaces $T^{k,\ell}(V) := V^{\otimes k} \otimes \check{V}^{\otimes \ell}$ inherit an action by φ; and the *Hodge tensors* $Hg^{k,\ell}(V) := T^{k,\ell}(V) \cap (T^{k,\ell}(V)_{\mathbb{C}})^{\varphi}$ (e.g. $Q \in Hg^{0,2}$)

are the tensors which, if (V, Q, φ) is motivic and the Hodge conjecture holds, are classes of algebraic cycles.

Now let φ be any polarizable HS on V.

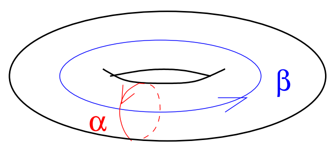
Definition 2.1. The *MT (Mumford-Tate) group* of φ is

$$G_\varphi \quad := \quad \text{the smallest } \mathbb{Q}\text{-algebraic subgroup of } SL(V) \text{ with group of } \mathbb{R}\text{-points containing } \varphi(S^1).$$

Theorem 2.2. (Deligne) *G_φ is: (i) the subgroup of $SL(V)$ fixing $\oplus_{k,\ell} Hg^{k,\ell}(V)$ pointwise; and (ii) reductive*[1] *if φ is polarizable.*

Moreover, the sub-HS $W \subset T^{k,\ell}$ are exactly the ($\mathbb{Q}$-)subspaces stabilized by G_φ.

Example 2.3. Let E be an elliptic curve



and write $[\omega] = [\alpha]^* + \tau[\beta]^*$ in $H^1(E)$, where $\omega \in \Omega^1(X)$. In $H^1(E)^{\otimes 2}$, $[\alpha]^* \wedge [\beta]^* = c \cdot [\omega] \wedge [\bar{\omega}]$ is a Hodge tensor. There exists an "extra" tensor in $Hg^{1,1}$ iff $\mu := [\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ (CM case), reflecting the existence of the endomorphism given by multiplcation by τ. (Exercise: construct this Hodge tensor for $\tau = \sqrt{-1}$.) So the M-T group is SL_2 if $\mu > 2$ and $U_{\mathbb{Q}(\tau)}$ (1-torus) if $\mu = 2$.

Remark 2.4. A HS is *effective* if $V_\varphi^{p,q} = \{0\}$ whenever $p < 0$ or $q < 0$. For us the *level* $\max\{p - q \,|\, \dim V_\varphi^{p,n-p} \neq 0 \neq \dim V_\varphi^{q,n-q}\}$ of a HS will matter more than the weight, since we shall make extensive use of non-effective HS below.

2.2. **Mumford-Tate domains.** Given V, Q, n, let $\underline{h} := (h^{p,n-p})_{p \in \mathbb{Z}}$ satisfy $h^{p,n-p} = h^{n-p,p}$ and $\sum h^{p,n-p} = \dim V$. The period domain

$$D_{\underline{h}} := \left\{ \varphi \;\middle|\; \begin{array}{c} (V, Q, \varphi) \text{ is a polarized HS of weight } n, \\ \text{with } \dim V_\varphi^{p,n-p} = h^{p,n-p} \end{array} \right\}$$

[1] that is, its finite-dimensional representations are reducible.

is a real analytic open subset in its compact dual

$$\check{D}_{\underline{h}} := \left\{ F^\bullet \;\middle|\; \begin{array}{c} F^\bullet \text{ is a flag on } V \text{ satisfying (HRI)} \\ \text{and } \dim(F^p/F^{p+1}) = h^{p,n-p} \end{array} \right\},$$

a complex projective variety. Writing $G = Aut(V, Q)$, $g.\varphi := g{\cdot}\varphi{\cdot}g^{-1} \in D_{\underline{h}}$ defines an action of $G(\mathbb{R})$ on $D_{\underline{h}}$.

Proposition 2.5. *(i)* $D_{\underline{h}} \cong G(\mathbb{R}).\varphi \cong G(\mathbb{R})/\mathscr{H}_\varphi$*, where the isotropy group* $\mathscr{H}_\varphi$ *is compact.*

(ii) $\check{D}_{\underline{h}} \cong G(\mathbb{C}).F^\bullet_\varphi \cong G(\mathbb{C})/\mathscr{P}_{F^\bullet_\varphi}$*, where* $\mathscr{P}_{F^\bullet_\varphi}$ *is parabolic.*

Proof. Exercise in bilinear forms using (HRII). □

Example 2.6. (i) For $n = 2m+1$ odd,

$$D_{\underline{h}} \cong Sp_n(\mathbb{R}) / \prod_{p \le m} U(h^{p,n-p}).$$

(ii) For $n = 2m$ even, writing $h_{\text{odd}} := \sum_{p \text{ odd}} h^{p,n-p}$ resp. $h_{\text{even}} := \sum_{p \text{ odd}} h^{p,n-p}$ we have

$$D_{\underline{h}} \cong SO(h_{\text{odd}}, h_{\text{even}}) / \left\{ SO(h^{m,m}) \times \prod_{p<m} U(h^{p,n-p}) \right\}.$$

(iii) $D_{\underline{h}}$ is a Hermitian symmetric domain (HSD) if and only if $\mathscr{H}_\varphi$ is maximally compact. This can happen when $G(\mathbb{R}) = SO(2,n)$ or $Sp_n(\mathbb{R})$, and the basic examples are $\underline{h} = (a,a)$ and $(1,b,1)$ (the others have "gaps" in the Hodge numbers).

The M-T domains are the subsets $D \subset D_{\underline{h}}$ obtained by choosing $\varphi \in D_{\underline{h}}$ and taking the orbit

$$\begin{array}{lcl} D & := & G_\varphi(\mathbb{R}).\varphi \;\cong\; G_\varphi(\mathbb{R})/H_\varphi \\ \cap & & \\ \check{D} & := & G_\varphi(\mathbb{C}).F^\bullet_\varphi \;\cong\; G_\varphi(\mathbb{C})/P_{F^\bullet_\varphi}. \end{array}$$

This produces homogeneous spaces of much greater variety, including for example Hermitian symmetric domains of type A_n, E_6 and E_7 and ones parametrizing (gap-free) Hodge structures of level > 2 [FL]. Note that $\check{D}$ is still a projective variety, and is in fact defined over $\bar{\mathbb{Q}}$.

Example 2.7. For easy examples in the spirit of the Picard 2-ball (type A_2), one can start with a HS $\varphi \in D_{\underline{h}}$ compatible with a cubic automorphism of V (and $V_{\mathbb{F}} = V_+ \oplus V_-$ as above). When $\underline{h} = (1, 2n, 1)$ [resp. $(n+1, n+1)$] and $\underline{h}_+ = (0, n, 1)$ [resp. $(n, 1)$] this yields embeddings of the n-ball $D \cong \mathbb{B}_n \cong U(1,n)/\{U(n) \times U(1)\}$ into type III [resp. IV] Hermitian-symmetric period domains.[2] There are rich relationships between (quotients of) hyperplane complements in such ball-subdomains and moduli of various objects in algebraic geometry: e.g. framed cubic surfaces [ACT1], cubic threefolds [ACT2], and non-hyperelliptic genus 4 curves [Ko]. We will give a much more general construction of Mumford-Tate domains (not from geometry) in §2.4.

2.3. **M-T group of a variation.** Let $\mathcal{V} = (\mathbb{V}, \mathcal{V}, \mathcal{F}^\bullet)$ be a holomorphic famiy of pure HS[3] over $\mathcal{S}$, $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathcal{S}}$ the flat connection with $\nabla(\mathbb{V}) = 0$, $\tilde{\mathcal{S}} \overset{\pi}{\to} \mathcal{S}$ the universal cover, $\tilde{\mathcal{V}} := \pi^* \mathcal{V}$. Fixing a point $s_0 = \pi(\tilde{s}_0) \in \mathcal{S}$, set $V := \mathbb{V}|_{s_0}$ and note that $\tilde{\mathbb{V}} = V \otimes \mathbb{Z}_{\tilde{\mathcal{S}}}$. (More generally, we will denote a fiber by subscript s.) Given $\mathsf{t} \in T^{m,n}(V)$ a Hodge (p,p)-tensor at $\tilde{s}_0$,

$$\tilde{\mathcal{S}}(\mathsf{t}) := \left\{ \tilde{s} \in \tilde{\mathcal{S}} \,|\, \mathsf{t} \in F^p_{\tilde{s}} \right\} \subset \tilde{\mathcal{S}}$$

is an analytic subvariety, and so also is $\mathcal{S}(\mathsf{t}) := \pi\left(\tilde{\mathcal{S}}(\mathsf{t})\right) \subset \mathcal{S}$. We assume s_0 has been chosen so as to belong to

$$\mathcal{S}^- := \mathcal{S} \backslash \cup_{\mathsf{t}: \mathcal{S}(\mathsf{t}) \neq \mathcal{S}} \mathcal{S}(\mathsf{t}),$$

so that any t' Hodge at s_0 is Hodge $\forall s \in \mathcal{S}$.

Theorem 2.8. (Deligne) *Let G_s denote the M-T group of V_s. Then G_s is locally constant ($= G$) off a countable union of proper analytic subvarieties, and $\leq G$ everywhere.*

Now assume that $\mathcal{V}$ is a polarized VHS (variation of HS), i.e. that $\nabla \mathcal{F}^\bullet \subset \mathcal{F}^{\bullet - 1} \otimes \Omega^1_{\mathcal{S}}$. Let $\Phi_{\underline{h}} : \mathcal{S} \to \Gamma \backslash D_{\underline{h}}$ be the associated period map.

[2]Of course, one can play the same game with $\underline{h} = (1, n, n, 1)$ and $\underline{h}_+ = (1, n, 0, 0)$ to embed $\mathbb{B}_n$ in a non-Hermitian period domain.

[3]Here $\mathbb{V}$ is a $\mathbb{Q}$-local system, $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_{\mathcal{S}}$ the [sheaf of sections of the] holomorphic vector bundle, and $\mathcal{F}^\bullet$ a filtration by [sheaves of sections of] holomorphic subbundles.

By Theorem 2.8, the $Hg^{m,n}(V_s)$ are invariant under ∇-flat continuation over $\mathcal{S}^-$. Since $Q > 0$ on Hodge tensors, monodromy $\rho : \pi_1(\mathcal{S}, s_0) \to Aut(V, Q)$ acts through an $SO_N(\mathbb{Z})$ on each $Hg^{m,n}(V_{s_0})$, which is to say by a finite group. This proves the first part of

Theorem 2.9. (Deligne/André) *(a) The geometric monodromy group* $\Pi := \left(\overline{\rho(\pi_1(\mathcal{S}))}^{Zar}\right)^\circ$ *is a subgroup of* G.

(b) $\Pi \trianglelefteq G^{der} := [G, G]$*, with equality if* $\mathcal{V}$ *has a CM point (i.e., some* V_s *has abelian M-T group).*

Proof. (of $\Pi \trianglelefteq G$) By the Theorem of the Fixed Part [Sc], $(T^{m,n}\mathbb{V})^\Pi$ underlies a sub-HS of $T^{m,n}\mathcal{V}$. Therefore, it is stabilized by G, and so every $g\Pi g^{-1}$ $(g \in G(\mathbb{Q}))$ fixes it. But a subgroup of $GL(V)$ is determined by its fixed tensors, and so then every conjugate $g\Pi g^{-1} \leq \Pi$. □

Remark 2.10. (i) Theorem 2.9(a) implies that $\Phi_{\underline{h}}$ factors through $\Phi : \mathcal{S} \to \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/H$. By a recent result [GRT], $\Gamma\backslash D$ is algebraic iff D fibers holomorphically (or antiholomorphically) over a HSD. One reason why the factoring result is important is that in higher weight/level, D may be a HSD when the "ambient" $D_{\underline{h}}$ is not.

(ii) Theorem 2.9(b) tells us that Π is semisimple, since G^{der} is.

2.4. **Which groups are Mumford-Tate?** Suppose given a HS $\varphi : S^1 \to G$ on V, polarized by Q, with M-T group $G \leq Aut(V, Q)$ and domain $D = G(\mathbb{R}).\varphi \cong G(\mathbb{R})/H$. Writing $\mathrm{Ad} : G \twoheadrightarrow G^{ad} \leq Aut(\mathfrak{g}, B)$ for the adjoint homomorphism, φ induces a HS of weight 0 on the $\mathbb{Q}$-vector spaces $\mathfrak{g} = T_e G$ and $\mathfrak{g}^{ad} = T_e G^{ad}$, and replacing G, V, φ, Q by $G^{ad}, \mathfrak{g}^{ad}, \mathrm{Ad} \circ \varphi, -B$ leaves the connected M-T domain $D^\circ = G(\mathbb{R})^\circ/H$ unchanged [KP1, KP2]. This motivates the slightly cheaper question

Which $\mathbb{Q}$*-simple* adjoint *algebraic groups are M-T groups?*

Theorem 2.11. [GGK2] *G is a M-T group* $\iff$ *$G(\mathbb{R})$ has a compact maximal torus.*

Proof. ($\Longrightarrow$) Let $T(\mathbb{R})$ be a maximal torus containing $\varphi(S^1)$, and write $H_\varphi = \{g \in G(\mathbb{R}) \,|\, g\varphi g^{-1} = \varphi\}$. Then

$$H_\varphi \subset Aut(\mathfrak{g}, B) \cap \times_j Aut(\mathfrak{g}^{j,-j}) \subset SO(\mathfrak{g}^{0,0}) \times \left(\times_{j>0} U(g^{j,-j})\right)$$

is compact, and contains $T(\mathbb{R})$.

($\Longleftarrow$) Let $\mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition with $\mathfrak{k} \supset \mathfrak{t}_\mathbb{R}$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus. Let $\Delta = \Delta_c \cup \Delta_n$ be the roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ (with c = compact, n = noncompact) and $\mathcal{R}$ the lattice they generate. We have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Delta_c} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus_{\beta \in \Delta_n} \mathfrak{g}_\beta \right); \tag{2.1}$$

note that $-B(X_\alpha, \overline{X_\alpha}) > 0$, while $-B(X_\beta, \overline{X_\beta}) < 0$.

Now the Cartan involution defined by $\theta|_\mathfrak{k} = \mathrm{id}_\mathfrak{k}$, $\theta|_\mathfrak{p} = -\mathrm{id}_\mathfrak{p}$ is a Lie-algebra homomorphism; so there exists $\Psi : \mathcal{R} \to 2\mathbb{Z}/4\mathbb{Z}$ with $\Psi(\alpha) \underset{(4)}{\equiv} 0$ and $\Psi(\beta) \underset{(4)}{\equiv} 2$. But since G is adjoint, $\mathcal{R}$ equals the weight lattice Λ, which (like any lattice) is free. So there is a lift $\tilde{\Psi} : \Lambda \to 2\mathbb{Z}$ of Ψ. Moreover, the co-character group $\mathrm{X}_*(T(\mathbb{C}))$ maps isomorphically to $Hom(\Lambda, 2\mathbb{Z})$ by $\varphi \mapsto \ell_\varphi := \frac{d\varphi}{dz}(1)$. So there exists a co-character $\varphi : S^1 \to T(\mathbb{R})$ with $\ell_\varphi \underset{(4)}{\equiv} \Psi$.

From $\mathrm{Ad}(\varphi(z))X_\alpha = z^{\langle \ell_\varphi, \alpha \rangle} X_\alpha$, we have:

- $(\mathrm{Ad} \circ \varphi)(i) = \theta \implies -B\left(\,\cdot\,, (Ad \circ \varphi)(i)\bar{\cdot}\,\right) > 0$ on $\mathfrak{g}_\mathbb{C} \implies (\mathfrak{g}, -B, \mathrm{Ad} \circ \varphi)$ is a polarized HS of weight 0; and
- $\mathfrak{g}_\mathbb{C} = \oplus_j \mathfrak{g}^{j,-j}$ with

$$\mathfrak{g}^{j,-j} = \begin{cases} \bigoplus_{\delta \in \Delta :\, \langle \ell_\varphi, \delta \rangle = 2j} \mathfrak{g}_\delta, & j \neq 0 \\ \mathfrak{t} \oplus \bigoplus_{\delta \in \Delta :\, \langle \ell_\varphi, \delta \rangle = 0} \mathfrak{g}_\delta, & j = 0. \end{cases}$$

Let $M \leq G$ be (a) the smallest $\mathbb{Q}$-algebraic subgroup such that $\mathrm{Ad} \circ (g\varphi g^{-1})$ factors through $M(\mathbb{R})$ $\forall g \in G(\mathbb{R})$; equivalently, M is (b) the M-T group of the family $\mathrm{Ad} \circ (g\varphi g^{-1})$ of HS. By Theorem 2.8, (b) $\Longrightarrow$ M is the M-T group of $\mathrm{Ad} \circ (g_0 \varphi g_0^{-1})$ for every g_0 in the complement of a countable union of proper analytic subvarieties. On the other hand, (a) $\implies M \trianglelefteq G$, and then G $\mathbb{Q}$-simple $\implies M = G$. So $G = G_{\mathrm{Ad}\circ(g_0 \varphi g_0^{-1})}$. □

Remark 2.12. (i) As a first consequence of Theorem 2.11, there is no M-T group G with $G(\mathbb{R}) = SL_n(\mathbb{R})$ for $n \geq 3$, since this has no compact maximal torus; but $U(p,q)$ does show up. In fact, real forms of all of the Cartan types do show up, including the exceptional ones (see Example 2.13 for $\mathbb{R}$-split G_2).

(ii) Theorem 2.11 remains true for semi-simple adjoint groups. The general case has recently been settled by Patrikis [Pa], who worked out the precise group-theoretic conditions on a connected reductive $\mathbb{Q}$-algebraic group which are necessary and sufficient for it to be the M-T group of some polarized HS.

The value of the above proof is that it leads to the following construction. Given a $\mathbb{Q}$-simple adjoint group G with $T \leq G(\mathbb{R})$ a sufficiently general compact maximal torus, let $\pi : \mathcal{R} \to \mathbb{Z}$ be any homomorphism with $\pi(\Delta_c) \in 2\mathbb{Z}$, $\pi(\Delta_n) \in 2\mathbb{Z}+1$. Then

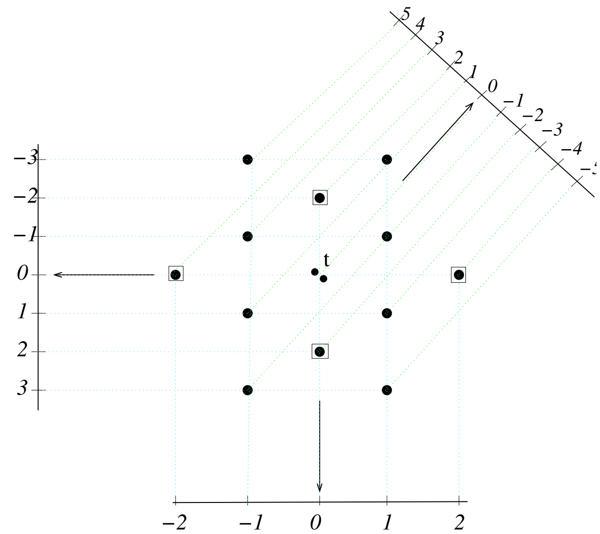
$$\mathfrak{g}^{j,-j} := \begin{cases} \bigoplus_{\delta \in \Delta :\, \pi(\delta) = j} \mathfrak{g}_\delta, & j \neq 0 \\ \mathfrak{t} \oplus \bigoplus_{\delta \in \Delta :\, \pi(\delta)=0} \mathfrak{g}_\delta, & j = 0, \end{cases} \tag{2.2}$$

produces a weight 0 HS $(\mathrm{Ad}\circ)\varphi$ on $\mathfrak{g}$, polarized by $-B$, with M-T group $G \underset{\mathrm{Ad}}{\subset} Aut(\mathfrak{g}, B)$. The M-T domain $D = G(\mathbb{R}).\varphi$ has dimension $\sum_{j<0} |\pi^{-1}(j) \cap \Delta|$, and the horizontal distribution $\mathcal{W} \subset TD$ has rank $|\pi^{-1}(-1) \cap \Delta|$.

So to determine the possible Hodge numbers (on $\mathfrak{g}$), we need to classify the projections π. A priori this list is infinite, but becomes finite (and short) if we impose the requirement that $F_\varphi^{-1}\mathfrak{g}$ bracket-generate $\mathfrak{g}$. There is a precise sense in which this merely eliminates redundancies [Ro, Prop. 3.10], so that there is really no loss of generality.

This kind of construction may or may not "lift" to other representations V of G, but if $\Lambda = \mathcal{R}$ as is the case for G_2, this isn't an issue.

Example 2.13. Let $G(\mathbb{R})$ be the $\mathbb{R}$-split form of G_2 (a subgroup of $SO(3,4)$) with 7-dimensional irrep V. There are three projections $\pi : \mathcal{R} \to \mathbb{Z}$ satisfying the parity and bracket conditions, and their restrictions to Δ look as follows:



The construction lifts to V, yielding 3 types of Hodge structures with M-T group G_2. The Hodge numbers $\underline{h}$ on V are given by restricting π to the 7 weights of V (0 together with the short roots); one reads off from the picture $(2,3,2)$, $(1,2,1,2,1)$, and $(1,1,1,1,1,1,1)$. One also reads off that the corresponding M-T domains D have dimensions 5, 5, and 6 while $\mathcal{W}$ has ranks 4, 2, and 2, respectively.

2.5. **What are the maximal dimensional VHS?.** Let a M-T domain $D = G(\mathbb{R})/H_\varphi$ be given, with compact dual $\check{D} = G(\mathbb{C})/P_{F^\bullet_\varphi}$. The tangent bundle $T\check{D} = G(\mathbb{C}) \times_P (\mathfrak{g}_\mathbb{C}/F^0\mathfrak{g}_\mathbb{C})$ contains the horizontal distribution $\mathcal{W} = G(\mathbb{C}) \times_P (F^{-1}\mathfrak{g}_\mathbb{C}/F^0\mathfrak{g}_\mathbb{C})$; at the point $\varphi \in D$ we have $T_\varphi D = \oplus_{j<0}\mathfrak{g}_\varphi^{j,-j} \supseteq \mathfrak{g}_\varphi^{-1,1} = \mathcal{W}_\varphi$.

For any period map $\Phi : \mathcal{S} \to \Gamma\backslash D$ arising from a VHS with M-T group G, Griffiths transversality implies that the local liftings of $\Phi(\mathcal{S})$ to D are integral manifolds of $\mathcal{W}$. So the dimension of $\Phi(\mathcal{S})$ is constrained by the maximal dimension of such integral manifolds, which on the surface is a hard problem in the theory of exterior differential systems. However, recent work of Robles has uncovered a different approach.

Fix $\varphi \in D$, write $\mathfrak{g}^{j,-j} := \mathfrak{g}_\varphi^{j,-j}$, $F^\bullet := F^\bullet_\varphi$, and $P := P_{F^\bullet}$. Let $T(\mathbb{R})$ be a maximal torus containing $\varphi(S^1)$, and Δ be the roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$.

Choose a positive root system of the form

$$\Delta^+ = \Delta(F^1\mathfrak{g}_{\mathbb{C}}) \cup \Delta^+(\mathfrak{g}^{0,0}),$$

and write $W := W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $W_0 := W(\mathfrak{g}^{0,0}_{\varphi}, \mathfrak{t}_{\mathbb{C}})$ for the Weyl group. Let W^P denote the set of minimal length representatives of the right-coset space $W_0 \backslash W$, or equivalently

$$W^P := \left\{ w \in W \mid w\Delta^+ \supset \Delta^+(\mathfrak{g}^{0,0}) \right\}.$$

For any $w \in W^P$,

$$\Delta_w := \Delta^- \cap w\Delta^+$$

is closed in Δ.

Writing $B \leq G(\mathbb{C})$ for the Borel subgroup with $\Delta(B) = \Delta^+$, the compact dual decomposes into Schubert cells

$$\check{D} = \coprod_{w \in W^P} C_w := \coprod_{w \in W^P} Bw^{-1}.F^{\bullet}$$

of dimension $\ell(w) := \dim(C_w) = |\Delta_w| = |w|$, where $|\Delta_w|$ is the number of elements in Δ_w and $|w|$ is the length of w. The *Schubert variety* $X_w := \overline{wC_w}$ has tangent space

$$T_{F^{\bullet}} X_w = \oplus_{\alpha \in \Delta_w} \mathfrak{g}_{\alpha} =: \mathfrak{n}_w$$

a sub-Lie algebra of $\mathfrak{g}$, since $\Delta(\mathfrak{n}_w) = \Delta_w$ is closed.

Definition 2.14. We say that X_w is a *Schubert VHS* if it is horizontal, that is, if $\mathfrak{n}_w \subset \mathfrak{g}^{-1,1}$.

In this case, $\mathfrak{n}_w$ is abelian since $[\mathfrak{g}^{-1,1}, \mathfrak{g}^{-1,1}] \subset \mathfrak{g}^{-2,2}$.

Put

$$W^P_{\mathscr{I}}(\ell) := \left\{ w \in W^P \mid \ell(w) = \ell, \, \Delta_w \subset \Delta(\mathfrak{g}^{-1,1}) \right\}.$$

Definition 2.15. An *IVHS (infinitesimal VHS)* through φ is an abelian subspace $\mathfrak{a} \subset \mathfrak{g}^{-1,1}$.

Write

$$\Sigma_{\ell} \subset Grass(\ell, \mathfrak{g}^{-1,1}) \subset \mathbb{P}(\wedge^{\ell} \mathfrak{g}^{-1,1})$$

for the variety of ℓ-dimensional IVHS through φ, $G^{0,0}$ for the subgroup of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}$ (and real points $G^{0,0}(\mathbb{R}) = H_{\varphi}$), and

$$\mathcal{I}_{\ell} := \sum_{w \in W^{P}_{\mathscr{I}}(\ell)} span_{\mathbb{C}} \left\{ G^{0,0}(\mathbb{C}). \wedge^{\ell} \mathfrak{n}_w \right\} \subset \wedge^{\ell} \mathfrak{g}^{-1,1}.$$

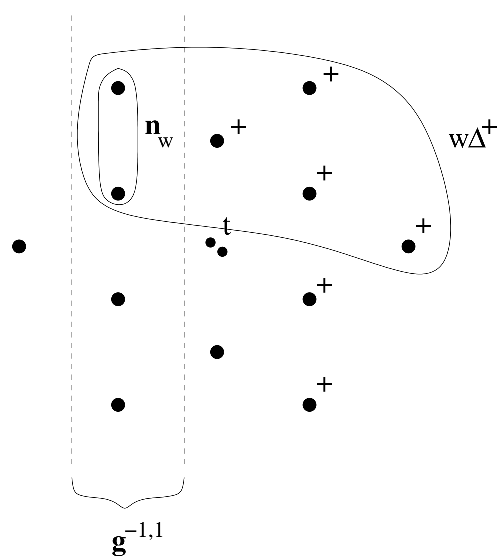
Theorem 2.16. [Ro] $\Sigma_{\ell} = Grass(\ell, \mathfrak{g}^{-1,1}) \cap \mathbb{P}\mathcal{I}_{\ell}$.

The idea of Robles's proof is to analyze $H_*(\oplus_{j<0}\mathfrak{g}^{j,-j})$ as a $\mathfrak{g}^{0,0}$-module using a theorem of Kostant on the decomposition of Lie algebra cohomology.

Theorem 2.16 leads at once to a very practical result, which puts an upper bound on the dimensions of integral manifolds of $\mathcal{W}$ hence images of period maps Φ.

Corollary 2.17. *The maximum possible dimension for an IVHS in D is the maximal dimension of Schubert VHS in $\check{D}$, which is given by* $\max \left\{ \ell \,|\, W^{P}_{\mathscr{I}}(\ell) \neq \emptyset \right\}$.

Example 2.18. To see how easy this is to use, consider the G_2 M-T domain D of dimension 5 (with $\mathcal{W}$ of rank 4) parametrizing HS on the 7-dimensional representation of type $(2,3,2)$. We choose a positive root system and find one element in each of $W^{P}_{\mathscr{I}}(1)$ and $W^{P}_{\mathscr{I}}(2)$, with the second displayed below:



The remaining $W^P_{\mathscr{I}}(\ell)$ $(\ell > 2)$ are empty, and the maximal possible dimension for an integral manifold of $\mathcal{W}$ is 2.

Example 2.19. [4] Consider the period domain $D_{(2,3,2)}$ containing the D of Example 2.18 as a subdomain. We have $G(\mathbb{R}) \cong SO(3,4)$ and $\check{D} \cong G(\mathbb{C})/P_2$, where Δ^+ is generated by simple roots $\sigma_1, \sigma_2, \sigma_3$ and $\Delta(P_2)$ by $\pm\sigma_1, \sigma_2, \pm\sigma_3$. We have $W^P_{\mathscr{I}}(\ell) = \emptyset$ for $\ell > 3$, and $W^P_{\mathscr{I}}(3) = \{w_2 w_3 w_1, w_2 w_3 w_2\}$, where w_i is the reflection in σ_i. Hence there are two Schubert VHS of dimension three, and dropping the "G_2-constraint" has increased the maximal dimension of an IVHS. This recovers a result of Carlson [Car, Rmk. 5.5(b)] (namely, that this maximal dimension is 3).

3. Arithmetic of periods

At the heart of current thought on the Hodge conjecture, two intertwined programs have emerged. On the one hand, by recent work of Green, Griffiths, and others [GG, BFNP], it can now be stated in terms of the existence of singularities for certain several-variable admissible normal functions obtained from Hodge classes. While this criterion pertains a priori to *degenerations* of normal functions, a recent result of Schnell [Sl2] reveals the importance of estimates on the dimension of their *zero-loci*, which have recently been proven algebraic [BP], generalizing a fundamental result on the locus of Hodge classes [CDK]. For an introduction to this circle of ideas, the reader may consult [KP3].

Another approach, championed by Voisin [Vo], is to break the Hodge conjecture into two pieces: first, to show that the locus of Hodge classes in a VHS arising from algebraic geometry over $\mathbb{Q}$ is defined over a number field; then second, to prove the Hodge conjecture for varieties defined over $\bar{\mathbb{Q}}$. Key to this approach is showing that a given family of Hodge classes is absolute, extending Deligne's theorem for abelian varieties [De]. The analogous question in the mixed case, regarding the field of definition of the zero-locus of a normal function, is tied to the

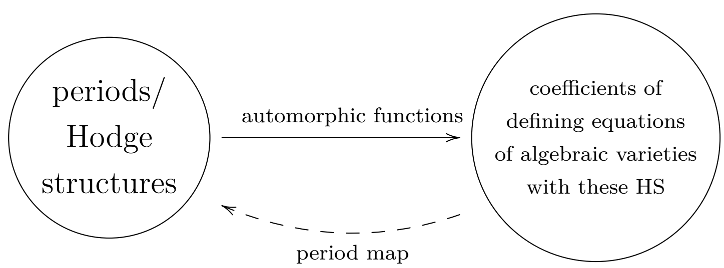
[4] We thank C. Robles for providing this example.

Bloch-Beilinson conjectures. It is this line of thought upon which we shall now briefly expand, touching as well on complex multiplication and transcendence theory.

3.1. **Spreads of period maps.** Let $D = G(\mathbb{R})/H$ be a M-T domain. The *infinitesimal period relation (IPR)* $\mathscr{I} \subset \Omega^\bullet(D)$ is the differential ideal generated by the 1-forms $\mathrm{ann}(\mathcal{W}) \subset \Omega^1(D)$ and their differentials. Essentially by definition, $\mathscr{I}$ pulls back to 0 under any local lifting of any VHS $\Phi : \mathcal{S} \to \Gamma\backslash D$. We consider two cases, which may roughly be thought of as "classical" and "non-classical".

Case 1: $\mathscr{I} = 0$. Then D is a HSD, and $\Gamma\backslash D$ is a quasi-projective algebraic variety (for any arithmetic subgroup $\Gamma \leq G(\mathbb{Q})$) by the Baily-Borel theorem. More precisely, $\Gamma\backslash D$ has a projective embedding by automorphic functions, and parametrizes a VHS which is known to be motivic (i.e. come from algebraic geometry) unless G is E_6 or E_7. In the motivic case, these automorphic functions provide the highly transcendental passage from

(3.1)



giving an inverse of the period map.

Example 3.1. The most basic example is that of elliptic curves, where $\Gamma\backslash D = SL_2(\mathbb{Z})\backslash\mathfrak{H}$ and (3.1) is

$$\tau \longmapsto g_4(\tau), g_6(\tau).$$

Other basic examples occur in the work of Holzapfel and Shiga on Picard curves [Ho, Sh] (where $\Gamma\backslash D$ is the 2-ball) and Clingher and Doran for lattice polarized $K3$ surfaces [CD] (where D is a type IV symmetric domain).

Case 2: $\mathscr{I} \neq 0$. Let $\pi : \mathcal{X} \to \mathcal{S}$ be a smooth, proper morphism of complex algebraic manifolds, and $s_0 \in \mathcal{S}$. Then $\mathbb{V} := R^n \pi_* \mathbb{Z}$ underlies a VHS $\mathcal{V}$, whose associated period map we may lift to the universal cover

$$\begin{array}{ccccc} & \mathcal{S}^{un} & \to & D \\ & \downarrow & & \downarrow \\ \Phi : & \mathcal{S} & \to & \Gamma\backslash D. \end{array}$$

If $\varphi_0 \in D$ and s_0 have the same image in $\Gamma\backslash D$, then the image of $\mathcal{S}^{un}$ gives an integral manifold of $\mathscr{I}$ through φ_0.

Now π is actually defined over some field K which is finitely generated over $\bar{\mathbb{Q}}$. There exists an affine variety $\mathrm{S}/\bar{\mathbb{Q}}$ and a very general[5] point $p \in \mathrm{S}(\mathbb{C})$ such that the evaluation map gives an isomorphism $ev_p : \bar{\mathbb{Q}}(\mathrm{S}) \overset{\cong}{\to} K$. Pulling back the defining equations under ev_p and clearing denominators yields the $\bar{\mathbb{Q}}$-*spread*

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{\tilde{\pi}} & \tilde{\mathcal{S}} \ /\bar{\mathbb{Q}} \\ & \searrow \quad \swarrow & \\ & \mathrm{S} & \end{array}$$

of π. The period map $\tilde{\Phi} : \tilde{\mathcal{S}} \to \Gamma\backslash D$ resulting from $\tilde{\pi}$ yields an integral manifold of $\mathscr{I}$ through φ_0, containing the original one, but still *proper in* D since $\mathscr{I} \neq 0$. Since there are only countably many families of algebraic varieties defined (as is $\tilde{\pi}$) over $\bar{\mathbb{Q}}$, we conclude that *only countably many integral manifolds of the IPR can come from algebraic geometry.* In particular, there can be nothing like (3.1) (although Movasati has some interesting work [Mv] on what one *does* have). This also leads to the open

Problem 3.2. This argument shows that the locus of motivic HS in D has measure 0. For any D, produce an explicit HS in the complement.

[5]That is, p is a point of maximal transcendence degree; equivalently, it lies in the complement of the complex points of countably many $\bar{\mathbb{Q}}$-subvarieties.

3.2. **Absoluteness of Hodge classes.** Take X to be a smooth projective variety over $k \subset \mathbb{C}$, and write

$$Hg^m(X) := F^m H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{C}) \cap H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{Q}(m)).$$

If we identify de Rham cohomology with algebraic differential forms

$$F^m H^{2m}_{(dR)}(X^{an}_{\mathbb{C}}, \mathbb{C}) \cong \mathbb{H}^{2m}_{Zar}(X_{\bar{k}}, \Omega^{\bullet \geq m}_{X_{\bar{k}}}) \otimes_{\bar{k}} \mathbb{C},$$

then putting $\sigma \in Aut(\mathbb{C})$ to work on the right hand side (including the coefficients of the defining equations of X) induces an action

$$\sigma_* : F^m H^{2m}(X^{an}_{\mathbb{C}}, \mathbb{C}) \to F^m(({}^{\sigma}X)^{an}_{\mathbb{C}}, \mathbb{C}).$$

Definition 3.3. The *absolute Hodge classes* of X are given by

$$AHg^m(X) := \{\xi \in Hg^m(X) \,|\, \sigma_*(\xi) \in Hg^m(X) \ (\forall \sigma \in Aut(\mathbb{C}))\}.$$

In general, we have known inclusions and conjectural equalities

$$cl(Z^m(X)) \ \subset \ AHg^m(X) \overset{\text{AHC}}{\overbrace{\ \subset\ }} Hg^m(X), \qquad cl(Z^m(X)) \overset{\text{HC}}{=} Hg^m(X)$$

Theorem 3.4. [De] *The absolute Hodge conjecture (AHC) holds if X is an abelian variety.*

This result was a crucial ingredient in Deligne's proof of the existence of canonical models for Shimura varieties of Hodge type.

Next we look at some elementary consequences of AHC. Let $\pi : \mathcal{X} \to \mathcal{S}$ be a smooth proper morphism of varieties defined over k (f.g. over $\bar{\mathbb{Q}}$), giving rise via $\mathbb{V} := R^n\pi_*\mathbb{Z}$ to a period map $\Phi : \mathcal{S} \to \Gamma\backslash D$; and let $D_M \subset D$ be a M-T subdomain, Γ_M its stabilizer in Γ.

Proposition 3.5. *If AHC holds, then any irreducible component $\mathscr{D}$ of $\pi^{-1}(\Gamma_M\backslash D_M)$ is defined over $\bar{k}$.*

Proof. Note that $\mathscr{D}$ is algebraic by [CDK]. Consider the (irreducible) $\bar{k}$-spread $\mathscr{P} \subset \mathcal{S}$ of an arbitrary $p \in \mathscr{D}(\mathbb{C})$. This is the Zariski closure of the set of points $q \in \mathcal{S}(\mathbb{C})$ such that $X_q = {}^{\sigma}X_p$ for some $\sigma \in Aut(\mathbb{C}/\bar{k})$. These $\{\sigma\}$ produce a continuous family of isomorphisms $H^n_{dR}(X_p) \overset{\cong}{\to}$

$H^n_{dR}(X_q)$, *inducing (by AHC) isomorphisms defined over* $\mathbb{Q}$ of spaces of Hodge tensors. Therefore, the Hodge tensor spaces are constant (with respect to the $\mathbb{Q}$-Betti structure), and $\mathscr{P} \subset \mathscr{D}$. We conclude that the $\bar{k}$-spread of $\mathscr{D}$ is $\mathscr{D}$. ☐

Corollary 3.6. *If AHC holds and* Φ *factors through* $\Gamma_M \backslash D_M$*, then so does the spread* $\tilde{\Phi}$.

Proof. Apply Prop. 3.5 to $\tilde{\pi}$, with $\bar{k} = \bar{\mathbb{Q}}$ (see §3.1). ☐

Taking $n = 2m$, some evidence for the conclusion of Proposition 3.5 is given by the following result:

Theorem 3.7. [Vo] *Suppose* $\mathcal{T} \subset \mathcal{S}$ *is an irreducible subvariety, defined over* $\mathbb{C}$*, such that:*

(i) $\mathcal{T}$ *is a component of the Hodge locus of some* $\alpha \in (\mathcal{F}^m \cap \mathbb{V}_{\mathbb{Q}})_{t_0}$*; and*

(ii) $\pi_1(\mathcal{T}, t_0)$ *fixes (under* ∇*-flat continuation in* $\mathbb{V}_{\mathbb{C}}$*) only the line generated by* α.

Then $\mathcal{T}$ *is defined over* $\bar{k}$.

Proof. (Sketch) Except in the trivial case, the hypotheses force $\dim(\mathcal{T}) > 0$. According to (ii), we may extend α to a ∇-flat family over $\mathcal{T}$. Given $\sigma \in Aut(\mathbb{C}/\bar{k})$, ${}^{\sigma}\alpha$ is a ∇-flat family over ${}^{\sigma}\mathcal{T}$ by algebraicity of ∇. Moreover, the fixed part of $\mathbb{V}_{\mathbb{C}}$ over ${}^{\sigma}\mathcal{T}$ must be of rank 1, since otherwise (applying σ^{-1} and algebraicity of ∇) its fixed part over $\mathcal{T}$ could not satisfy (ii). So ${}^{\sigma}\alpha = \lambda\beta$, where $\lambda \in \mathbb{C}$ and β is $\mathbb{Q}$-Betti; but then β is Hodge, since ${}^{\sigma}\alpha \in \mathcal{F}^m$.

As in the proof of Prop. 3.5, varying σ yields a continuum of conjugates ${}^{\sigma}\mathcal{T}$ on which the line $\mathbb{C}\langle {}^{\sigma}\alpha \rangle$ remains rational; hence it is constant. Since the polarization is algebraic, $Q(\alpha, \alpha) = Q({}^{\sigma}\alpha, {}^{\sigma}\alpha) = \lambda^2 Q(\beta, \beta) \implies \lambda^2 \in \mathbb{Q}$, and again by continuity $\lambda = 1$. Therefore ${}^{\sigma}\alpha$ remains Hodge, and α extends to a Hodge class on the $\bar{k}$-spread of $\mathcal{T}$, which must then (by (i)) be $\mathcal{T}$ itself. ☐

3.3. **Zero loci of normal functions.** We turn next to a mixed-Hodge analogue of the Hodge locus. Let H be a pure $\mathbb{Z}$-HS of weight -1, and

$$0 \to H \to V \to \mathbb{Z}(0) \to 0$$

an extension. The vanishing of its class in $Ext^1_{\mathrm{MHS}}(\mathbb{Z}(0), H)$ is equivalent to the existence of a splitting $\mathbb{Z}(0) \to V$, and thus to the presence of an integral Hodge $(0,0)$ class in V.

Take $\pi : \mathcal{X} \to \mathcal{S}$ as in §3.2, with fibers $\{X_s\}$; and let $\mathcal{J} \to \mathcal{S}$ be the intermediate Jacobian bundle of $\mathbb{V} := R^{2p-1}\pi_*\mathbb{Z}$. Consider an algebraic cycle $\mathfrak{z} \in Z^p(\mathcal{X})$ (defined over k) meeting the fibers properly, with $Z_s := \mathfrak{z} \cdot X_s \underset{\text{hom}}{\equiv} 0$, and let $\nu_{\mathfrak{z}} : \mathcal{S} \to \mathcal{J}$ be the normal function defined by $\nu_{\mathfrak{z}}(s) = AJ_{X_S}(Z_s)$. By [BP], the zero-locus $\mathcal{T}(\nu_{\mathfrak{z}})$ is algebraic.

Proposition 3.8. [Ch] *Assume the local system $\mathbb{V}_{\mathbb{C}}$ has no nonzero global sections over $\mathcal{T}(\nu_{\mathfrak{z}})$. Then $\mathcal{T}(\nu_{\mathfrak{z}})$ is defined over $\bar{k}$.*

Proof. (Sketch) Let $\mathcal{T}_0 \subset \mathcal{T}(\nu_{\mathfrak{z}})$ be an irreducible component, and put $\mathfrak{z}_0 := \mathfrak{z}|_{\mathcal{T}_0}$. Given $\sigma \in Aut(\mathbb{C}/k)$, we have ${}^{\sigma}\mathfrak{z}_0 = \mathfrak{z}|_{{}^{\sigma}\mathcal{T}_0}$.

The infinitesimal invariant of a normal function is algebraic: so its vanishing for $\nu_{\mathfrak{z}_0}(=0)$ implies its vanishing for $\nu_{({}^{\sigma}\mathfrak{z}_0)}$. Thus $\nu_{({}^{\sigma}\mathfrak{z}_0)}$ lives in the fixed part of $\mathcal{J}|_{{}^{\sigma}\mathcal{T}_0}$.

By the algebraicity of ∇ and the nonexistence of global sections of $\mathbb{V}_{\mathbb{C}}|_{\mathcal{T}_0}$, the fixed part of $\mathbb{V}_{\mathbb{C}}|_{{}^{\sigma}\mathcal{T}_0}$ hence of $\mathcal{J}|_{{}^{\sigma}\mathcal{T}_0}$ is trivial. So $\nu_{({}^{\sigma}\mathfrak{z}_0)} = 0$ and ${}^{\sigma}\mathcal{T}_0$ belongs to $\mathcal{T}(\nu_{\mathfrak{z}})$.

Since $\mathcal{T}(\nu_{\mathfrak{z}})$ is algebraic, it has only finitely many components. Hence $\mathcal{T}_0$ has only finitely any conjugates, and is defined over a finite extension of k. □

When $k \subset \bar{\mathbb{Q}}$, the field of definition of $\mathcal{T}(\nu_{\mathfrak{z}})$ is related to a basic question regarding filtrations on Chow groups. If X and $Z \in Z^m(X)$ are defined over a field K (finitely generated $/\bar{\mathbb{Q}}$), then spreading out yields $\rho : \mathcal{X} \to \mathcal{S}$ and $\mathfrak{z} \in Z^m(\mathcal{X})$ defined over $\bar{\mathbb{Q}}$. This leads to a cycle map Ψ, given by the composition

$$CH^m(X/K) \overset{\cong}{\to} \mathrm{im}\{CH^m(\bar{\mathcal{X}}/\bar{\mathbb{Q}}) \to \varinjlim_{\substack{U \subset S/\bar{\mathbb{Q}} \\ \text{Zar. op.}}} CH^m(\mathcal{X}_U)\}$$

$$\to \mathrm{im}\{H^{2m}_{\mathcal{D}}(\bar{\mathcal{X}}^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(m)) \to \varinjlim_{U} H^{2m}_{\mathcal{H}}((\mathcal{X}_U)^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(m))\}. \tag{3.2}$$

Now there exists a Leray filtration $\mathcal{L}^\bullet$ on (3.2), and we define[6] a filtration F^i_{BB} on $CH^m(X/K)$ by $\Psi^{-1}(\mathcal{L}^i)$. Then each graded piece $Gr^i_{F_{BB}} CH^m(X/K)$ is captured by a Hodge theoretic invariant in $Gr^i_{\mathcal{L}}$ of (3.2). For $i = 0$ this is the fundamental class $[Z]$, and for $i = 1$ it is equivalent to the normal function $\nu_{\mathfrak{z}}$.

Therefore, for the class of Z to be in F^2_{BB} is equivalent to having $\nu_{\mathfrak{z}}$ identically zero. Clearly this implies $AJ(Z) = 0$, since $AJ(Z) = \nu_{\mathfrak{z}}(s_0)$.

Proposition 3.9. [KP3] *The converse (i.e. $F^2_{BB} CH^m \supseteq \ker(AJ)$) holds in general if, and only if, $\mathcal{T}(\nu_{\mathfrak{z}})$ is defined over $\bar{\mathbb{Q}}$ whenever $\mathfrak{z}$ is.*

The proof is in the spirit of those above: spreading out a point in the zero locus should remain in the zero locus!

3.4. **CM points.** These are, roughly speaking, Hodge structures (i.e., points in a M-T domain D) with "lots of endomorphisms".

Definition 3.10. A HS (V, φ) is *CM* if its M-T group M_φ is abelian (i.e. an algebraic torus).

The CM Hodge structures $\varphi \in D$ are precisely the 0-dimensional M-T subdomains, and they are analytically dense in D.

Here is a construction of CM HS. Let L be a CM field – that is, a totally imaginary extension of a totally real field – and choose a formal partition of its complex embeddings into (p, q) types

$$Hom(L, \mathbb{C}) = \{\theta_1, \ldots, \theta_g; \bar{\theta}_1, \ldots, \bar{\theta}_g\} = \amalg_{p+q=n} \Pi^{p,q}$$

subject to the condition that $\overline{\Pi^{p,q}} = \Pi^{q,p}$. (If $n = 1$ and only $(1, 0)$ and $(0, 1)$ are allowed, this is called a "CM type".) Viewing L as a $\mathbb{Q}$-vector space of dimension $2g$, we put $V := L$ and let $\ell \in L$ act by

[6] While there is nothing conjectural about our construction of $F^\bullet_{BB}$, the existence of a "Bloch-Beilinson filtration" is conjectural. Our $F^\bullet_{BB}$ only qualifies as one if $\cap_i F^i_{BB} = \{0\}$; this depends on the injectivity of Ψ, which is sometimes called the "arithmetic Bloch-Beilinson conjecture".

multiplication. The complexification decomposes into eigenspaces

$$V_{\mathbb{C}} \cong L \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\theta \in Hom(L,\mathbb{C})} E_\theta(V_{\mathbb{C}})$$

on which ℓ acts as multiplication by $\theta(\ell)$, and we define the HS $V_{(L,\Pi)}$ by

$$V_\varphi^{p,q} := \bigoplus_{\theta \in \Pi^{p,q}} E_\theta(V_{\mathbb{C}}).$$

Theorem 3.11. *(i)* [GGK2] *Any HS of this form is polarizable, and any polarizable CM HS decomposes as a direct sum of these (and copies of* $\mathbb{Q}(-\frac{n}{2})$*, if* n *is even).*

(ii) [Ab] *Any polarized CM HS is motivic.*

The idea of (ii) is that $V_{(L,\Pi)}$ is a sub-HS of $H^n(A)$ for a CM abelian variety A, which is constructed from the set $\Theta(\Pi)$ of CM types refined by Π via

$$A := \times_{\Theta \in \Theta(\Pi)} A_{(L,\Theta)}^{\times m_\Theta},$$

where $A_{(L,\Theta)} = J(V_{(L,\Theta)})$. Note that the Hodge conjecture is not known in general for "degenerate" CM abelian varieties $A_{(L,\Theta)}$, i.e. those whose M-T group has dimension $< g$; these include the so-called Weil abelian fourfolds.

On the other hand, according to Theorem 3.4, the AHC is known for *all* abelian varieties, and CM ones play a special role in Deligne's proof. In rough outline:

(1) Start with a family $\mathcal{A} \to \mathcal{S}$ of abelian varieties over a connected Shimura variety of Hodge type.
(2) The CM points are dense in $\mathcal{S}$. By algebraicity of ∇, the AHC for generic A_s reduces to the AHC for CM abelian varieties.
(3) Focusing on a CM A (with HS φ on $H^1(A)$), define an "absolute M-T group" $G_\varphi^{abs} \geq G_\varphi$ whose fixed tensors are the AH tensors. If there are AH tensors cutting out G_φ, this will force $G_\varphi^{abs} = G_\varphi$.
(4) Weil Hodge classes (the ones for which HC isn't known) are absolute.

One can imagine an extension of this beyond the abelian variety setting, where the Shimura variety $\mathcal{S}$ parametrizes a VHS of higher weight arising from a family $\mathcal{X} \overset{\pi}{\to} \mathcal{S}$ via $R^n\pi_*\mathbb{Q}_{\mathcal{X}}$. (In particular, the motivic cases amongst the CY variations studied in [FL] would be candidates.) It is of course still true that the $s \in \mathcal{S}$ with $H^n(X_s)$ CM are dense. If Abdulali's inclusion of $H^n(X_s)$ in H^n of a CM abelian variety were (for each such s) induced by an AH class in $H^{2n}(X_s \times A)$, then the AHC would hold for the $T^{k,\ell}(H^n)$ of every fiber.

While density of CM points in a M-T domain is a done deal, their distribution in a VHS is another story. Dropping the Shimura variety assumption, $\Gamma\backslash D$ may be non-algebraic, but the period map $\Phi : \mathcal{S} \to \Gamma\backslash D$ associated to $R^n\pi_*\mathbb{Q}$ has quasi-projective image [So]. If π is defined over $\bar{\mathbb{Q}}$, then in the spirit of André-Oort one can state the following:

Conjecture 3.12. *The Zariski closure of the set of CM HS in* $\Phi(\mathcal{S})$ *is a union of Shimura varieties.*

Example 3.13. For the mirror quintic VHS $\Phi : \mathbb{P}^1\backslash\{0,1,\infty\} \to \Gamma\backslash D_{(1,1,1,1)}$ and others like it (cf. [DM]), Conjecture 3.12 would imply that there are finitely many CM points. This is an open problem.

3.5. **Transcendence of periods.** Let E be an elliptic curve *defined over* $\bar{\mathbb{Q}}$, with period ratio $\tau \in \mathfrak{H}$. Then by a theorem of Schneider (see below), we have that $[\mathbb{Q}(\tau) : \mathbb{Q}]$ is either 2 (and $H^1(E)$ is CM) or ∞ (and the M-T group of $H^1(E)$ is SL_2). Put differently: if the Hodge structure $H^1(E)$ is not contained in a proper subdomain of $\mathfrak{H} = D$, then it gives a period point whose spread is *all* of D.

To formulate the expected generalization of this result, again let $\pi : \mathcal{X} \to \mathcal{S}$ be defined over $\bar{\mathbb{Q}}$, $\Phi : \mathcal{S} \to \Gamma\backslash D$ be an associated period map, and write $\rho : D \twoheadrightarrow \Gamma\backslash D$.

Conjecture 3.14. (Grothendieck, André [An2]) *Given* $p \in \mathcal{S}(\bar{\mathbb{Q}})$ *and* $\varphi \in D$ *satisfying* $\rho(\varphi) = \Phi(p)$*,* φ *is very general in* $D_{G_\varphi} = G_\varphi(\mathbb{R}).\varphi$*, i.e. it is a point of maximal transcendence degree in the projective variety* $\check{D}_{G_\varphi}$.

Remark 3.15. (i) The transcendental periods occurring should have arithmetic meaning, due to countability of the image. For example, the conifold mirror quintic has (after resolving singularities) H^3 of type $(1,0,0,1)$, with period ratio the quotient of two $\mathbb{Q}(e^{\frac{\pi i}{10}}, \{\Gamma(\frac{k}{5})\}_{k=1}^4)$-linear combinations of ${}_4F_3$ special values.

(ii) A (more precise) mixed-Hodge theoretic analogue of this assertion is given by Beilinson's conjectures relating extension classes arising from generalized algebraic cycles to special values of L-functions.

The evidence for Conjecture 3.14 is given by Schneider's result and a generalization due to Tretkoff, Shiga and Wolfart:

Theorem 3.16. (Schneider [Sc]) *Given an elliptic curve $E/\bar{\mathbb{Q}}$ with period ratio $\tau \in \bar{\mathbb{Q}}$, E has CM (equivalently, $[\mathbb{Q}(\tau):\mathbb{Q}] = 2$).*

So when an elliptic curve is defined over a number field, the period ratio cannot be (for example) a cubic irrationality.

Theorem 3.17. [Co, SW] *Given a family $\mathcal{A} \to \mathcal{S} = \Gamma\backslash D$ of abelian varieties over a Shimura variety of PEL type,*[7] *defined over $\bar{\mathbb{Q}}$. If $\rho(\varphi) =: s \in \mathcal{S}(\bar{\mathbb{Q}})$ ($\iff$ $A_s/\bar{\mathbb{Q}}$) and $\varphi \in \check{D}(\bar{\mathbb{Q}})$, then A_s has CM.*

Tretkoff has generalized this to some families of Calabi-Yau varieties over Shimura varieties.

What is behind all this (at least, the more general Theorem 3.17) is the mysterious *Analytic Subgroup Theorem* of Wüstholz. A corollary of his powerful result is:

Theorem 3.18. [Wu] *Let G be a connected $\bar{\mathbb{Q}}$-algebraic group, $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ a proper subspace, defined over $\bar{\mathbb{Q}}$, with $0 \neq v \in \mathfrak{h} \cap \ker(\exp)$. Then there exists a closed connected algebraic subgroup $G_0 \leq G$, defined over $\bar{\mathbb{Q}}$, such that $v \in \mathfrak{g}_{0,\mathbb{C}} \subset \mathfrak{h}$.*

To see how this is used, we shall show that it implies Theorem 3.16. Given $E/\bar{\mathbb{Q}}$, and $\omega \in H^0(E, \Omega^1_{E/\bar{\mathbb{Q}}})$ with period lattice $\Lambda := \mathbb{Z}\langle \pi_1, \pi_0 \rangle$,

[7]This is just (an arithmetic quotient of) a M-T domain for HS of level one cut out by 2-tensors.

assume $\tau := \frac{\pi_1}{\pi_0} \in \bar{\mathbb{Q}}$. By the short-exact sequence

$$0 \to \Lambda^2 \to \mathbb{C}^2 \to E^2 =: G \to 0,$$

we have $v := (\pi_0, \pi_1) \in \ker(\exp)$. Put $\mathfrak{h} := \mathbb{C}\langle v \rangle \subset \mathfrak{g}_{\mathbb{C}}$. By Wüstholz, there exists a closed subgroup $G_0 \subset E \times E$, defined over $\bar{\mathbb{Q}}$, such that $v \in \mathfrak{g}_{0,\mathbb{C}} \subset \mathbb{C}\langle v \rangle$. But then $\mathfrak{g}_{0,\mathbb{C}} = \mathbb{C}\langle v \rangle$, so that $\exp(\mathfrak{h})$ is closed. Thus multiplication by τ gives a correspondence, and E has CM.

4. Asymptotics: limits of VHS and normal functions

There are two complementary aspects to the study of degenerating variations of Hodge structure. For those arising from an algebro-geometric degeneration, Steenbrink's approach to the limiting mixed Hodge structure (LMHS) uses logarithmic structures and the nearby cycle functor to construct a cohomological mixed Hodge complex on the singular fiber. (See the masterful presentation in Chapter 11 of [PS].) However, the LMHS exists for an arbitrary degenerating VHS, and representation-theoretic techniques have long been central to the work of Cattani, Kaplan, Pearlstein, Schmid and others from this perspective.

In this section we shall not attempt to do justice to either of these stories; we shall only briefly recall the more general definition, and in the geometric case state the relation to the cohomology of the singular fiber. Rather, our aim is to illustrate the influence of the symmetries of §2 upon LMHS via the technology of boundary components [KP1] (which indeed gets into some mild representation theory). We also briefly describe an "analogue of Steenbrink" for limits of geometric normal functions [GGK3], and hint at the interaction of limits (of both VHS and normal functions) with arithmetic.

4.1. **Limits of period maps.**

4.1.1. *Existence of the LMHS.* Let $\mathcal{V} = (\mathcal{V}, \mathbb{V}_{\mathbb{Z}}, Q, \mathcal{F}^\bullet)$ be a polarized $\mathbb{Z}$-VHS of weight n over the punctured unit disk Δ^*, with M-T group

G and associated period map

$$\Phi : \Delta^* \to \Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/H.$$

Denoting a fiber $\mathbb{V}_{\mathbb{Z},s_0}$ by $V_{\mathbb{Z}}$,[8] assume the monodromy operator $T \in G(\mathbb{Z}) \leq Aut(V_{\mathbb{Z}}, Q)$ is unipotent and define

$$N := \log(T) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k}(T-I)^k \in \mathfrak{g}_{\mathbb{Q}} \subset End(V,Q).$$

Now choose a local holomorphic parameter $s \in \mathcal{O}(\Delta)$, vanishing at the origin to first order. The "untwisting"

$$\tilde{\mathbb{V}} := e^{-\frac{\log(s)}{2\pi i}N}\mathbb{V} \tag{4.1}$$

of $\mathbb{V}$ is clearly a trivial local system (over Δ^*) which extends (as $\tilde{\mathbb{V}}_e$) to Δ, and we set $\mathcal{V}_e := \tilde{\mathbb{V}}_e \otimes \mathcal{O}_\Delta$ and $V_{lim} := \tilde{\mathbb{V}}_e|_0$. Moreover, there is a unique filtration $W(N)_\bullet$ (which we shall denote $W_\bullet$) on V such that $N(W_\bullet) \subset W_{\bullet-2}$ and $N^\ell : Gr^W_{n+\ell} \to Gr^W_{n-\ell}$ is an isomorphism ($\forall \ell$).

We have the following respective consequences of Schmid's Nilpotent and SL_2 orbit theorems:

Theorem 4.1. [Sc] *(i) The $\{\mathcal{F}^i\}$ extend to holomorphic sub-bundles $\mathcal{F}^i_e \subset \mathcal{V}_e$.*

(ii) Writing $F^i_{lim} := \mathcal{F}^i_e|_0 \subset V_{lim,\mathbb{C}}$, $(V_{lim}, W_\bullet, F^\bullet_{lim})$ is a mixed Hodge structure, called the LMHS.

4.1.2. *Deligne bigradings.* For *any* MHS $(V, W_\bullet, F^\bullet)$ we have the following result, due to Deligne:

Theorem 4.2. [CKS, Thm. 2.13] *There exists a unique bigrading $I^{p,q}$ of $V_{\mathbb{C}}$ such that*

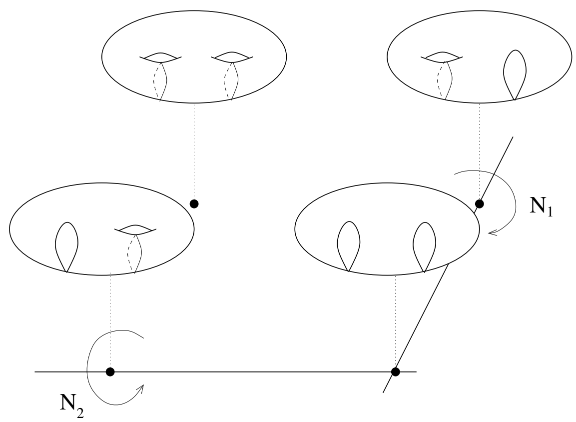
$$F^\bullet V_{\mathbb{C}} = \bigoplus_{\substack{p,q \\ p\geq \bullet}} I^{p,q}, \qquad W_\bullet V_{\mathbb{C}} = \bigoplus_{\substack{p,q \\ p+q\leq \bullet}} I^{p,q},$$

$$\text{and} \quad I^{p,q} \equiv \overline{I^{q,p}} \mod \bigoplus_{\substack{a<p \\ b<q}} I^{a,b}.$$

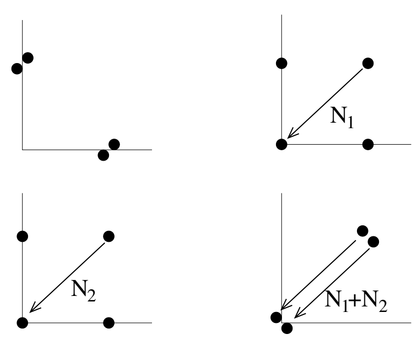
[8] Note: dropping the $\mathbb{Z}$ will mean $\mathbb{Q}$-coefficients.

In the case of a LMHS, $N : \oplus I^{p,q} \to \oplus I^{p-1,q-1}$ may be completed to an $\mathfrak{sl}_{2,\mathbb{C}}$-representation,[9] which then decomposes $V_{lim,\mathbb{C}}$ into isotypical components, compatibly with the bigrading. We can visualize all this by using dots to depict the dimensions of the $\{I^{p,q}\}$ and arrows for the action of N.

Example 4.3. Consider the 2-parameter VHS over $(\Delta^*)^2$ obtained from relative H^1 of a family $\{C_{s,t}\}$ of degenerating genus-2 curves over Δ^2:



with M-T group Sp_4. The respective $I^{p,q}$-pictures for the LMHS of the 1-parameter slices $t \mapsto (1,1)$, $(t,1)$, $(1,t)$, (t,t) are:



(Here for example the upper right LMHS has $I^{0,0}, I^{1,0}, I^{0,1}, I^{1,1}$ each of rank 1, with $N_1 : I^{1,1} \overset{\cong}{\to} I^{0,0}$. Two dots at the same (p,q) spot means $I^{p,q}$ has rank 2; we think of the dots as elements of a basis.) The lower right LMHS can also be viewed as the limit of the 2-parameter VHS.

[9]This can be done over $\mathbb{R}$ precisely when the LMHS is $\mathbb{R}$-split, i.e. $I^{p,q} = \overline{I^{q,p}}$ exactly.

Remark 4.4. In what follows, Example 4.3 will be continued as a running example. A good exercise is to work out the analogous results for (the transcendental H^2 of) $K3$ surfaces of Picard ranks 17, 18, and 19 corresponding to semistable degenerations of Kulikov type II and III.

4.1.3. *Nilpotent orbits.* In the situation of §4.1.1, the nilpotent orbit attached to $\mathcal{V}$ is the polarized variation

$$\mathcal{V}_{nilp} := (\mathcal{V}, \mathbb{V}, Q, e^{-\frac{\log(s)}{2\pi i}N} F^{\bullet}_{lim}) \tag{4.2}$$

defined over Δ^* after possibly shrinking the radius. One might think of this as the "most trivial PVHS having the same LMHS as $\mathcal{V}$". That this still yields a period map $\Phi_{nilp} : \Delta^* \to \Gamma\backslash D$ is guaranteed by:

Proposition 4.5. *The M-T group of $\mathcal{V}_{nilp}$ is contained in that of $\mathcal{V}$.*

Proof. (Sketch) By §2.3, monodromy acts on the Hodge tensors of a polarized VHS through a finite group. Since T is unipotent, it must therefore fix the Hodge tensors in each $\mathcal{V}^{\otimes k} \otimes \check{\mathcal{V}}^{\otimes \ell}$. Since the process of "computing the LMHS followed by taking the nilpotent orbit" is compatible with linear-algebraic operations on VHS (including tensors, duals, and inclusions) and does nothing to a constant variation, the Hodge tensors remain Hodge in the LMHS and in $\mathcal{V}_{nilp}$. □

The LMHS depends upon the choice of local coordinate s: if $F^{\bullet}_{lim}$ is written with respect to a basis of V_{lim}, rescaling $s \mapsto e^{2\pi i \alpha} s$ transforms the latter by $e^{-\alpha N}$, in effect replacing the former by $e^{\alpha N} F^{\bullet}_{lim}$. (More generally, in several variables the effect of a reparametrization is to send $F^{\bullet}_{lim} \mapsto e^{\sum \alpha_i N_i} F^{\bullet}_{lim}$.) The nilpotent orbit lacks this well-definedness issue, in two senses: first, the effects of the rescaling in (4.1) and (4.2) cancel out. Second, and more importantly, we will redefine the nilpotent orbit below as the full *set* of flags $e^{\mathbb{C}N} F^{\bullet}_{lim}$, i.e. in essence as the "LMHS modulo reparametrization".

Example 4.6. For the 2-parameter LMHS in Ex. 4.3, the extension class of $\mathbb{Q}(-1)^{\oplus 2}$ by $\mathbb{Q}(0)^{\oplus 2}$ has (in this setting, because of the polarization) three degrees of freedom, two of which are killed by $e^{\alpha_1 N_1 + \alpha_2 N_2}$.

What remains is the cross-ratio of the 4 points (in the preimage of the singular locus) in a resolution $\mathbb{P}^1 \twoheadrightarrow \frac{\mathbb{P}^1}{a\equiv b,\, c\equiv d} = C_{0,0}$ (cf. [Ca]).

4.1.4. *Clemens-Schmid sequence.* Suppose $\mathcal{V}$ arises from the relative cohomology of a semi-stable degeneration (SSD)

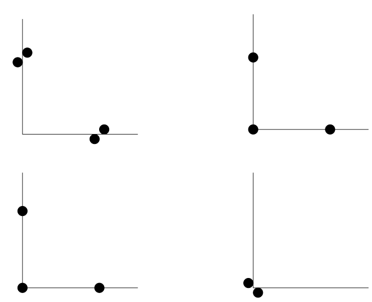
$$\begin{array}{ccccc} & & \mathcal{X} & \to & \Delta \\ & & \cup & & \cup \\ \cup Y_i & = & X_0 & \to & \{0\}, \end{array}$$

where $\mathcal{X}$ is smooth and X_0 is a reduced strict normal crossings divisor (with $\{Y_i\}$ smooth). Then we have a long-exact sequence

$$H^m(X_0) \to H^m_{lim}(X_s) \stackrel{N}{\to} H^m_{lim}(X_s)(-1) \to H_m(X_0)(-m-1) \to \cdots$$

of MHS in which the first arrow is often an injection. $H^m(X_0)$ is computed in a standard way using double complexes: e.g., for the underlying $\mathbb{Q}$-Betti structure one considers $C^j(Y^{[i]})$, where $Y^{[i]} := \amalg_{|K|=i+1} Y_{k_1} \cap \cdots \cap Y_{k_i}$ and C^j denotes real analytic "cochains with pullback" (i.e. meeting components of $Y^{[i+1]}$ properly).

Example 4.7. In the four LMHS of Example 4.3, $\ker(N)$ ($= H^1$ of a singular fiber) takes the respective forms:



In particular, $H^1(C_{0,0}) \cong \mathbb{Q}(0)^{\oplus 2}$ completely misses the extension class associated to the cross ratio described in Example 4.6.

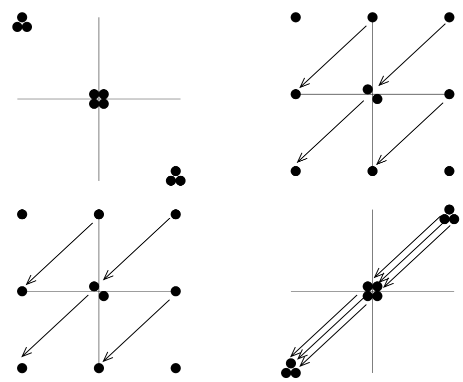
4.1.5. *Adjoint reduction.* We remind the reader of a fact mentioned in §2.4. Without changing D, we may replace:

- the M-T group G by $M := G^{ad}$;
- the underlying $\mathbb{Q}$-vector space (and M-T representation) V by $\mathfrak{m} := Lie(M)$ (= subquotient of $End(V) \cong V \otimes \check{V}$);

- Hodge structures $\varphi : S^1 \to G(\mathbb{R})$ ("on V") by $\mathrm{Ad} \circ \varphi : S^1 \to M(\mathbb{R})$ ("on $\mathfrak{m}$", of weight 0).

The $I^{p,q}$ decomposition of the LMHS is compatible with linear algebra operations on representations of the M-T group, and so carries over to $\mathfrak{m}$ along with N (in the form $\mathrm{ad}(N)$).

Example 4.8. The LMHS's on V in our running example induce the following LMHS's on $\mathfrak{m} = \mathfrak{sp}_4$:



The arrows denote the action of $\mathrm{ad}N$.

4.1.6. *Boundary components.* To systematically compactify the images of period maps, it turns out that what is needed are partial compactifications of $\Gamma\backslash D$ by spaces $\bar{B}(\sigma)$ which "classify LMHS modulo rescaling and change of basis". The description of these *boundary components* in [KP1] also answers the question: *what algebraic constraints on the LMHS* (for instance, on which extensions can be nontrivial) *result from the VHS having a given M-T group?* Below we shall focus on the components $\bar{B}(N)$ needed for 1-parameter degenerations.

Given $N \in \mathfrak{m}_{\mathbb{Q}}$ nilpotent,

$$\tilde{B}(N) := \left\{ F^\bullet \in \check{D} \;\middle|\; \begin{array}{c} \mathrm{Ad}(e^{\tau N})F^\bullet \in D \text{ for } \Im(\tau) \gg 0 \\ \text{and } NF^\bullet \subset F^{\bullet-1} \end{array} \right\}$$

comprises the possible limiting Hodge flags for a period map $\Phi : \Delta^* \to \Gamma\backslash D$ with monodromy logarithm N, and hence the possible LMHS (via $F^\bullet \mapsto (V, W(N)_\bullet, F^\bullet)$). Passing (modulo rescalings) to nilpotent orbits yields the left quotient $B(N) := e^{\mathbb{C}N}\backslash\tilde{B}(N)$; and taking $\Gamma_N \leq \Gamma$ to be the largest subgroup stabilizing the line generated by N, we have:

Definition 4.9. The *boundary component* associated to N is $\bar{B}(N) := \Gamma_N\backslash B(N)$.

Let Z_N denote the centralizer of N in M. In the bigrading diagrams (cf. §4.1.2), we may visualize its Lie algebra $\mathfrak{z}_N = \ker(\mathrm{ad}N) \leq \mathfrak{m}$ as the bottoms of all the $\mathfrak{sl}_2$-strings. The grading of the weight filtration given by $\oplus_{p+q=m}I^{p,q}$ induces a Levi decomposition $Z_N = G_N M_N$, where $\mathfrak{g}_N := Lie(G_N)$ is pure of weight 0 and $\mathfrak{m}_N := Lie(M_N) \subset W_{-1}\mathfrak{m}$, and $G_N(\mathbb{R})M_N(\mathbb{C})$ acts transitively on $B(N)$.

In fact, the structure of $\bar{B}(N)$ can be read off from the $I^{p,q}/N$-diagrams. In [KP1], it is shown that there is a tower of fibrations

$$B(N) \twoheadrightarrow \cdots \twoheadrightarrow B(N)_{(k)} \overset{\pi_{(k)}}{\twoheadrightarrow} B(N)_{(k-1)} \twoheadrightarrow \cdots \overset{\pi_{(1)}}{\twoheadrightarrow} D(N),$$

where the *primitive M-T domain* $D(N) \cong G_N(\mathbb{R})/H_N$ parametrizes the HS on the (primitive parts of the) associated graded $\oplus Gr_i^W\mathfrak{m}$. Moreover, the tangent space to a fiber of $\pi_{(k)}$ is $\frac{Gr^W_{-k}\mathfrak{z}_{N,\mathbb{C}}}{F^0Gr^W_{-k}\mathfrak{z}_{N,\mathbb{C}}}$ (or $\frac{Gr^W_{-2}\mathfrak{z}_{N,\mathbb{C}}}{F^0Gr^W_{-2}\mathfrak{z}_{N,\mathbb{C}}+\mathbb{C}N}$ for $k=2$) and to $D(N)$ is $\frac{Gr^W_0\mathfrak{z}_{N,\mathbb{C}}}{F^0Gr^W_0\mathfrak{z}_{N,\mathbb{C}}} = \frac{\mathfrak{g}_{N,\mathbb{C}}}{F^0(\cdots)}$. The tower passes to the quotient by Γ_N, whereupon the fibers become (for $k > 1$, generalized) intermediate Jacobians, assuming Γ is neat.

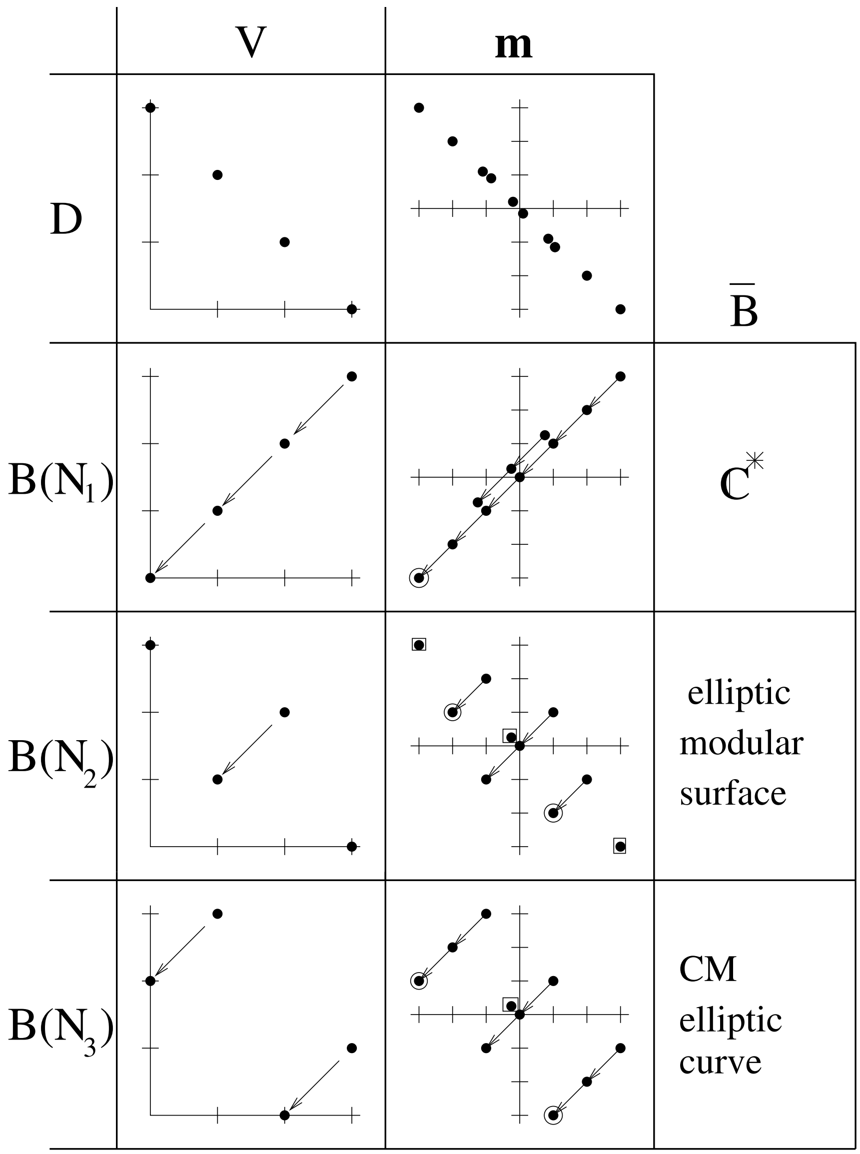
We now look at a few special cases, with $G = Sp_4$, $U(2,1)$, and G_2 ($\implies M = PSp_4$, $SU(2,1)^{ad}$, resp. G_2).

Example 4.10. We conclude our running example. From the $I^{p,q}$ diagrams on $\mathfrak{sp}_4$ in Ex. 4.8, we read off that for $N = N_1$ or N_2, $\bar{B}(N) \twoheadrightarrow \Gamma_N\backslash D(N)$ has fibers $\frac{Gr^W_{-1}\mathfrak{m}_{\mathbb{C}}}{F^0(\cdots)+(\cdots)_{\mathbb{Z}}}$, which are elliptic curves. Moreover, $G_N \cong SL_2$ and $D(N) \cong \mathfrak{H}$, so $\bar{B}(N)$ is essentially an elliptic modular surface.

If $N = N_1 + N_2$, then $\bar{B}(N) = \frac{W_{-2}\mathfrak{m}_{\mathbb{C}}}{\mathbb{C}N+(\cdots)_{\mathbb{Z}}} \cong (\mathbb{C}^*)^{\times 2}$. One may define boundary components for nilpotent cones such as $\sigma = \mathbb{R}_{\geq 0}\langle N_1, N_2\rangle$; in this case, $\bar{B}(\sigma) = \frac{W_{-2}\mathfrak{m}_{\mathbb{C}}}{\mathbb{C}N_1+\mathbb{C}N_2} \cong \mathbb{C}^*$ records the cross ratio of Ex. 4.6.

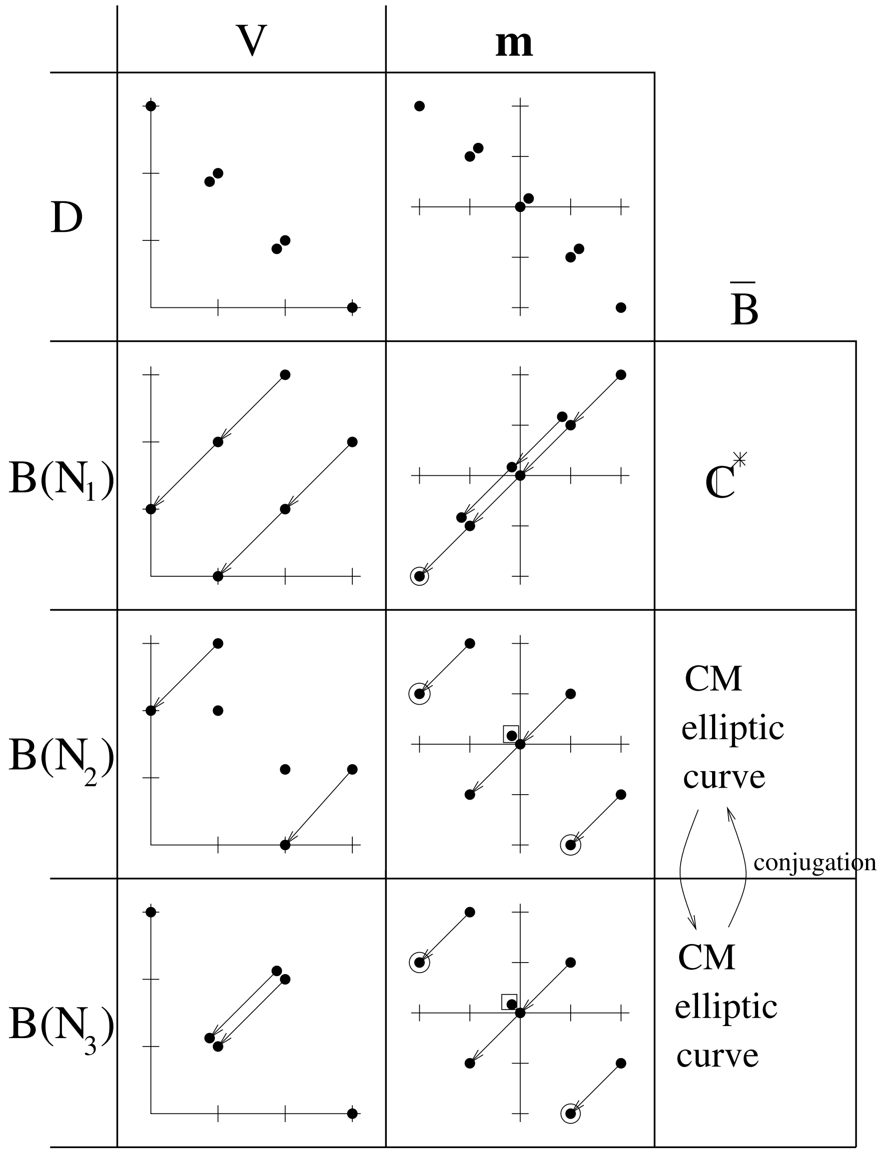
In the next three examples, the dots (= basis vectors) representing $\mathfrak{g}_{N,\mathbb{C}}$ [resp. $\mathfrak{m}_{N,\mathbb{C}}/\mathbb{C}N$] are boxed [resp. circled]. For a description of what the N_1,N_2,N_3 actually are in each case, the reader may consult §8 of [KP1].

Example 4.11. We begin with a period domain case: $\dim V = 4$, weight 3, $\underline{h} = (1,1,1,1)$, $D \cong Sp_4(\mathbb{R})/U(1)^{\times 2}$.

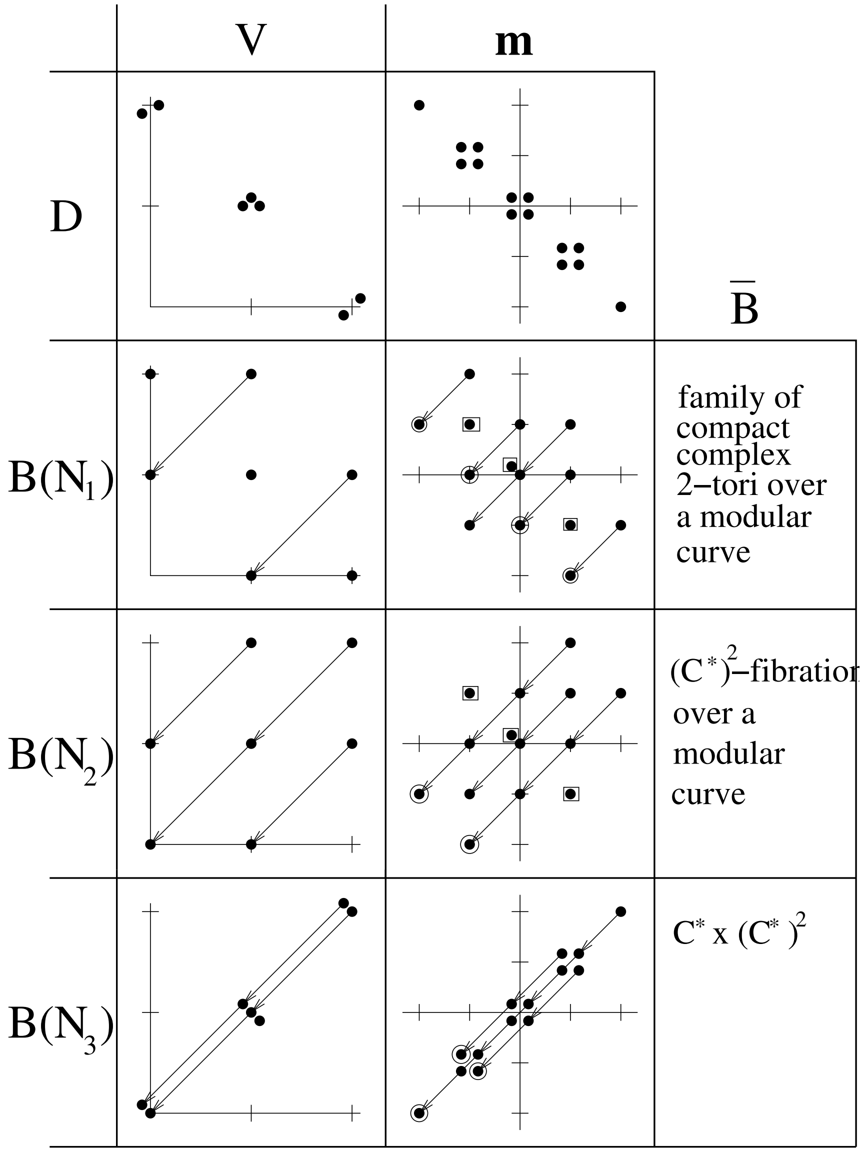


We remark that $\bar{B}(N_3)$ is specifically a CM elliptic curve because the diagram implies that G_{N_3} (which contains the M-T group of each pure HS $Gr_k^W \mathfrak{m}$) is a 1-torus.

Example 4.12. Carayol's M-T domain: $\dim V = 6$, weight 3, $\underline{h} = (1,2,2,1)$; $D \cong U(2,1)/U(1)^{\times 3}$ parametrizing HS with endomorphisms by $\mathbb{Q}(\sqrt{-d})$, and induced decomposition $V_{\mathbb{Q}(\sqrt{-d})} = V_+ \oplus V_-$ with $\underline{h}_+ = (1,1,1,0)$.



Example 4.13. An exceptional M-T domain: $\dim V = 7$, weight 2, $\underline{h} = (2,3,2)$, $D \cong G_2(\mathbb{R})/U(2)$ parametrizing HS with a distinguished Hodge 3-tensor.



4.1.7. *Arithmetic of limiting periods.* Here we just give a quick idea and refer the reader to [GGK1] for philosophy and [DKP] for computations.

Let k be a number field, and suppose that $\mathcal{V}$ arises from a *semistable degeneration over* k: this means that $\mathcal{X} \to \Delta$ belongs to a larger family over $\mathbb{P}^1$, defined over k, and with the $\{Y_I\}$ defined over k.

Conjecture 4.14. *The LMHS is the Hodge realization of a (mixed) motive defined over* k*. In particular, extension classes belonging to* $Ext^1_{MHS}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong \mathbb{C}/\mathbb{Q}$ *are essentially Borel regulators of elements of* $K_{2n-1}(k)$*. If* $k = \mathbb{Q}$*, these classes are all therefore rational multiples of* $\frac{\zeta(n)}{(2\pi i)^n}$.

Example 4.15. [CDGP, GGK1] Let X_s denote the minimal smooth toric compactification of the hypersurface defined by

$$1 - \xi \left(\sum_{i=1}^{4} x_i + \prod_{i=1}^{4} x_i^{-1} \right) = 0$$

in $(\mathbb{C}^*)^4$, where $s = \xi^5$. The mirror quintic VHS is given by $H^3(X_s)$, and has LMHS at $s = 0$ in $B(N_1)$ of Example 4.11. (This degeneration can be given the structure of an SSD/$\mathbb{Q}$.) Writing $I^{3,3} = \mathbb{C}e_3$, and $\gamma_3, \gamma_2, \gamma_1, \gamma_0$ for a $\mathbb{Q}$-symplectic basis, we have $e_3 = \gamma_3 - \frac{200\zeta(3)}{(2\pi i)^3}\gamma_0$. The extension class (given by the coefficient of γ_0) obviously satisfies the conjecture.

Example 4.16. [DR, KP1] If we consider the G_2-domain where $\underline{h} = (1, 1, 1, 1, 1, 1, 1)$ (instead of the $(2, 3, 2)$ of Example 4.13), then the Hodge-Tate boundary component takes the form $\mathbb{C}^* \cong Ext^1(\mathbb{Q}(-5), \mathbb{Q}(0))$. Katz's middle convolution algorithm is used in [DR] to produce a family of motives (defined over $\mathbb{Q}$) producing a VHS of this type, with LMHS at $s = 0$ in the H-T boundary. So we expect the extension class to be a rational multiple of $\frac{\zeta(5)}{(2\pi i)^5}$.

4.2. **Limits of normal functions.** Start with an SSD

$$\begin{array}{ccccccc} \mathcal{X}^* & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_0 & = & \cup_i Y_i \\ \downarrow \pi & & \downarrow \bar{\pi} & & & & \\ \Delta^* & \overset{\jmath}{\hookrightarrow} & \Delta & \hookleftarrow & \{0\} & & \end{array}$$

as in §4.1.4 (π smooth, $\bar{\pi}$ proper holomorphic), with $\dim \mathcal{X} = 2m$. Consider an algebraic cycle $\mathfrak{z} \in Z^m(X)$ properly intersecting fibers, so that for each $s \in \Delta$ (including 0)

$$Z_s := \mathfrak{z} \cdot X_s \in Z^m(X_s)$$

is defined. Assuming that $0 = [\mathfrak{z}] \in H^{2m}(\mathcal{X})$, we have $0 = [Z_s] \in H^{2m}(X_s)$ ($\forall s \in \Delta$). We ask: *is there as sense in which*

$$\lim_{s \to 0} AJ_{X_s}(Z_s) = AJ_{X_0}(Z_0)? \tag{4.3}$$

4.2.1. *Meaning of the left-hand side of* (4.3). $AJ_{X_s}(Z_s)$ yields (as in §3.3) a section $\nu_{\mathfrak{z}} \in \Gamma(\Delta^*, \mathcal{J})$, where $\mathcal{J} \to \Delta^*$ is the Jacobian bundle associated with $\mathbb{V} := R^{2m-1}\pi_* \mathbb{Z}_{\mathcal{X}^*}$. There exists a non-Hausdorff extension of $\mathcal{J}$ to Δ, defined by the short-exact sequence (cf. §4.1.1 for notation)

$$0 \to \jmath_* \mathbb{V} \to \frac{\mathcal{V}_e}{\mathcal{F}^m_e} \to \mathcal{J}_e \to 0,$$

and an extension $\bar{\nu}_{\mathfrak{z}} \in \Gamma(\Delta, \mathcal{J}_e)$ of the normal function $\nu_{\mathfrak{z}}$ due to El Zein and Zucker [EZ].[10] Set

$$\lim_{s \to 0} AJ_{X_s}(Z_s) := \bar{\nu}_{\mathfrak{z}}(0).$$

4.2.2. *Meaning of the right-hand side of* (4.3). The singular variety $X_0 = \cup Y_i \overset{\imath_0}{\hookrightarrow} \mathcal{X}$ has substrata (of codimensions $\ell = 0, \ldots, 2m-1$)

$$Y^{[\ell]} := \coprod_{|I| = \ell+1} Y_I,$$

where $Y_I := \cap_{i \in I} Y_i$. Its motivic cohomology $H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))$ is the $2m^{\text{th}}$ cohomology of a double complex constructed (essentially) from Bloch's higher Chow complexes on the substrata. Up to torsion, one may think of this as being built out of (subquotients of) the K-groups of substrata $K^{alg}_{\ell}(Y^{[\ell]})$. An explicit map of double complexes described

[10]Their theorem applies to the more general setting $[\mathfrak{z}|_{\mathcal{X}^*}] = 0$ in $H^{2m}(\mathcal{X}^*)$.

in [KL, GGK3] induces an Abel-Jacobi homomorphism
(4.4)
$$AJ^m_{X_0} : H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{hom} \to J^m(X_0) := \frac{H^{2m-1}(X_0, \mathbb{C})}{F^m + H^{2m-1}(X_0, \mathbb{Z}(m))},$$

defined on the cohomologically-trivial classes. Again up to torsion, one may consider (4.4) to be induced from the Chern-class (or "regulator") maps on $K^{alg}_{\ell}(Y^{[\ell]})$.

Example 4.17. For the reader familiar with higher Chow cycles, for $m = 2$ the double complex computing $H^4_{\mathcal{M}}(X_0, \mathbb{Z}(2))$ is

$$\begin{array}{ccccccc}
Z^2_\sharp(Y^{[0]}) & \overset{\delta}{\to} & Z^2_\sharp(Y^{[1]}) & & & & \\
\uparrow & & \uparrow \partial & & & & \\
 & \to & Z^2_\sharp(Y^{[1]}, 1) & \overset{\delta}{\to} & Z^2_\sharp(Y^{[2]}, 1) & & \\
 & & \uparrow & & \uparrow \partial & & \\
 & & & \to & Z^2_\sharp(Y^{[2]}, 2) & \overset{\delta}{\to} & Z^2(Y^{[3]}, 2) \\
 & & & & \uparrow & & \uparrow \partial \\
 & & & & & \to & \boxed{Z^2(Y^{[3]}, 3)}.
\end{array}$$

Here ∂ is Bloch's boundary map restricted to a quasi-isomorphic subcomplex $Z^2_\sharp(Y^{[i]}, \bullet) \subset Z^2(Y^{[i]}, \bullet)$ which consists of higher cycles *meeting substrata properly*, and δ is the alternating sum of pullbacks thus enabled.

4.2.3. *Meaning of equality in* (4.3). Now intersection with the Y_i yields a map
$$\imath_0^* : CH^m(\mathcal{X})_{hom} \to H^{2m}_{\mathcal{M}}(X_0, \mathbb{Z}(m))_{hom},$$
which intuitively sends "$\mathfrak{z} \mapsto Z_0$". Writing
$$\Psi : J^m(X_0) \to (\mathcal{J}_e)_0$$
for the map induced by $H^{2m-1}(X_0) \to H^{2m-1}_{lim}(X_s)$ in Clemens-Schmid, we can state

Theorem 4.18. [GGK3] (4.3) *holds with the right-hand side replaced by* $\Psi(AJ_{X_0}(\imath_0^* \mathfrak{z}))$.

Let $\hat{\mathcal{J}}_e$ be the modification of $\mathcal{J}_e$ produced by replacing the fiber $(\mathcal{J}_e)_0$ over 0 by $J^m(X_0)$. It was shown by M.Saito [Sa] that $\hat{\mathcal{J}}_e$ is Hausdorff.[11]

Corollary 4.19. [GGK3] *The extension $\bar{\nu}_{\mathfrak{z}}$ is actually a section of $\hat{\mathcal{J}}_e$.*

4.2.4. *Arithmetic implications.* Thinking back to §3.1, it turns out that the IPR implies a rigidity result for limits of certain kinds of normal functions. The limits of this type, for $\mathfrak{z}$ defined over $\mathbb{C}$, therefore *already come from algebraic geometry over* $\bar{\mathbb{Q}}$, leaving only countably many possible values. This suggests

Corollary 4.20. [GGK3] *The regulators in* (4.4) *imply arithmetic behavior for the limit of the AJ map.*

Example 4.21. Rather than making a precise statement, we discuss what the last Corollary looks like when $\mathcal{X}$ is a certain 1-parameter family of quintic threefolds. After carrying out semistable reduction, X_0 is a union of five $\mathbb{P}^3$'s blown up along Fermat quintic curves. The Abel-Jacobi map relevant for codimension-2 cycles sits in the diagram

$$\begin{array}{ccccc} H^4_{\mathcal{M}}(X_0,\mathbb{Z}(2)) & \stackrel{AJ^2_{X_0}}{\to} & J^2(X_0) & \stackrel{\Psi}{\hookrightarrow} & (\mathcal{J}_e)_0 \\ \cup & & \cup & & \\ K_3^{ind}(\mathbb{C}) & \stackrel{R}{\to} & \mathbb{C}/\mathbb{Z}(2) & & \\ & & \downarrow \Im & & \\ & & \mathbb{R} & & \end{array}$$

in which the regulator map R has countable image. In particular, the image of $K_3^{ind}(\mathbb{C})$ is just that of $K_3^{ind}(\bar{\mathbb{Q}})$, with imaginary part related to special values of L-functions of number fields. In §4 of [op. cit.], it is shown how to construct families of cycles $\mathfrak{z}$ with $\imath_0^*\mathfrak{z}$ in the $K_3^{ind}(\mathbb{C})$ part of motivic cohomology, and compute the limits $\nu_0 := \lim_{s\to 0}\nu_{\mathfrak{z}}(s)$ of the associated normal functions. In particular, one has $\nu_0 \in \mathbb{C}/\mathbb{Z}(2)$, and

[11] There is a related important construction of Schnell which leads to a very natural proof of the algebraicity of 0-loci of normal functions [Sl1]. Also note that, while Hausdorff, $\hat{\mathcal{J}}_e$ may not be a complex analytic space: the fiber over 0 usually has lower dimension than the other fibers (cf. §4.2.5).

$\Im(\nu_0)$ can be written as a $\mathbb{Z}$-linear combination of values of the Bloch-Wigner function at algebraic arguments. (For instance, one limit takes the value $D_2(\sqrt{-3})\,(\neq 0)$.)

Referring to Example 4.17, here is a rough sketch of the method used in [op. cit.] to construct local families of cycles $\mathfrak{z}$ with this property. Begin with a ∂-cocycle $\mathcal{W}$ in one summand (say, $Z^2(Y_{2345},3)$) of the boxed term of the double complex (which gives a class in $CH^2(Spec(k),3) \underset{\otimes\mathbb{Q}}{\cong} K_3^{ind}(k)$). The main problem is to find a class of such $\mathcal{W}$'s which can be moved by the total differential $D = \partial \pm \delta$ to (say) a cycle $\mathcal{Z}_5$ in $Z^2_\sharp(Y_5)$ in the upper left term, satisfying additional intersection conditions ($\mathcal{Z}_5$ does not meet the Y_{j5}, and meets the Fermat curve blowups only along the proper transform of the Fermat quintic surface) which allow it to deform to the smooth fibers $\{X_s\}$. Explicit computation in the double complex shows that cycles of the form $\mathcal{W}_{f,g} := \{(u, f(u), g(u)) \,|\, u \in \mathbb{P}^1\} \cap (\mathbb{P}^1 \backslash \{1\})^3$ (where $f \equiv 1$ on $|(g)| \cup \{0\}$, $g \equiv 1$ on $|(f)| \cup \{\infty\}$, and poles of f are allowed to have order 3 at $u = 1$ and order 2 elsewhere) give such a class.

A number of examples arising from open mirror symmetry, which gives another source for such cycle families $\mathfrak{z}$, have been computed in [JW, LW].

4.2.5. *Hausdorffness of $\hat{\mathcal{J}}_e$.* To conclude, we shall explain why $\mathcal{J}_e$ is not in general Hausdorff, and try to convey the flavor of the estimates used by M. Saito [Sa] to show that $\hat{\mathcal{J}}_e$ avoids this fate. We shall do this in the context of a nilpotent orbit over Δ^* with ($\mathbb{Q}$-split) LMHS of the type parametrized by $B(N_3)$ in Example 4.11.

Consider then the rank 4 local system $\mathbb{V}_{\mathbb{Z}}$ with fiber $V_{\mathbb{Z}} = \mathbb{Z}\langle \alpha, \beta, \gamma, \delta\rangle$ over s_0, and monodromy logarithm N sending $\alpha \mapsto \beta \mapsto 0$ and $\gamma \mapsto \delta \mapsto 0$. Writing $\ell(s) = \frac{\log(s)}{2\pi i}$ ($s \in \Delta^*$), a basis of $\tilde{\mathbb{V}}$ is given by $\tilde{\alpha} = \alpha - \ell(s)\beta$, $\tilde{\beta} = \beta$, $\tilde{\gamma} = \gamma - \ell(s)\delta$, and $\tilde{\delta} = \delta$. We may take our variation to have $\mathcal{F}^2_s = \langle \tilde{\alpha} - i\tilde{\gamma}, \tilde{\beta} - i\tilde{\delta}\rangle$. Then

$$\mathcal{J}_e = \frac{\mathcal{O}_\Delta\langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\rangle}{\mathcal{O}_\Delta\langle \tilde{\alpha} - i\tilde{\gamma}, \tilde{\beta} - i\tilde{\delta}\rangle + \mathbb{Z}\langle \alpha, \beta, \gamma, \delta\rangle}$$

$$\cong \frac{\mathcal{O}_\Delta\langle\tilde{\gamma},\tilde{\delta}\rangle}{\mathbb{Z}\langle i\tilde{\gamma}+i\ell(s)\tilde{\delta}, i\tilde{\delta}, \tilde{\gamma}+\ell(s)\tilde{\delta}, \tilde{\delta}\rangle}$$

has fibers

$$(\mathcal{J}_e)_s \cong \frac{\mathbb{C}^2}{\Gamma_s} := \frac{\mathbb{C}^2}{\mathbb{Z}\left\langle \begin{pmatrix} i \\ i\ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ \ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle}$$

over $s \neq 0$ and

$$(\mathcal{J}_e)_0 \cong \frac{\mathbb{C}^2}{\mathbb{Z}\left\langle \begin{pmatrix} 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle} = \mathbb{C} \oplus \mathbb{C}/\mathbb{Z}[i],$$

whereas

$$(\hat{\mathcal{J}}_e)_0 \cong \frac{\mathbb{C}\langle\beta,\delta\rangle}{\mathbb{C}\langle\beta - i\delta\rangle + \mathbb{Z}\langle\beta,\delta\rangle} = \mathbb{C}/\mathbb{Z}[i].$$

Now to see that $\mathcal{J}_e$ is not Hausdorff, choose $a \in \mathbb{Z}\backslash\{0\}$, $b \in \mathbb{C}$. The sequence of points $(s_n, \underline{v}_n) :=$

$$\left(s_n, \begin{pmatrix} a \\ b \end{pmatrix}\right) := \left(\exp\left(2\pi i\left(\frac{b+ni}{a}\right)\right), a\begin{pmatrix} 1 \\ \ell(s_n) \end{pmatrix} - n\begin{pmatrix} 0 \\ i \end{pmatrix}\right)$$

in $\mathcal{O}_\Delta\langle\tilde{\gamma},\tilde{\delta}\rangle$ clearly approaches $\left(0, \begin{pmatrix} a \\ b \end{pmatrix}\right)$ as $n \to \infty$. But in $\mathcal{J}_e$, we have $\underline{v}_n \in \Gamma_{s_n} \implies (s_n, \underline{v}_n) \equiv (s_n, \underline{0})$, and so the sequence approaches both $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$ (which are distinct points of $(\mathcal{J}_e)_0$) in the quotient topology. Hence these two points cannot be separated.

In spite of this, $\hat{\mathcal{J}}_e$ is Hausdorff. Let $a = 0$, $b \in \mathbb{C}\backslash\mathbb{Z}[i]$. Then there exist

- $\epsilon > 0$ sufficiently small, and
- $M \gg 0$ sufficiently large,

that $\Im(z) > M \implies$

$$\left\| n_1\begin{pmatrix} i \\ i\ell(s) \end{pmatrix} + n_2\begin{pmatrix} 0 \\ i \end{pmatrix} + n_3\begin{pmatrix} 1 \\ \ell(s) \end{pmatrix} + n_4\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix} \right\| > \epsilon$$

for all $\underline{n} \in \mathbb{Z}^4$. This can be checked by hand, and similar norm estimates lead to M. Saito's result.

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