Reciprocity Laws on Algebraic Surfaces via Iterated Integrals

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(with an Appendix by Ivan Horozov and Matt Kerr)

Abstract

In this paper we introduce new local symbols, which we call 4-function local symbols. We formulate reciprocity laws for them. These reciprocity laws are proven using a new method - multidimensional iterated integrals. Besides providing reciprocity laws for the new 4-function local symbols, the same method works for proving reciprocity laws for the Parshin symbol. Both the new 4-function local symbols and the Parshin symbol can be expressed as a finite product of newly defined bi-local symbols, each of which satisfies a reciprocity law. The K-theoretic variant of the first 4-function local symbol is defined in the Appendix. It differs by a sign from the one defined via iterated integrals. Both the sign and the K-theoretic variant of the 4-function local symbol satisfy reciprocity laws, whose proof is based on Milnor K-theory (see the Appendix). The relation of the 4-function local symbols to the double free loop space of the surface is given by iterated integrals over membranes.

Key words: reciprocity laws, complex algebraic surfaces, iterated integrals
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0 Introduction

This paper is the second one in a series of papers on reciprocity laws on varieties via iterated integrals (after [H1]). We construct and prove reciprocity laws for both classical and new symbols. First, we define new *bi-local symbols* and prove the corresponding reciprocity laws. Using them we introduce *new* 4*-function local symbols* on surfaces and prove their reciprocity laws. Using the same methods, we find a new proof of the reciprocity laws for the Parshin symbol. We recall that the Parshin symbol was defined by Parshin in [P1] and [P2] up to a sign. The sign was computed by Fesenko and Vostokov in [FV] and a K-theoretic proof of the reciprocity laws for the Parshin symbol was given by Kato in [Ka].

The present paper uses many ideas from the preprint "Refinement of the Parshin symbol for surfaces" [H2]. In an email to the author [D2], Deligne pointed out that the refinement of the Parshin symbol was not independent of choices of local uniformizers. After examining carefully the origin of the refinement - namely, iterated integrals of differential forms over membranes - we realized that the refinement becomes independent of local uniformizers by introducing bi-local symbols. A key property of the bi-local symbols is that they resemble the tame symbol on a curve; however, they are defined over surfaces.

We introduce the new 4-function local symbols as a product of simpler bi-local symbols that satisfy reciprocity laws. Moreover, such a presentation in terms of bi-local symbols provides proofs of the reciprocity laws for the local symbols. In the same manner we construct a refinement of the Parshin symbol in the sense that the latter is a product of bi-local symbols each of which satisfies a reciprocity law. Another reason for using bi-local symbols is that they are computationally effective.

Iterated integrals over membranes

In our paper [H1], we used iterated integrals over paths for reciprocity laws on a complex curve. Here, we define a higher dimensional analogue of iterated path integrals, which we call *iterated integrals over membranes* for extending this technique to complex algebraic surfaces. The idea for *iterated integrals over membranes* had its genesis in a generalization of Manin's non-commutative modular symbol [M1] to a non-commutative Hilbert modular symbol [H3], [H5]. Iterated integrals over membranes give functions on the double free loop space of a surface.

Our approach is based on examining a cocycle on the loop space of the surface or on the double free loop space of a surface. In this way both the 4-function symbol and the Parshin symbol occur naturally. A homotopy invariant function on the double free loop space of a complex algebraic surface is a 0-cocycle such as I_2, I_3 and I_4 from Section 1.4. The Parshin symbol is expressed in terms of the cocycles I_2 and I_3, while the first 4-function local symbol is expressed in terms of the cocycle I_4. We define such functions on the double free loop space of the surface, using iterated integrals over membranes, which resembles Chen's construction of functions on the loop space of a manifold [Ch1], [Ch2].

A geometric proof of the reciprocity laws uses the following observation: a closed form on a loop space (with a base) is homotopy invariant with respect to homotopy variations of the loop. Integrating over certain loops gives us essentially a logarithm of a local symbol. If a composition of such loops is homotopy trivial then the integral over that loop will vanish. Moreover, the integral over this homotopy trivial loop is equal to the sum of the logarithm of the symbols. After exponentiating, we obtain that a product of local symbols is equal to 1.

Usually, reciprocity laws are proved by providing a cocycle of a Galois group. One uses them to examine a portion of the Galois group, for example, the abelian Galois group. Instead of a Galois group one can consider the fundamental group. (An intermediate object is the étale fundamental group.) The cocycle on the fundamental group can be replaced by a cocycle on the loop space. Both the fundamental group of a non-simply connected space (see [Ch1]) and the loop space of a simply connected space (see [Ch2]) were studied by iterated integrals. This paper gives a new direction for reciprocity laws. Here we exhibit the need to study the loop space and the double free loop space of a non-simply connected variety. In particular the Parshin symbol is given by a 1-cocycle on a loop space of a (possibly) non-simply connected surface and the new 4-function local symbol is given by a 0-cocycle on a double free loop space of such a surface. Hopefully, the relation of local symbols to the loop space and to the double free loop space can give more structure both in reciprocity laws and in study of loop spaces. For example, one question is: what portion of the double free loop space is captured by such reciprocity laws?

The Parshin symbol is interpreted in this paper as a 1-cocycle of the loop space of a surface. Alternatively, it gives a closed 0-cocycle on the double loop space of the surface. The first 4-function local symbol naturally occurs as a closed 0-cocycle on the double free loop space. Equivalently, the first 4-function local symbol is a homotopy invariant integral over a torus with respect to homotopy variations of the torus (Subsection 1.4).

The use of iterated integrals over membranes has other applications in number theory and algebraic geometry. For example, they provide a new approach to the two-dimensional Contou-Carrère symbol [H6]. We also use them to construct multiple Dedekind zeta values [H4] and a non-commutative Hilbert modular symbol [H5].

The sources of new symbols in our approach are iterated integrals. More precisely, every iterated integral leads to a reciprocity law. In [H1] Theorems 2.9 and 3.3, we use higher order iteration on a complex curve. Then the reciprocity laws are complicated. One can do the same for surfaces. However, we have chosen to consider at most double iterated integrals, which lead to relatively simple reciprocity laws. Over a surface there are three such (iterated) integrals, involving:

(i) a 2-form - leading to an analogue of "the sum of the residues is zero";

(ii) an iteration of a 2-form with a 1-form - leading to the Parshin symbol;

(iii) an iteration of a 2-form with a 2-form - leading to both 4-*function local symbols.*

Higher order iterations will lead again to reciprocity laws. They will be considered in a follow up paper, since they capture more properties from the cohomology of the double free loop space of a complex algebraic surface.

Some known reciprocity laws

There are several interesting formulas that we would like to bring to the attention of the reader. For explaining the formulas defining the reciprocity laws, it would be instructive to make a comparison with the Weil reciprocity law stated in terms of the tame symbol (defined in the Appendix).

A divisor of a non-zero rational function f on a complex smooth projective curve is the formal sum

$$(f) = \sum_i a_i P_i,$$

such that P_i's are points where f has zeros or poles and the coefficients $a_i \in \mathbb{Z}$ are the orders of vanishing of f at the points P_i. Let also

$$(g) = \sum_j b_j Q_j.$$

Assume for the moment that the $\{Q_j\}$ and $\{P_i\}$ are disjoint. Following Weil [W], we define

$$f((g)) = \prod_j f(Q_j)^{b_j} \quad \text{and} \quad g((f)) = \prod_i g(P_i)^{a_i}.$$

Denote by $|(f)|$ the support of the divisor of a non-zero rational function f.

Theorem 0.1. *(Weil reciprocity law) If $|(f)|$ and $|(g)|$ are disjoint then*

$$f((g)) = g((f)).$$

Weil reciprocity can be expressed in terms of the tame symbol, in order to include the cases when the support of f and g have common points. The tame symbol on a curve C is defined as

$$\{f, g\}_P = (-1)^{ab}\left(\frac{f^b}{g^a}\right)(P),$$

where $a = \mathrm{ord}_P(f)$ and $b = \mathrm{ord}_P(g)$.

Theorem 0.2. *(Weil reciprocity law in terms of the tame symbol) The tame symbol satisfies the following reciprocity law*

$$\prod_P \{f, g\}_P = 1,$$

where the product is taken over all points P of the smooth projective curve C.

If Q is in the support of g but not in the support of f then

$$\{f, g\}_Q = f(Q)^b,$$

where $b = \mathrm{ord}_Q(g)$. As a consequence, if $|(f)| \cap |(g)| = \emptyset$ then

$$\prod_{Q \in |(g)|} \{f, g\}_Q = f((g)),$$

from which one recovers Theorem 0.1.

We give a different proof of the Weil reciprocity law for the tame symbol in Subsection 1.2 (Theorem 1.9), using iterated integrals. This proof is essential for the paper. Based on it we derive a reciprocity law for the bi-local symbols, for the Parshin symbol for surfaces, and for the newly defined 4-function local symbols. Moreover, the use of iterated integrals gives a geometric meaning to the symbols as homotopy invariant functions on double free loop space of a complex algebraic surface.

Before we present the definition of the Parshin symbol and the two 4-function local symbols, we need to introduce the following notation.

Let X be a smooth complex projective surface, let C be a smooth curve on the surface X and let P be a point on the curve C. For a non-zero rational function f_k on the surface X, let

$$a_k = \mathrm{ord}_C(f_k) \tag{0.1}$$

be the order of vanishing of f_k on the curve C. Let also x be a rational function on the surface X, representing a uniformizer at the curve C, such that any two irreducible components of the support of the divisor of x do not intersect at the point P. Let

$$b_k = \mathrm{ord}_P((x^{-a_k} f_k)|_C). \tag{0.2}$$

Definition 0.3. *The Parshin symbol for the surface X with respect to a curve C on X and a point P on C is defined as*

$$\{f_1, f_2, f_3\}_{C,P} = (-1)^K \left(f_1^{D_1} f_2^{D_2} f_3^{D_3}\right)(P), \tag{0.3}$$

where

$$D_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

and

$$K = a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2 + b_1b_2a_3 + b_2b_3a_1 + b_3b_1a_2.$$

The Parshin symbol was defined by Parshin in [P1] and [P2].

Definition 0.4. *We are going to use the terminology strict normal crossing divisor, which means that the irreducible components of the divisor are assumed smooth, meet transversally, and no three components of the divisor meet at a point.*

Theorem 0.5. *(Reciprocity laws of the Parshin symbol) Let f_1, f_2, f_3 be non-zero rational functions on a smooth complex projective surface X, then the following reciprocity laws hold:*

(a)

$$\prod_P \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is taken over all points P over a fixed curve C. Here we assume that the union of the divisors $\bigcup_{i=1}^3 |(f_i)|$ in X is a strict normal crossing divisor.

(b)

$$\prod_C \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is taken over all curves C passing through a fixed point P. Here we assume that the the divisor $\bigcup_{i=1}^3 |(f_i)|$ in $\tilde{X}$ is a strict normal crossing divisor. We denote by $\tilde{X}$ the blow-up of X at the point P.

Remark: The assumptions can be removed using invariance of the Parshin symbol under blow-ups, which allows to extend the definition of the Parshin symbol to any complex projective surface and any divisor $\bigcup_{i=1}^3 |(f_i)|$.

A K-theoretic proof of the reciprocity laws for the Parshin symbol was given by Kato [Ka]. We give an alternative proof (of Theorem 0.3), based on iterated integrals over membranes (Theorems 2.10 and 3.7). The usefulness of this proof is that it leads to relations of the Parshin symbol to double free loop space of a complex algebraic surface.

Main results

The main result of this paper is the construction of the new 4-function local symbols, their reciprocity laws and their relation to the double free loop space of the surface. A K-theoretic definition of the first 4-function local symbol (up to a sign) is given in the Appendix.

Definition 0.6. *(4-function local symbols) Using the notation from Equations (0.1) and (0.2), we define two new 4-function local symbols:*

$$\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = (-1)^L \frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4 - b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2 - b_1a_2}}(P).$$

and

$$\{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = (-1)^L \frac{\left(\frac{f_1^{a_2+b_2}}{f_2^{a_1+b_1}}\right)^{-(a_3b_4 - b_3a_4)}}{\left(\frac{f_3^{a_4+b_4}}{f_4^{a_3+b_3}}\right)^{-(a_1b_2 - b_1a_2)}}(P),$$

where $L = (a_1b_2 - b_1a_2)(a_3b_4 - b_3a_4)$.

Theorem 0.7. *(Reciprocity laws for the new 4-function local symbols) Let f_1, f_2, f_3, f_4 be non-zero rational functions on a smooth complex projective surface X, then the following reciprocity laws hold:*

(a)

$$\prod_P \{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = 1,$$

where the product is taken over all point P of a fixed curve C. Here we assume that the divisor $\bigcup_{i=1}^4 |(f_i)|$ in X has strict normal crossings.

(b)

$$\prod_C \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = 1,$$

where the product is taken over all curves C passing through a fixed point P. Here we assume that the divisor $\bigcup_{i=1}^4 |(f_i)|$ in $\tilde{X}$ is a strict normal crossing divisor after a single blow-up $\tilde{X} \to X$ at the point $P \in X$.

Remark: In part (a), for convenience for the analytic construction, we assume that $\bigcup_{i=1}^4 |(f_i)|$ is a strict normal crossing divisor. That assumption can be removed using invariance of the first 4-function local symbol under blow-ups, which allows to extend the definition of the 4-function local symbol to any complex projective surface and any divisor $\bigcup_{i=1}^4 |(f_i)|$. For part (b), we use a single blow-up $\tilde{X} \to X$ at the point $P \in X$. Then, we apply part (a) of the Theorem 0.7 for the first 4-function local symbol to the divisor $\bigcup_{i=1}^4 |(f_i)|$ on the surface $\tilde{X}$ (see Definition 3.8 and Lemma 3.9) and relate it to the second 4-function local symbol on x from part (b).

Our approach is based on new types of symbols which we call bi-local symbols. They allow us to refine the local symbols that we study (the Parshin symbol, the 4-function symbols) in the sense that the local symbols of interest are presented as products of the bi-local symbols and then reciprocity laws are proven for the latter.

Our reciprocity laws have a K-theoretic interpretation that can be found in the Appendix.

We learned from Pablos Romo that recently a third proof of the reciprocity laws for the 4-function local symbols, as well as new results about refinements of the Parshin symbol were obtained [PR2].

The work in this paper is related to other results in the area. Brylinski and McLaughlin (see [BrMcL]) used gerbes to define the Parshin symbol. Here we give an alternative, more analytic approach, based on iterated integrals over membranes. We should mention a few other approaches to tame symbols and to the Parshin symbol, for example, [D1], [OZh], [PR1].

Structure of the paper

In Subsection 1.1, we recall basic properties of iterated integrals over paths. Then, in Subsection 1.2, we prove Weil reciprocity, using iterated integrals, by establishing first a reciprocity law for a bi-local symbol, and then removing the dependence on the base point, we recover the Weil reciprocity for the tame symbol on a curve. Subsection 1.3 gives a construction of two foliations. They are needed for the definition of iterated integrals on membranes, presented in Subsection 1.4. Such integrals are the key technical ingredient in this paper.

Section 2 contains the first type of reciprocity laws for the Parshin symbol (Theorem 2.9) and for the first 4-function local symbol (Theorem 2.13), where the product of the symbols is over all points P of a fixed curve C on a surface X. The proofs are based on the reciprocity laws for bi-local symbols (Theorem 2.6) expressed as iterated integrals on membranes. Certain products of bi-local symbols become local symbols such as the Parshin symbol or the first 4-function local symbol. We call such products a refinement of the Parshin symbol or a refinement of the first 4-function local symbol. Moreover, the product of bi-local symbols that express the Parshin symbol or the first 4-function local symbol are homotopy invariant functions on a double free loop space (see Definition 2.5).

Section 3 is about the second type of reciprocity laws, where the product of the symbols is taken over all curves C on X passing though a fixed point P. We also define bi-local symbols suitable for the second type of reciprocity law. Then the corresponding reciprocity laws are proven for the bi-local symbols (Theorem 3.2), the Parshin symbol (Theorem 3.7), and the second 4-function local symbol (Theorem 3.11). The bi-local symbols in Section 3 provide a second type of refinement of the Parshin symbol and of the second 4-function local symbol.

For convenience of the reader, in the Appendix we give an alternative proof of the reciprocity laws of the 4-function local symbols based on Milnor K-theory.

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1 Geometric and analytic background

1.1 Iterated path integrals on complex curves

This Subsection contains a definition and properties of iterated integrals, which will be used for the definition of bi-local symbols and for another proof of the Weil reciprocity law in Subsection 1.2.

Let $C \subset \mathbb{P}^k$ be a smooth complex curve. Let f_1 and f_2 be two non-zero rational functions on C. Let

$$\gamma : [0, 1] \to C$$

be a path, which is a continuous, piecewise differentiable function on the unit interval.

Definition 1.1. *We define the following iterated integral*

$$\int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{0<t_1<t_2<1} \gamma^* \left(\frac{df_1}{f_1}\right)(t_1) \wedge \gamma^* \left(\frac{df_2}{f_2}\right)(t_2).$$

The two Lemmas below are due to K.-T. Chen [Ch1].

Lemma 1.2. *An iterated integral over a path γ on a smooth curve C is homotopy invariant with respect to a homotopy, fixing the end points of the path γ.*

Lemma 1.3. *If $\gamma = \gamma_1\gamma_2$ is a composition of two paths, where the end of the first path γ_1 is the beginning of the second path γ_2, then*

$$\int_{\gamma_1\gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\gamma_1} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma_1} \frac{df_1}{f_1} \int_{\gamma_2} \frac{df_2}{f_2} + \int_{\gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.$$

Let σ be a simple loop around a point P on C with a base point Q. Let $\sigma = \gamma\sigma_0\gamma^{-1}$, where σ_0 is a small loop around P, with a base the point R and let γ be a path joining the point Q with R.

The following Lemma is essential for the proof of the Weil reciprocity (see also [H1]).

Lemma 1.4. *With the above notation, the following holds*

$$\int_\sigma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_\gamma \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2}.$$

Proof. First, we use Lemma 1.3 for the composition $\gamma\sigma_0\gamma^{-1}$. We obtain

$$\begin{aligned} \textstyle\int_\sigma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} &= \textstyle\int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_\gamma \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \\ &\textstyle\quad + \int_\gamma \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \end{aligned} \tag{1.1}$$

Then, we use the homotopy invariance of iterated integrals, Lemma 1.2, for the path $\gamma\gamma^{-1}$. Thus,

$$0 = \int_{\gamma\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.$$

Finally, we use Lemma 1.3 for the composition of paths $\gamma\gamma^{-1}$. That gives

$$0 = \int_{\gamma\gamma^{-1}} \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = \int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_\gamma \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}. \tag{1.2}$$

The Lemma 1.4 follows from Equations (1.1) and (1.2). □

1.2 Weil reciprocity via iterated path integrals

Here, we present a proof of the Weil reciprocity law, based on iterated integrals and bi-local symbols. This method will be generalized in the later Subsections in order to prove reciprocity laws on complex surfaces. Similar ideas about the Weil reciprocity law are contained in [H1], however, without bi-local symbols.

Let x be a rational function on a curve $C \subset \mathbb{P}^k$, representing a uniformizer at P. Let

$$a_i = \mathrm{ord}_P(f_i).$$

and let

$$g_i = x^{-a_i} f_i.$$

Then

$$\frac{df_i}{f_i} = a_i \frac{dx}{x} + \frac{dg_i}{g_i}.$$

Let σ_0^ϵ be a small loop around the point P, whose points are at most at distance ϵ from the point P. One can take the metric inherited from the Fubini-Study metric on $\mathbb{P}^k$. Put $\sigma_0^\epsilon = \sigma_0$ in Lemma 1.4, then

$$\int_\gamma \frac{df_1}{f_1} \int_{\sigma_0^\epsilon} \frac{df_2}{f_2} = 2\pi i a_2 \int_\gamma \frac{df_1}{f_1} = 2\pi i a_2 \left(a_1 \int_\gamma \frac{dx}{x} + \int_\gamma \frac{dg_1}{g_1} \right).$$

Similarly,

$$\int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} = 2\pi i a_1 \left(-a_2 \int_\gamma \frac{dx}{x} - \int_\gamma \frac{dg_2}{g_2} \right).$$

From [H1], we have that

$$\lim_{\epsilon \to 0} \int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \frac{(2\pi i)^2}{2} a_1 a_2. \tag{1.3}$$

Using Lemma 1.4, we obtain

$$\int_\sigma \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = 2\pi i \left(a_2 \log(g_1) - a_1 \log(g_2) + \pi i a_1 a_2 \right) |_Q^P.$$

After exponentiation, we obtain

Lemma 1.5. *With the above notation the following holds*

$$\exp\left(\frac{1}{2\pi i} \int_\sigma \frac{df_1}{f_2} \circ \frac{df_2}{f_2} \right) = (-1)^{a_1 a_2} \frac{g_1^{a_2}}{g_2^{a_1}}(P) \left(\frac{g_1^{a_2}}{g_2^{a_1}}(Q) \right)^{-1} = (-1)^{a_1 a_2} \frac{f_1^{a_2}}{f_2^{a_1}}(P) \left(\frac{f_1^{a_2}}{f_2^{a_1}}(Q) \right)^{-1}$$

Definition 1.6. *(Bi-local symbol on a curve) With the above notation, we define a bi-local symbol*

$$\{f_1, f_2\}_P^Q = (-1)^{a_1 a_2} \frac{f_1^{a_2}}{f_2^{a_1}}(P) \left(\frac{f_1^{a_2}}{f_2^{a_1}}(Q) \right)^{-1}. \tag{1.4}$$

Let the curve C be of genus g and let $P_1, \ldots, P_n$ be the points of the union of the support of the divisors of f_1 and f_2. Let $\sigma_1, \ldots, \sigma_n$ be simple loops around the points $P_1, \ldots, P_n$, respectively. Let also $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the $2g$ loops on the curve C such that

$$\pi_1(C, Q) =< \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n > / \sim,$$

where $\delta \sim 1$, for

$$\delta = \prod_{i=1}^{n} \sigma_i \prod_{j=1}^{g} [\alpha_j, \beta_j].$$

From Theorem 3.1 in [H1], we have

Lemma 1.7.

$$\int_{\alpha\beta\alpha^{-1}\beta^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_\alpha \frac{df_1}{f_1} \int_\beta \frac{df_2}{f_2} - \int_\alpha \frac{df_2}{f_2} \int_\beta \frac{df_1}{f_1}.$$

Using the above result, we obtain that

$$0 = \int_\delta \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \in (2\pi i)^2 \mathbb{Z} + \sum_{i=1}^{n} \int_{\sigma_i} \frac{df_1}{f_1} \cdot \frac{df_2}{f_2},$$

where the sum is over simple loops σ_i around each of the points P_i. Then we obtain:

Theorem 1.8. *(Reciprocity law for the bi-local symbol* (1.4)*) With the above notation, the following holds*

$$\prod_P \{f_1, f_2\}_P^Q = 1.$$

If we want to make the above reciprocity law into a reciprocity law for a local symbol we have to remove the dependency on the base point Q. This can be achieved in the following way: In the reciprocity law for the bi-local symbol, the dependency on Q is

$$\prod_P f_1(Q)^{a_2} f_2(Q)^{-a_1} = f_1(Q)^{\sum_P Res_P \frac{df_2}{f_2}} \times f_2(Q)^{-\sum_P Res_P \frac{df_1}{f_1}} =$$

$$= f_1(Q)^0 f_2(Q)^0 = 1.$$

Thus, we recover Weil reciprocity:

Theorem 1.9. *(Weil reciprocity) The local symbol*

$$\{f_1, f_2\}_P = (-1)^{a_1 a_2} \frac{f_1^{a_2}}{f_2^{a_1}}(P).$$

satisfies the following reciprocity law

$$\prod_P \{f_1, f_2\}_P = 1,$$

where the product is over all points P in C.

1.3 Two foliations on a surface

The goal of this Subsection is to construct two foliations on a complex projective algebraic surface X in $\mathbb{P}^k$. This algebro-geometric material is needed for the definition of iterated integrals on membranes presented in Subsection 1.4.

Let f_1, f_2, f_3 and f_4 be four non-zero rational functions on the surface X. Let

$$C \cup C_1 \cup \cdots \cup C_n = \bigcup_{i=1}^{4} |div(f_i)|,$$

where we fix one of the irreducible components C. Let

$$\{P_1, \ldots, P_N\} = C \cap (C_1 \cup \cdots \cup C_n).$$

We can assume that the curves $C, C_1, \ldots, C_n$ are smooth and that they form a strict normal crossing divisor on X, by allowing blow-ups on the surface X. For the second type of reciprocity laws, where the product of the symbols is over all curves passing through a point P, we require that strict transforms of the curves $C, C_1, \ldots, C_n$ under a single blow-up at the point P together with the exceptional curve form a strict normal crossing divisor.

The two foliations have to satisfy the following

Conditions:

1. There exists a foliation F'_v such that
   (a) $F'_v = (f - v)_0$ are the level sets of a rational function
   $$f : X \to \mathbb{P}^1,$$
   for small values of v, (that is, for $|v| < \epsilon$ for a chosen ϵ);
   (b) F'_v is smooth for all but finitely many values of v;
   (c) F'_v has only nodal singularities;
   (d) $ord_C(f) = 1$;
   (e) $R_i \notin C_j$, for $i = 1, \ldots, M$ and $j = 1, \ldots, n$, where
   $$\{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m)$$
   and
   $$F'_0 = (f)_0 = C \cup D_1 \cup \cdots \cup D_m.$$
2. There exists a foliation G_w such that
   (a) $G_w = (g - w)_0$ are the level sets of a rational function
   $$g : X \to \mathbb{P}^1;$$
   (b) G_w is smooth for all but finitely many values of w;
   (c) G_w has only nodal singularities;
   (d) $g|_C$ is non constant.

3. Coherence between the two foliations F' and G:
   (a) All but finitely many leaves of the foliation G are transversal to the curve C.
   (b) $G_{g(P_i)}$ intersects the curve C transversally, for $i = 1, \ldots, N$. (For definition of the points P_i see the beginning of this Subsection.)
   (c) $G_{g(R_i)}$ intersects the curve C transversally, for $i = 1, \ldots, M$. (For definition of the points R_i see condition 1(e).)

The existence of $f \in \mathbb{C}(X)^\times$ satisfying properties 1(a-d) is a direct consequence from the following result, which follows immediately from (a special case of) a result of Thomas ([Th], Theorem 4.2).

Theorem 1.10. *Consider a smooth curve C in a smooth projective surface X, with hyperplane section H_X. There exists a large constant $N \in \mathbb{N}$ and a pencil in $|NH_X|$, given as the level sets $(f-x)_0$ of some rational function f such that $(f-x)_0$ is smooth for all but finitely many values of x, at which it has only nodal singularities, and $C \subset (f)_0$.*

Moreover, a general choice of $g \in \mathbb{C}(X)^\times$ will satisfy 2(a-d) and 3(a-c). (For instance, the quotient of two generic linear forms on $\mathbb{P}^k$ restricted to C will not have branch points in $\{P_i\} \cup \{R_j\}$.)

It remains to examine property 1(e). The proof of Theorem 4.2 in [op. cit.] contains the basic

Observation: *The base locus of the linear system $H^0(I_C(N))$ is the smooth curve C for $N \gg 0$. So by Bertini's theorem the general element of the linear system is smooth away from C.*

Consider $C \subset X$. By the Observation, there exists $\mathcal{F} \in H^0(X, \mathcal{O}(N))$ such that $\mathrm{ord}_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$, where D is a second smooth curve on X, meeting C transversally (if at all).

Claim: *We may choose $\mathcal{F}$ so that condition 1(e) holds, that is, $R_i \notin C_j$ for each i, j, where $\{R_1, \ldots, R_M\} = C \cap D$. Equivalently, $C \cap D \cap C_j = \emptyset$.*

Proof. Define $H^0(I_C(N))^{reg}$ to be the subset of $H^0(X, I_C(N))$ whose elements $\mathcal{F}$ satisfy $\mathrm{ord}_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$ as above. Assume that for every $N >> 0$ and $\mathcal{F} \in H^0(I_C(N))^{reg}$ we have $D \cap C \cap C_j \neq \emptyset$ for some particular j. If we obtain a contradiction (for some N) then the claim is proved, since this is a closed condition for each j.

According to our assumption, $(\mathcal{F})$ always has an ordinary double point at the intersection $\Delta := C \cap C_j \neq \emptyset$. In the exact sequence

$$0 \to H^0(X, I_C^2(N)) \to H^0(X, I_C(N)) \to H^0(C, \mathcal{N}^*_{C/X}(N)) \to H^1(X, I_C^2(N)),$$

the last term vanishes by ([GH], Vanishing Theorem B) for N sufficiently large. Hence, every section over C of the twisted conormal sheaf $\mathcal{N}^*_{C/X}(N)$ has a zero along $\Delta = C \cap C_j$.

Next consider the exact sequence

$$0 \to H^0(C, I_\Delta \otimes \mathcal{N}^*_{C/X}(N)) \to H^0(C, \mathcal{N}^*_{C/X}(N)) \to \mathbb{C}^{|\Delta|} \to H^1(C, I_\Delta \otimes \mathcal{N}^*_{C/X}(N)).$$

The last term vanishes again by [loc. cit.]. Denote the third arrow by ev_Δ. Then we can take a section of $\mathcal{N}^*_{C/X}(N)$ not vanishing on Δ simply by taking an element in the preimage of $ev_\Delta(1,\ldots,1)$. This produces the desired contradiction. $\square$

Consider a metric on the projective surface X, which respects the complex structure. For example, we can take the metric inherited from the Fubini-Study metric on $\mathbb{P}^k$ via the embedding $X \hookrightarrow \mathbb{P}^k$. Let $U_1^\epsilon,\ldots,U_M^\epsilon$ be disks of radii ϵ on C, centered respectively at $R_1,\ldots,R_M$. Let

$$C_0 = C - \bigcup_{j=1}^{M} U_j^\epsilon - \{P_1,\ldots,P_N\}.$$

Definition 1.11. *With the above notation, let F_v be the connected component of*

$$F'_v - \left(\bigcup_{i=1}^{M} G_{g(U_i^\epsilon)}\right) \cap F'_v,$$

containing C_0, for $|v| << \epsilon$, where

$$G_{g(U_i^\epsilon)} = \bigcup_{w\in U_i^\epsilon} G_{g(w)}$$

Lemma 1.12. *With the above notation, for small values of $|v|$, we have that each leaf F_v is a continuous deformation of $F_0 = C_0$, preserving homotopy type.*

Proof. From Property 3(c), it follows that C and D_i meet R_j transversally (if at all). At the intersection R_i, locally we can represent the curves by $xy = 0$. The deformation leads to $v - xy = 0$, which is a leaf of F', locally near R_i. Consider a disk U of radius ϵ_i at $(x,y) = (0,0)$ in the xy-plane. Then for $|v| << \epsilon_i$ we have that U separates F'_v into 2 components, one close to the x-axis and the other close to the y-axis. We do the same for each of the points $R_1,\ldots,R_M$ and we take the minimum of the bounds ϵ_i. Then F_v will consist of points close to the curve C_0. $\square$

1.4 Iterated integrals on a membrane. Definitions and properties

In this Subsection, we define types of iterated integrals over membranes, needed in most of this manuscript.

Let τ be a simple loop around C_0 in $X - C_0 - \left(\bigcup_{i=1}^{M} G_{g(U_i^\epsilon)}\right)$. Let σ be a loop on the curve C_0. We define a *membrane* m_σ associated to a loop σ in C_0 by

$$m_\sigma : [0,1]^2 \to X$$

and

$$m_\sigma(s,t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}.$$

Note that for fixed values of s and t, we have that

$$F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$

consists of finitely many points, where F and G are foliations satisfying the Conditions in Subsection 1.3 and Lemma 1.12.

Claim: The image of m_σ is a torus.

Indeed, consider a tubular neighborhood around a loop σ on the curve C_0. One can take the following tubular neighborhood:

$$\bigcup_{|v|<\epsilon} F_v \cap G_{g(\sigma)}$$

of σ. Its boundary is $F_{f(\tau)} \cap G_{g(\sigma)}$, where τ is a simple loop around C_0 on $X - \bigcup_{i=1}^{n} C_i - \bigcup_{j=1}^{m} D_j$ and $|f(\tau(t))| = \epsilon$.

We shall define the simplest type of iterated integrals over membranes. Also, we are going to construct local symbols in terms of iterated integrals I_1, I_2, I_3, I_4 on membranes, defined below.

We define the following differential forms

$$A(s,t) = m^*\left(\frac{\mathrm{d}f_1}{f_1} \wedge \frac{\mathrm{d}f_2}{f_2}\right)(s,t)$$

$$b(s,t) = m^*\left(\frac{\mathrm{d}f_3}{f_3}\right)(s,t)$$

and

$$B(s,t) = m^*\left(\frac{\mathrm{d}f_3}{f_3} \wedge \frac{\mathrm{d}f_4}{f_4}\right)(s,t).$$

The first diagram

$$\begin{array}{cc} t & \boxed{A} \\ & s \end{array}$$

denotes

$$I_1 = \int_0^1 \int_0^1 A(s,t).$$

The second diagram

$$\begin{array}{cc} t_2 & \boxed{b} \\ t_1 & \boxed{A} \\ & s \end{array}$$

denotes

$$I_2 = \int_0^1 \int\int_{0<t_1<t_2<1} A(s,t_1) \wedge b(s,t_2).$$

Note that the iteration happens with respect to t_1 and t_2. In other word,

$$\int\int_{0<t_1<t_2<1} A(s,t_1) \wedge b(s,t_2)$$

is a differential 1-form on the loop space of X(see [Ch2]). These differential 1-forms on the loop space of X are closed, since $dA = db = 0$ and $A(s,t) \wedge b(s,t) = 0$.

The third diagram

$$\begin{array}{c|c|c|}
\cline{2-3}
t & A & b \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{s_1} & \multicolumn{1}{c}{s_2}
\end{array}$$

denotes

$$I_3 = \int\int_{0<s_1<s_2<1}\int_0^1 A(s_1,t)\wedge b(s_2,t).$$

And the fourth diagram

$$\begin{array}{c|c|c|}
\cline{2-3}
t_2 & & B \\
\cline{2-3}
t_1 & A & \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{s_1} & \multicolumn{1}{c}{s_2}
\end{array}$$

denotes

$$I_4 = \int\int_{0<s_1<s_2<1}\int\int_{0<t_1<t_2<1} A(s_1,t_1)\wedge B(s_2,t_2).$$

The integral I_4 is a homotopy invariant function with variable the torus of integration m. The proof of the homotopy invariance for iterated integrals over membranes, such at I_4, can be found in [H4] and in more general form in [H5].

Local symbols will be defined via the above four types of iterated integrals. The integrals that we define below, used for defining bi-local symbols, are a technical tool for proving reciprocity laws for the local symbols. Bi-local symbols also satisfy reciprocity laws.

Consider the dependence of $\log(f_i(m(s,t)))$ on the variables s and t via the parametrization of the membrane m.

Definition 1.13. *Let*

$$l_i(s,t) = \log(f_i(m(s,t)))$$

We have

$$\mathrm{d}l_i(s,t) = \frac{\partial l_i(s,t)}{\partial s}\mathrm{d}s + \frac{\partial l_i(s,t)}{\partial t}\mathrm{d}t.$$

$$b(s,t) = \mathrm{d}l_3(s,t)$$

$$A(s,t) = \frac{\partial l_1(s,t)}{\partial s}\frac{\partial l_2(s,t)}{\partial t}\mathrm{d}s\wedge\mathrm{d}t - \frac{\partial l_1(s,t)}{\partial t}\frac{\partial l_2(s,t)}{\partial s}\mathrm{d}s\wedge\mathrm{d}t \tag{1.5}$$

$$B(s,t) = \frac{\partial l_3(s,t)}{\partial s}\frac{\partial l_4(s,t)}{\partial t}\mathrm{d}s\wedge\mathrm{d}t - \frac{\partial l_3(s,t)}{\partial t}\frac{\partial l_4(s,t)}{\partial s}\mathrm{d}s\wedge\mathrm{d}t \tag{1.6}$$

The above equations express the differential forms A, B and b is terms of monomials in terms of first derivatives of l_1, l_2, l_3, l_4. We are going to define bi-local symbols associated to monomials in first derivatives of l_1, l_2, l_3, l_4, which occur in

$$A(s,t),\; A(s,t_1)\wedge b(s,t_2),\; A(s_1,t)\wedge b(s_2,t),\; \text{and } A(s_1,t_2)\wedge B(s_2,t_2)$$

Definition 1.14. *(Iterated integrals on membranes) Let $f_1, \dots, f_{k+l}$ be rational functions on X, where the pairs (k,l) will be superscripts of the integrals. Let m be a membrane as above. We define:*

(a) $I^{(1,1)}(m; f_1, f_2) =$

$$= \int_0^1 \int_0^1 \left(\frac{\partial l_1(s,t)}{\partial s}\mathrm{d}s\right) \wedge \left(\frac{\partial l_2(s,t)}{\partial t}\mathrm{d}t\right)$$

(b) $I^{(1,2)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\le s\le 1;\ 0\le t_1\le t_2\le 1} \left(\frac{\partial l_1(s,t_1)}{\partial s}\frac{\partial l_2(s,t_1)}{\partial t_1}\mathrm{d}s \wedge \mathrm{d}t_1\right) \wedge \left(\frac{\partial l_3(s,t_2)}{\partial t_2}\mathrm{d}t_2\right)$$

(c) $I^{(2,1)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\le s_1\le s_2\le 1;\ 0\le t\le 1} \left(\frac{\partial l_1(s_1,t)}{\partial s_1}\frac{\partial l_2(s_1,t)}{\partial t}\mathrm{d}s_1 \wedge \mathrm{d}t\right) \wedge \left(\frac{\partial l_3(s_2,t)}{\partial s_2}\mathrm{d}s_2\right)$$

(d) $I^{(2,2)}(m; f_1, f_2, f_3, f_4) =$

$$= \int\int\int\int_{0\le s_1\le s_2\le 1;\ 0\le t_1\le t_2\le 1} \left(\frac{\partial l_1(s_1,t_1)}{\partial s_1}\frac{\partial l_2(s_1,t_1)}{\partial t_1}\mathrm{d}s_1 \wedge \mathrm{d}t_1\right) \wedge$$
$$\wedge \left(\frac{\partial l_3(s_2,t_2)}{\partial s_2}\frac{\partial l_4(s_2,t_2)}{\partial t_2}\mathrm{d}s_2 \wedge \mathrm{d}t_2\right)$$

Proposition 1.15. *(a)* $I_1 = I^{(1,1)}(m; f_1, f_2) - I^{(1,1)}(m; f_2, f_1)$*;*
(b) $I_2 = I^{(1,2)}(m; f_1, f_2, f_3) - I^{(1,2)}(m; f_2, f_1, f_3)$*;*
(c) $I_3 = I^{(2,1)}(m; f_1, f_2, f_3) - I^{(2,1)}(m; f_2, f_1, f_3)$*;*
(d) $I_4 = I^{(2,2)}(m; f_1, f_2, f_3, f_4) - I^{(2,2)}(m; f_2, f_1, f_3, f_4) - I^{(2,2)}(m; f_1, f_2, f_4, f_3) + I^{(2,2)}(m; f_2, f_1, f_4, f_3)$*;*

Consider a metric on the projective surface X inherited from the Fubini-Study metric on $\mathbb{P}^k$. Let τ be a simple loop around the curve C of distance at most ϵ from C. We are going to take the limit as $\epsilon \to 0$. Informally, the radius of the loop τ goes to zero. Then we have the following lemma.

Lemma 1.16. *With the above notation the following holds:*
(a)

$$\lim_{\epsilon\to 0} I^{(1,1)}(m_\sigma, f_1, f_2) = (2\pi i) Res \frac{df_2}{f_2} \int_\sigma \frac{df_1}{f_1}$$

(b)

$$\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3) = \frac{(2\pi i)^2}{2} Res \frac{df_2}{f_2} Res \frac{df_3}{f_3} \int_\sigma \frac{df_1}{f_1}$$

(c)

$$\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3) = -(2\pi i) Res \frac{df_2}{f_2} \int_\sigma \frac{df_1}{f_1} \circ \frac{df_3}{f_3}$$

(d)
$$\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4) = -\frac{(2\pi i)^2}{2} Res\frac{df_2}{f_2} Res\frac{df_4}{f_4} \int_\sigma \frac{df_1}{f_1} \circ \frac{df_3}{f_3}$$

Proof. First, we consider the integrals in parts (a) and (c), where there is integration with respect to the variable t in the definition of the membrane m. Let $m(s,\cdot)$ denote the loop obtained by fixing the first variable s and varying the second variable t.Then, there is no iteration along the loop $m(s,\cdot)$ around the curve C, for fixed value of s. Using Properties 1(d) and e(b), the integration over the loop $m(s,\cdot)$ gives us a single residue. This process is independent of the base point of the loop $m(s,\cdot)$. That proves parts (a) and (c).

For parts (b) and (d), we have a double iteration along the loop $m(s,\cdot)$ around the curve C, where the value of s is fixed and the second argument varies. After taking the limit as ϵ goes to 0, the integral along $m(s,\cdot)$, with respect to t_1 and t_2, becomes a product of two residues (see Equation (1.3)), which are independent of a base point. That proves parts (b) and (d). $\square$

2 First type of reciprocity laws

2.1 Reciprocity laws for bi-local symbols

In this Subsection, we define bi-local symbols and prove their reciprocity laws. Using them, in the following two Sections, we establish the first type of reciprocity laws for the Parshin symbol and for a new 4-function symbol. By a first type of reciprocity law, we mean that the product of the local symbols is taken over all points P of a fixed curve C on the surface X.

Consider the fundamental group of C_0. We recall that C_0 is essentially the curve C without several intersection points and without several open neighborhoods. More precisely,
$$C_0 = C - \left(\bigcup_{j=1}^{m} G_{U_j^\epsilon}\right) \cap C - \left(\bigcup_{i=1}^{n} C_i\right) \cap C.$$
where U_j^ϵ is a small neighborhood of R_j on the complex curve C. We recall the notation for the intersection points
$$\{P_1,\dots,P_N\} = C \cap (C_1 \cup \dots \cup C_n),$$
$$\{R_1,\dots,R_M\} = C \cap (D_1 \cup \dots \cup D_m),$$
Let
$$\pi_1(C_0, Q) =< \sigma_1,\dots,\sigma_n, \alpha_1, \beta_1,\dots,\alpha_g,\beta_g > / \sim$$
be a presentation of the fundamental group, where
$$\delta \sim 1,$$
for
$$\delta = \prod_{i=1}^{n} \sigma_i \prod_{j=1}^{g} [\alpha_j, \beta_j].$$

We are going to drop the indices i and j. Thus, we are going to write P instead of P_i or R_j and σ instead of σ_i. Consider the definition of a membrane m_σ, associated to a loop σ, given in the beginning of Subsection 1.4. Let $m_\sigma(s,\cdot)$ be the loop obtained by fixing the variable s and letting the second argument vary. Similarly, $m_\sigma(\cdot,t)$ denotes the loop obtained by fixing the variable t and letting the first argument vary.

Definition 2.1. *Let* $a_k = \mathrm{ord}_C(f_k)$ *and* $b_k = \mathrm{ord}_P((x^{-a_k}f_k)|_C)$, *where* x *is a rational function, representing an uniformizer such that* $\mathrm{ord}_C(x) = 1$ *and* P *is not an intersection of any two of the components of the divisor of* x.

It is straightforward to represent the order of vanishing as residues, given by the following:

Lemma 2.2. *We have*

$$a_k = \frac{1}{2\pi i}\int_{m_\sigma(s,\cdot)} \frac{df_k}{f_k} \qquad \textit{and} \qquad b_k = \frac{1}{2\pi i}\int_{m_\sigma(\cdot,t)} \frac{df_k}{f_k}.$$

Using properties 1(d) and 3(b), we should think of $m_\sigma(\cdot,t)$ and $m_\sigma(s,\cdot)$ as translates of σ and of τ, respectively. Then the above integrals are residues, which detect the order of vanishing. For example a_k is the order of vanishing of f_k along a generic point of C. Then the following theorem holds, whose proof is immediate from Lemmas 1.16 and 2.2.

Theorem 2.3. *(a)*

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,1)}(m_\sigma, f_1, f_2) = a_2 b_1,$$

(b)

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3) = (\pi i) a_2 a_3 b_1,$$

(c)

$$\exp\Big((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3)\Big) = \Big(\{f_1, f_3\}_P^Q\Big)^{-a_2},$$

(d)

$$\exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4)\right) = \Big(\{f_1, f_3\}_P^Q\Big)^{-a_2 a_4}.$$

Let us denote by α the loop α_j and by β the loop β_j. Then the following lemma holds

Lemma 2.4. *(a)*

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,1)}(m_{[\alpha,\beta]} f_1, f_2) = 0,$$

(b)

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_{[\alpha,\beta]}, f_1, f_2, f_3) = 0,$$

(c)

$$\exp\Big((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_{[\alpha,\beta]}, f_1, f_2, f_3)\Big) = 1,$$

(d)

$$\exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_{[\alpha,\beta]}, f_1, f_2, f_3, f_4)\right) = 1.$$

Proof. It follows from Lemmas 1.16 and 1.7. A more modern proof follows from the well-definedness of the integral Beilinson regulator on K_2 on the level of homology (see [Ke1].)

Definition 2.5. *(Bi-local symbols on a surface) For a simple loop σ around a point P in C_0, based at Q, let*

$$Log^{(i,j)}[f_1,\ldots,f_{i+j}]^{(1),Q}_{C,P} = \lim_{e\to 0} I^{(i,j)}(m_\sigma, f_1,\ldots,f_{i+j}),$$

$$^{1,2}[f_1,f_2,f_3]^{(1),Q}_{C,P} = \exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma,f_1,f_2,f_3)\right),$$

$$^{2,1}[f_1,f_2,f_3]^{(1),Q}_{C,P} = \exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma,f_1,f_2,f_3)\right),$$

$$^{2,2}[f_1,f_2,f_3,f_4]^{(1),Q}_{C,P} = \exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma,f_1,f_2,f_3,f_4)\right).$$

The following reciprocity laws hold for the above bi-local symbols.

Theorem 2.6. *(a)* $\sum_P Log^{1,1}[f_1,f_2]^{(1),Q}_{C,P} = 0.$
(b) $\prod_P {}^{1,2}[f_1,f_2,f_3]^{(1),Q}_{C,P} = 1.$
(c) $\prod_P {}^{2,1}[f_1,f_2,f_3]^{(1),Q}_{C,P} = 1.$
(d) $\prod_P {}^{2,2}[f_1,f_2,f_3,f_4]^{(1),Q}_{C,P} = 1.$

Proof. Parts (b), (c) and (d) follow directly from Theorem 2.3 and from Weil reciprocity. Part (a) follows again from Theorem 2.3 and the theorem that the sum of the residues of a differential form on a curve is zero. □

2.2 Parshin symbol and its first reciprocity law.

In this Subsection, we construct a refinement of the Parshin symbol in terms of six bi-local symbols. Using this presentation of the Parshin symbol, Definition 2.7 and Theorem 2.8, we prove the first reciprocity of the Parshin symbol (Theorem 2.9).

Definition 2.7. *We define the following bi-local symbol*

$$\begin{aligned} Pr^Q_{C,P} = &\left({}^{1,2}[f_1,f_2,f_3]^{(1),Q}_{C,P}\right)\left({}^{1,2}[f_2,f_3,f_1]^{(1),Q}_{C,P}\right)\left({}^{1,2}[f_3,f_1,f_2]^{(1),Q}_{C,P}\right)\times \\ &\times\left({}^{2,1}[f_1,f_2,f_3]^{(1),Q}_{C,P}\right)\left({}^{2,1}[f_2,f_3,f_1]^{(1),Q}_{C,P}\right)\left({}^{2,1}[f_3,f_1,f_2]^{(1),Q}_{C,P}\right)\end{aligned}$$

at the points $P = P_i \in C\cap(C_1\cup\cdots\cup C_n)$ and a fixed point Q in $C - C\cap(C_1\cup\cdots\cup C_n)$.

Using Theorem 2.3 parts (b) and (c), we obtain:

Theorem 2.8. *(Refinement of the Parshin symbol) We have the following explicit formula*

$$Pr^Q_{C,P} = (-1)^K\frac{\left(f_1^{D_1}f_2^{D_2}f_3^{D_3}\right)(P)}{\left(f_1^{D_1}f_2^{D_2}f_3^{D_3}\right)(Q)},$$

where
$$D_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \ D_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \ D_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
and
$$K = a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2 + b_1b_2a_3 + b_2b_3a_1 + b_3b_1a_2.$$

Note that $Pr^Q_{C,P}$ is essentially the Parshin symbol given in Definition (0.3). Recall, the Parshin symbol is

$$\{f_1, f_2, f_3\}_{C,P} = (-1)^K \left(f_1^{D_1} f_2^{D_2} f_3^{D_3}\right)(P).$$

The only difference between the two symbols is the constant factor in $Pr^Q_{C,P}$, depending only on the base point Q (the denominator of $Pr^Q_{C,P}$). Rescaling by that constant leads to the Parshin symbol.

Theorem 2.9. *(First reciprocity law for the Parshin symbol) For the Parshin symbol, the following reciprocity law holds*

$$\prod_P \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is taken over points P in $C \cap (C_1 \cup \cdots \cup C_n)$. (When P is another point of C then the symbol is trivial.)

Proof. Without loss of generality, we can assume that the divisor $\bigcup_{i=1}^3 |(f_i)|$ in X is a strict normal crossing divisor. The assumption can be achieved, first, by considering successive blow-ups of the surface X to obtain a strict normal crossing divisor. Then we can use the invariance of the Parshin symbol under blow-ups. Indeed, if $\bigcup_{i=1}^3 |(f_i)|$ is a strict normal crossing divisor then a blow-up will change the parameters from (a_i, b_i) to $(a_i, a_i + b_i)$ for $i = 1, 2, 3$. Note that the Parshin symbol is invariant under such transformation. If we have two resolutions of singularities via blow-ups of X, $X_1 \to X$ and $X_2 \to X$, we can take their fiber product $X_1 \times_X X_2 \to X$, which will be a third resolution of singularity. Using the above change of coordinates we obtain that the Parshin symbols on X_1 and on $X_1 \times_X X_2$, will be the same. Similarly, the Parshin symbols on X_2 and on $X_1 \times_X X_2$. Thus, the Pashin symbol is independent of blow-ups.

The reciprocity law for the bi-local symbol Pr^Q_P follows at once from Theorem 2.6 parts (b) and (c). It is related to the Parshin symbol by the formula

$$\{f_1, f_2, f_3\}_{C,P} = Pr^Q_{C,P} \left(f_1^{D_1} f_2^{D_2} f_3^{D_3}\right)(Q).$$

Now, we remove the dependence of Pr^Q_P on the base point Q. In order to do that, note that

$$\prod_P f_1(Q)^{D_1} = g_1(Q)^{\sum_P D_1}.$$

Here $g_1 = x^{-a_1} f_1$, where x is a rational function on the surface X, representing an uniformazer at the curve C, such that the components of the divisor of x do not intersect at the points P or Q. Moreover,

$$D_1 = (2\pi i)^{-2} \left(Log^{1,1}[f_2, f_3]^{(1),Q}_P - Log^{1,1}[f_3, f_2]^{(1),Q}_P\right)$$

by Theorem 2.3 part (a) and Proposition 1.15 part (a). Using Theorem 2.6 part (a), for the above equality, we obtain

$$\sum_P D_1 = 0.$$

Therefore,

$$\prod_P g_1(Q)^{D_1} = 1.$$

Similarly,

$$\prod_P g_2(Q)^{D_2} = 1 \text{ and } \prod_P g_3(Q)^{D_3} = 1,$$

where $g_k = x^{-a_k} f_k$. □

2.3 First 4-function local symbol and its reciprocity law

In this Subsection, we define a new 4-function local symbol on a surface. We also express the new 4-function local symbol as a product of bi-local symbols (Definition 2.10 and Proposition 2.11), which serves as a refinement similar to the refinement of the Parshin symbol in Subsection 2.2. Using the reciprocity laws for bi-local symbols established in Subsection 2.1, we obtain the first type of reciprocity law for the new 4-function local symbol (Theorem 2.13).

Definition 2.10. *We define the following bi-local symbol, which will lead to the 4-function local symbol on a surface.*

$$PR_{C,P}^{Q} = \left({}^{2,2}[f_1, f_2, f_3, f_4]_P^{(1),Q}\right) \left({}^{2,2}[f_1, f_2, f_4, f_3]_P^{(1),Q}\right)^{-1} \times$$
$$\times \left({}^{2,2}[f_2, f_1, f_3, f_4]_P^{(1),Q}\right)^{-1} \left({}^{2,2}[f_2, f_1, f_4, f_3]_P^{(1),Q}\right).$$

Using Theorem 2.3, part (d), we obtain:

Proposition 2.11. *Explicitly, the bi-local symbol $PR_{C,P}^{Q}$ is given by*

$$PR_{C,P}^{Q} = (-1)^L \frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(P) \cdot \left(\frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(Q) \right)^{-1}, \tag{2.1}$$

where

$$L = (a_1b_2 - a_2b_1)(a_3b_4 - a_4b_3).$$

Definition 2.12. *(4-function local symbol) With the above notation, we define a 4-function local symbol*

$$\{f_1, f_2, f_3, f_4\}_{C,P}^{(1)} = (-1)^L \frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(P).$$

It is an easy exercise to check that the symbol $\{f_1,f_2,f_3,f_4\}^{(1)}_{C,P}$ is independent of the choices of local uniformizers. See also the Appendix for K-theoretical approach for the 4-function local symbol. Note that the relation between the bi-local symbol $PR^C_{C,P}$ and the local symbol $\{f_1,f_2,f_3,f_4\}^{(1)}_{C,P}$ is only a constant factor depending on the base point Q. There is a similar relation between the bi-local symbol $Pr^Q_{C,P}$ and the Parshin symbol $\{f_1,f_2,f_3\}_{C,P}$.

Theorem 2.13. *(Reciprocity law for the first 4-function local symbol) The following reciprocity law for the 4-function local symbol on a surface holds*

$$\prod_P \{f_1,f_2,f_3,f_4\}^{(1)}_{C,P} = 1,$$

where the product is taken over points P on a fixed curve C.

Proof. Without loss of generality, we can assume that the divisor $\bigcup_{i=1}^4 |(f_i)|$ in X is a strict normal crossing divisor. The assumption can be achieved, first, by considering successive blow-ups of the complex projective surface X to a obtain strict normal crossing divisor.

Similar to the Parshin symbol the 4-function local symbol is invariant under blow-ups. Indeed, if $\bigcup_{i=1}^4 |(f_i)|$ is a strict normal crossing divisor then a blow-up will change the parameters from (a_i,b_i) to (a_i,a_i+b_i) for $i=1,2,3,4$. Note that the first 4-function local symbol is invariant under such transformation.

Note also that both a torus of integration and the two foliations can push-forward with respect to a blow-up map. That allows to transfer analytic tools to a general complex projective surface.

Using Theorem 2.6 part (d), we obtain that the bi-local symbol $PR^Q_{C,P}$ satisfies a reciprocity law, namely,

$$\prod_P PR^Q_{C,P} = 1, \tag{2.2}$$

where the product is over all points P in $C\cap(C_1\cup\cdots\cup C_n)$. In order to complete the proof of Theorem 2.13, we proceed similarly to the proof of the first Parshin reciprocity law. Namely,

$$\prod_P g_1(Q)^{a_2(a_3b_4-a_4b_3)} = g_1(Q)^{a_2\sum_P a_3b_4-a_4b_3} = g(Q)^{b_2\cdot 0} = 1, \tag{2.3}$$

where $g_1 = x^{-a_1}f_1$ and x is a rational function representing an uniforminzer at the curve C, such that the components of the divisor of x do not intersect at the points P or Q. The last equality of (2.3) holds, because

$$a_3b_4 - a_4b_3 = (2\pi i)^{-2}\left(Log^{1,1}[f_3,f_4]^{(1),Q}_{C,P} - Log^{1,1}[f_4,f_3]^{(1),Q}_{C,P}\right) = 0$$

and

$$\sum_P (2\pi i)^{-2}\left(Log^{1,1}[f_3,f_4]^{(1),Q}_{C,P} - Log^{1,1}[f_4,f_3]^{(1),Q}_{C,P}\right) = 0,$$

by Theorem 2.3 (a) and Theorem 2.6 (a), respectively. $\square$

There is one more interesting relation for the 4-function symbol, whose is a direct consequence of the explicit formula of the symbol.

Theorem 2.14. *Let*

$$R_{ijkl} = \{f_i, f_j, f_k, f_l\}_{C,P}.$$

Then R_{ijkl} has the same symmetry as the symmetry of a Riemann curvature tensor with respect to permutations of the indices, namely

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = -R_{klij}.$$

3 Second type of reciprocity laws

3.1 Bi-local symbols revisited

In this Subsection, we define bi-local symbols, designed for proofs of the second type of reciprocity laws for local symbols. These bi-local symbols also satisfy reciprocity laws. Using them, in the following two sections, we establish the second type of reciprocity laws for the Parshin symbol and for a new 4-function new symbol. By a second type of reciprocity law, we mean that the product of the local symbols is taken over all curves C on the surface X, passing through a fixed point P.

Let $C_1, \dots, C_n$ be curves in X intersecting at a point P. Assume that $C_1, \dots, C_n$ are among the divisors of the rational functions $f_1, \dots, f_4$. Let $\tilde{X}$ be the blow-up of X at the point P. Assume that after the blow-up the curves above $C_1, \dots, C_n$ meet transversally the exceptional curve E and no two of them intersect at a point on the exceptional curve E.

Let D be a curve on $\tilde{X}$ such that D intersects E in one point. Setting

$$\tilde{P}_k = E \cap \tilde{C}_k,$$

where $\tilde{C}_k$ is the strict transform of C_k under the blow-up, and

$$Q = E \cap D,$$

Definition 3.1. *We define the following bi-local symbols*

$${}^{i,j}[f_1, \dots, f_{i+j}]^{(2),D}_{C_k,P} :={}^{i,j} [f_1, \dots, f_{i+j}]^{(1),Q}_{E,\tilde{P}_k}.$$

Theorem 3.2. *The following reciprocity laws for bi-local symbols hold:*

(a)

$$\prod_{C_k} {}^{1,2}[f_1, f_2, f_3]^{(2),D}_{C_k,P} = 1,$$

(b)

$$\prod_{C_k} {}^{2,1}[f_1, f_2, f_3]^{(2),D}_{C_k,P} = 1,$$

(c)

$$\prod_{C_k} {}^{2,2}[f_1, f_2, f_3, f_4]^{(2),D}_{C_k,P} = 1,$$

where the product is over the curves C, among the divisors of at least one of the rational functions $f_1, \dots, f_4$, which pass through the point P.

The proof is reformulation of Theorem 2.6, where the triple (C_k, D, P) in the above Theorem correspond to the triple (P, Q, C) with $P = C_k \cap E$ and $Q = D \cap E$ in Theorem 2.6, where the curve C in Theorem 2.6 corresponds to the curve E.

3.2 Parshin symbol and its second reciprocity law.

In this Subsection, we present an alternative refinement of the Parshin symbol in terms of bi-local symbols (Definition 3.3). This implies the second reciprocity law for the Parshin symbol, since each of the bi-local symbols satisfy the second type of reciprocity laws (see Subsection 3.1).

Definition 3.3. *We define the following bi-local symbol, useful for the proof of the second reciprocity law of the Parshin symbol*

$$\begin{aligned} Pr^D_{C,E} = &\left({}^{1,2}[f_1, f_2, f_3]^{(2),D}_{C,P}\right)\left({}^{1,2}[f_2, f_3, f_1]^{(2),D}_{C,P}\right)\left({}^{1,2}[f_3, f_1, f_2]^{(2),D}_{C,P}\right) \times \\ &\times \left({}^{2,1}[f_1, f_2, f_3]^{(2),D}_{C,P}\right)\left({}^{2,1}[f_2, f_3, f_1]^{(2),D}_{C,P}\right)\left({}^{2,1}[f_3, f_1, f_2]^{(2),D}_{C,P}\right), \end{aligned}$$

Let $\tilde{P} = \tilde{C} \cap E$, $Q = D \cap E$. Then

$$Pr^D_{C,E} = Pr^Q_{E,\tilde{P}}.$$

Similarly to the proof of Theorem 2.9, we can remove the dependence of the bi-local symbol $Pr^Q_{E,\tilde{P}}$ on the base point Q.

Definition 3.4. *The second Parshin symbol $\{f_1, f_2, f_3\}^{(2)}_{C,E}$ is the symbol, explicitly given by*

$$\{f_1, f_2, f_3\}^{(2)}_{C,E} = (-1)^K \left(f_1^{D_1} f_2^{D_2} f_3^{D_3}\right)(\tilde{P}),$$

where

$$D_1 = \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix}, \; D_2 = \begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix}, \; D_3 = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}$$

and

$$K = c_1c_2d_3 + c_2c_3d_1 + c_3c_1d_2 + d_1d_2c_3 + d_2d_3c_1 + d_3d_1c_2,$$

with $c_k = \mathrm{ord}_E(f_k)$ and $d_i = \mathrm{ord}_{\tilde{P}}((y^{-c_k} f_k)|_E)$. Here y is a rational function representing an uniformizer at E such that the components of the divisor of y do not intersect at the point $\tilde{P}$.

Proposition 3.5. *The second Parshin symbol is equal to the inverse of the Parshin symbol. More precisely,*

$$\{f_1, f_2, f_3\}^{(2)}_{C,E} = (\{f_1, f_2, f_3\}_{C,P})^{-1}$$

Let

$$a_i = \mathrm{ord}_C(f_i)$$

and

$$b_i = \mathrm{ord}_P((x^{-a_i} f_i)|_C),$$

where x is a rational function representing a uniformizer at C, whose support does not contain other components passing through the point P.

Lemma 3.6. *With the above notation, the following holds*

$$\mathrm{ord}_E(f_i) = c_i = a_i + b_i.$$

Proof. We still assume that after the blow-up the union of the support of the rational functions f_1, f_2, f_3 have normal crossings and no three curves intersect at a point. Before the blow-up, let $C_1, \ldots, C_n$ be all the components of the union of the support of the three rational functions that meet at the point P. And let E be the exceptional curve above the point P. Then for $C = C_1$, we have

$$b_i = \sum_{j=2}^{n} \mathrm{ord}_{C_j}(f_i)$$

and

$$\mathrm{ord}_E(f_i) = \sum_{j=1}^{n} \mathrm{ord}_{C_j}(f_i).$$

That proves the Lemma. □

Proof. (of Proposition 3.5) Consider the pairs (C, P) on the surface X and $(E, \tilde{C})$ on the blow-up $\tilde{X}$. Then by the above Lemma, we have

$$\begin{bmatrix} c_i \\ d_i \end{bmatrix} = \begin{bmatrix} \mathrm{ord}_E(f_i) \\ \\ \mathrm{ord}_{\tilde{C}}(f_i) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ \\ b_i \end{bmatrix} \tag{3.1}$$

The Parshin symbol is invariant under change of variables given by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Also the Parshin symbol is sent to its reciprocal when we change the variables by a matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. That proves the Proposition. □

Theorem 3.7. *(Second reciprocity law for the Parshin symbol) We have*

$$\prod_C \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is over the curves C from the divisor $\bigcup_{i=1}^{3} |(f_i)|$, which pass through the point P. (For all other choices of curves C, the Parshin symbol will be equal to 1.)

Proof. We can use Proposition 3.5 and the first reciprocity law for the Parshin symbol given in Theorem 2.9. Then Theorem 3.7 follows. □

3.3 The second 4-function local symbol and its reciprocity law

In this Subsection, We define a second type of 4-function local symbol (Definition 3.10), which satisfies the second type reciprocity laws. By a second reciprocity law, we mean that the product of the local symbols is taken over all curves C on the surface X, which pass through a fixed point P. The 4-function local symbol has a refinement (see Definition 3.8, which provides a proof of the second reciprocity law (Theorem 3.11).

Definition 3.8. *We define a bi-local symbol, useful for the second reciprocity law for a new 4-function local symbol. Let*

$$PR^D_{C,P} = \left({}^{2,2}[f_1,f_2,f_3,f_4]^{(2),D}_{C,E}\right)\left({}^{2,2}[f_1,f_2,f_4,f_3]^{(2),D}_{C,E}\right)^{-1}\times$$
$$\times\left({}^{2,2}[f_2,f_1,f_3,f_4]^{(2),D}_{C,E}\right)^{-1}\left({}^{2,2}[f_2,f_1,f_4,f_3]^{(2),D}_{C,E}\right).$$

Let

$$L = (c_1d_2 - c_2d_1)(c_3d_4 - c_4d_3),$$

where

$$c_i = \mathrm{ord}_E(f_i),$$
$$d_i = \mathrm{ord}_{\tilde{P}}((x^{-a_i}f_i)|_E),$$

for a rational function x, representing a uniformizer at E, whose support does not contain other components passing through the point $\tilde{P} = E \cap \tilde{C}$.

Lemma 3.9.

$$PR^D_{C,E} = (-1)^L \left(\frac{\left(\frac{f_1^{c_2}}{f_2^{c_1}}\right)^{c_3d_4-c_4d_3}}{\left(\frac{f_3^{c_4}}{f_4^{c_3}}\right)^{c_1d_2-c_2d_1}}(\tilde{P})\right)^{-1}\frac{\left(\frac{f_1^{c_2}}{f_2^{c_1}}\right)^{c_3d_4-c_4d_3}}{\left(\frac{f_3^{c_4}}{f_4^{c_3}}\right)^{c_1d_2-c_2d_1}}(Q),$$

where $Q = D \cap E$.

It follows directly from Equation 2.1 and Lemma 3.6.

Definition 3.10. *The second 4-function local symbol has the following explicit representation:*

$$\{f_1,f_2,f_3,f_4\}^{(2)}_{C,P} = (-1)^L \left(\frac{\left(\frac{f_1^{a_2+b_2}}{f_2^{a_1+b_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4+b_4}}{f_4^{a_3+b_3}}\right)^{a_1b_2-b_1a_2}}(P)\right)^{-1}.$$

Theorem 3.11. *(Reciprocity law for the second 4-function local symbol) We have the following reciprocity law*

$$\prod_C \{f_1,f_2,f_3,f_4\}^{(2)}_{C,P} = 1,$$

where the product is over the curves C from the support $\bigcup_{i=1}^4 |(f_i)|$, which pass through the point P.

Proof. Without loss of generality, we can assume that the the divisor $\bigcup_{i=1}^4 |(f_i)|$ in $\tilde{X}$ is a strict normal crossing divisor, where $\pi : \tilde{X} \to X$ is a single blow-up at the point P. For any complex projective surface X we can blow-up at a point $P \in X$ and use the first 4-function local symbol with respect to the exceptional curve. See Lemma 3.9 for comparison between the first and the second 4-function local symbols and the Remark after Theorem 2.13 for the first 4-function local symbol on a general complex projective surface.

Using Theorem 3.2, we obtain a reciprocity law for the bi-local symbol $PR^{(2),D}_{C,E}$. Multiplying each symbol by the same constant, depending only on Q, we can remove the dependence on Q. Explicitly, the separation between the dependence on D and the second 4 function local symbol are given in Lemma 3.9. Then we can use Lemma 3.6 in order to express the coefficients c_i and d_i in terms of a_i and b_i, which implies the reciprocity law stated in the Theorem 3.11. $\square$

A A K-theoretic perspective on the 4-function symbol

Ivan Horozov and Matt Kerr

In this Appendix, we will give an alternative construction of the 4-function local symbol (valid *up to sign*) using Milnor K-theory and the Tame symbol. This leads to a complementary proof of the reciprocity laws. To begin, we shall recall the K-theoretic approach to the Parshin symbol (Definition 0.3) and its reciprocity laws, due to Kato [Ka].

Preliminaries on the Tame symbol

For a field $\mathbb{F}$, let $\mathcal{S}_{\mathbb{F}^*}$ denote the set of elements of $\mathbb{F}^*$. The n^{th} Milnor K-group $K_n^M(\mathbb{F})$ of $\mathbb{F}$ is the quotient of the abelian group $\otimes^n\mathbb{Z}[\mathcal{S}_{\mathbb{F}^*}]$ by the subgroup generated by all permutations of

(i) $\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n+\beta_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n-\alpha_1\beta_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n$,
(ii) $\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_n+\alpha_2\otimes\alpha_1\otimes\cdots\otimes\alpha_n$, and
(iii) $\alpha_1\otimes(1-\alpha_1)\otimes\alpha_3\otimes\cdots\otimes\alpha_n$.

We shall write the resulting abelian group multiplicatively, as products of Milnor symbols $\{\alpha_1,\ldots,\alpha_n\}$ (= image of $\alpha_1\otimes\cdots\otimes\alpha_n$ in the quotient). So for example ($n=3$), by (ii) we have $\{\alpha_1,\alpha_2,\alpha_3\}\{\alpha_2,\alpha_1,\alpha_3\}=1$. An easy consequence of (i) is that $\{\alpha_1,\ldots,\alpha_n\}=1$ if any $\alpha_i=1$. We remark that $K_2^M(\mathbb{F})=K_2(\mathbb{F})$, and $K_1^M(\mathbb{F})=K_1(\mathbb{F})=\mathbb{F}^*$.

Specializing henceforth to the case $\mathbb{F}=\mathbb{C}(\mathrm{X})$, for X a smooth complex projective variety, we shall use the notations: $(f)=(f)_0-(f)_\infty\in Z^1(\mathrm{X})$ for the divisor of a function $f\in\mathbb{C}(\mathrm{X})^*$, where $(f)_0$ and $(f)_\infty$ are effective; $|Z|$ for the support of $Z\in Z^1(X)$; $x\in\mathrm{X}^1$ for a point of codimension 1; and a bar for Zariski closure (for instance, $\bar{x}$).

Let $\xi=\{f_1,\ldots,f_n\}\in K_n^M(\mathbb{C}(\mathrm{X}))$ be given. One can show (cf. Prop. 1.2.12 of [Ke2]) that ξ may be expressed as $\prod_{i=1}^N\{g_{i1},\ldots,g_{in}\}^{m_i}$, where the $\{g_{ij}\}\subset\mathbb{C}(\mathrm{X})^*$ satisfy

$$\operatorname{codim}_{\mathrm{X}_{(i)}}\left(\bigcap_{j\in J}\left|(g_{ij})_{\epsilon(j)}\right|\right)\geq|J| \tag{A.1}$$

for each $i\in\{1,\ldots,N\}$, multi-index $J\subset\{1,\ldots,n\}$, and function $\epsilon:J\to\{0,\infty\}$, and where $\mathrm{X}_{(i)}:=\mathrm{X}\setminus\cup_{j=1}^n\overline{g_{ij}^{-1}(1)}$. The point is that we may decompose an arbitrary Milnor symbol ξ into a product of better behaved symbols, within each of which we have control over intersections of divisors of functions away from the locus where one or more functions are 1.

Now fix a point $x \in \mathrm{X}^1$ (of codimension 1), and write $\mathbb{C}(x)$ for the residue field. The pullback of a function $g \in \mathbb{C}(X)^*$ with $\mathrm{ord}_x(g) = 0$ will be denoted by $g(x) \in \mathbb{C}(x)^*$. The Tame symbol is a homomorphism of abelian groups

$$\mathrm{Tame}_x : K_n^M(\mathbb{C}(\mathrm{X})) \to K_{n-1}^M(\mathbb{C}(x)), \tag{A.2}$$

which we shall define on an arbitrary $\xi = \{f_1, \ldots, f_n\}$ by defining it on the factors $\{g_{i1}, \ldots, g_{in}\}$ described above. Namely, for each such factor (i.e. choice of $i \in \{1, \ldots, N\}$), there are two possibilities: either (1) $\bar{x}$ is contained in the closed subset $\mathrm{X} \setminus \mathrm{X}_{(i)}$ of X; or (2) we have $x \in \mathrm{X}^1_{(i)}$. In case (1), or if (in case (2)) $\mathrm{ord}_x(g_{ij}) = 0\,(\forall j)$, we set $\mathrm{Tame}_x\{g_{i1}, \ldots, g_{in}\} = 1$. Otherwise, by (A.1) there is exactly one $j \in \{1, \ldots, n\}$ such that $\mathrm{ord}_x(g_{ij}) \neq 0$; denoting this by $j(i)$, we set

$$\mathrm{Tame}_x\{g_{i1}, \ldots, g_{in}\} = \{g_{i1}(x), \ldots, \widehat{g_{i,j(i)}}, \ldots, g_{in}(x)\}^{(-1)^{j(i)}\mathrm{ord}_x(g_{i,j(i)})}. \tag{A.3}$$

One may check that relations map to relations, so that (A.2) is well-defined. Moreover, given $y \in \mathrm{X}^2$ (of codimension 2), a short computation using (A.1) and (A.3) shows that the composition

$$K_n^M(\mathbb{C}(\mathrm{X})) \xrightarrow{\oplus \mathrm{Tame}_x} \oplus_{x \in \mathrm{X}^1} K_{n-1}^M(\mathbb{C}(x)) \xrightarrow{\sum \mathrm{Tame}_y} K_{n-2}^M(\mathbb{C}(y)) \tag{A.4}$$

is trivial. Finally, if $\bar{\tilde{x}}$ is the proper transform of $\bar{x}$ under a blow-up $\tilde{\mathrm{X}} \xrightarrow{\beta} \mathrm{X}$, then $\mathbb{C}(x) \xrightarrow{\cong} \mathbb{C}(\tilde{x})$ identifies $\mathrm{Tame}_{\tilde{x}}\beta^*\xi = \mathrm{Tame}_x\xi$.

We remark that Prop. 1.2.12 of [Ke2] relies on the Nesterenko-Suslin-Totaro result $K_n^M(\mathbb{C}(\mathrm{X})) \cong CH^n(Spec(\mathbb{C}(\mathrm{X})), n)$ together with Bloch's moving lemma (cf. [op. cit.], Cor. 1.2.6), which also gives the well-definedness of (A.2). Once one has used this to rewrite ξ, triviality of (A.4) is nothing but the statement that $\partial \circ \partial = 0$ in Bloch's higher Chow complex. The triviality of (A.4) may alternatively be deduced from the Gersten sequence for Milnor K-theory (proved by Gabber; cf. §6 of [Ro]).

We also note that a direct formula for $\mathrm{Tame}_x\{f_1, \ldots, f_n\}$ is known, cf. §1.2.3 of [Ke2]. For $n = 2$, this is $(-1)^{\mathrm{ord}_x(f_1)\mathrm{ord}_x(f_2)} \left(\frac{f_1^{\mathrm{ord}_x(f_2)}}{f_2^{\mathrm{ord}_x(f_1)}}\right)(x)$; if $n > 2$ and $\mathrm{ord}_x(f_j) = 0$ for $j > 2$, then $\mathrm{Tame}_x\{f_1, \ldots, f_n\} = \{\mathrm{Tame}_x\{f_1, f_2\}, f_3(x), \ldots, f_n(x)\}$.

Alternate description of the Parshin symbol

Henceforth we shall take $\dim_{\mathbb{C}}(\mathrm{X}) = 2$. Let $x \in \mathrm{X}^1$ be a point of codimension 1, C be the normalization of the complete curve $\bar{x}$, and $P \in C(\mathbb{C})$ be a (closed) point on C.

Definition A.1. *$\mathcal{P}_{C,P}$ is the composition*

$$K_3^M(\mathbb{C}(\mathrm{X})) \xrightarrow{\mathrm{Tame}_C} K_2(\mathbb{C}(C)) \xrightarrow{\mathrm{Tame}_p} \mathbb{C}^*. \tag{A.5}$$

Clearly $\mathcal{P}_{C,P}$ is invariant under blow-up, when $\bar{x}$ is replaced by its proper transform. The first reciprocity law $\prod_{P \in C(\mathbb{C})} \mathcal{P}_{C,P}\{f_1, f_2, f_3\} = 1$ follows from Weil reciprocity, and the second law $\prod_{C(\mathbb{C}) \ni P} \mathcal{P}_{C,P}\{f_1, f_2, f_3\} = 1$ from (A.4). Alternatively, after a series of blow-ups one has that $\cup|(f_i)|$ is a strict normal crossing divisor, with only C' and C'' through P'. Then the second reciprocity law follows from the special case

$$\mathcal{P}_{C',P'}\{f_1,f_2,f_3\} = \left(\mathcal{P}_{C'',P'}\{f_1,f_2,f_3\}\right)^{-1} \tag{A.6}$$

of (A.4) together with Weil reciprocity on the irreducible components of the exceptional divisor.

To check that $\mathcal{P}_{C,P}$ is the Parshin symbol (0.3), it will suffice to restrict to the case where only two of the $|(f_i)|$ contain P, by (A.1) and the fact that both symbols are homomorphisms. (This means that both annihilate relations (i)-(iii) above for $n = 3$. This is true of $\mathcal{P}_{C,P}$ by construction, and is an easy explicit check for (0.3) which we leave to the reader.) Moreover, we may assume $\cup|(f_i)|$ is a strict normal crossing divisor, due to invariance of both under blow-up.

Let $U \ni P$ be an analytic open neighborhood with $U \cap (\bigcup_i |(f_i)|) = U \cap (C \cup C')$, and $x, y \in \mathbb{C}(\mathrm{X})^*$ be such that $(x)|_U = C \cap U$ and $(y)|_U = C' \cap U$. Then for $i = 1, 2, 3$ we have $f_i = x^{a_i} y^{b_i} g_i$ where $g_i|_U \in \mathcal{O}^*(U)$; the above assumptions give $a_j = b_j = 0$ for some j, say $j = 3$. We compute $\mathcal{P}_{C,P}\{f_1,f_2,f_3\} =$

$$\begin{aligned} &= \mathrm{Tame}_P \mathrm{Tame}_C \{x^{a_1} y^{b_1} g_1, x^{a_2} y^{b_2} g_2, f_3\} \\ &= \mathrm{Tame}_P \left\{ (-1)^{a_1 a_2} y^{b_1 a_2 - b_2 a_1} \left. \frac{g_1^{a_2}}{g_2^{a_1}} \right|_C, f_3|_C \right\} \\ &= f_3(P)^{a_1 b_2 - a_2 b_1} \end{aligned}$$

which coincides with (0.3). The reader may easily check the cases $j = 1, 2$.

The K-theoretic 4-function local symbol

Turning to our main subject, let x, C, and P be as above.

Definition A.2. *$\mathcal{Q}_{C,P}$ is the composition*

$$K_2(\mathbb{C}(\mathrm{X}))^{\otimes 2} \xrightarrow{\mathrm{Tame}_C^{\otimes 2}} (\mathbb{C}(C)^*)^{\otimes 2} \twoheadrightarrow K_2(\mathbb{C}(C)) \xrightarrow{\mathrm{Tame}_p} \mathbb{C}^*. \tag{A.7}$$

Again, invariance under blow-up follows from the definition, and the first reciprocity law from Weil reciprocity on C. On the other hand, the second reciprocity law fails dramatically for $\mathcal{Q}_{C,P}$, even in the setting of (A.6) above.

Example A.3. *Let $\mathrm{X} = \mathbb{P}^2$ with coordinates $[X_0 : X_1 : X_2]$, and put $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$ on $U = \mathrm{X} \setminus \{X_0 = 0\}$. Working on U, set $C = \{y = 0\}, C' = \{x = 0\}$, $f_1 = x, f_2 = -\alpha y, f_3 = -\frac{x}{\beta}, f_4 = y$, with $\alpha, \beta \in \mathbb{C}^*$. Then $Tame_C\{f_1, f_2\} = x$, $Tame_{C'}\{f_1, f_2\} = \frac{-1}{\alpha y}$, $Tame_C\{f_3, f_4\} = -\frac{x}{\beta}$, $Tame_{C'}\{f_3, f_4\} = y^{-1}$, and so*

$$\mathcal{Q}_{C,P}\left(\{f_1,f_2\} \otimes \{f_3,f_4\}\right) = Tame_P \left\{ x, -\frac{x}{\beta} \right\} = \beta$$

$$\mathcal{Q}_{C',P}\left(\{f_1,f_2\} \otimes \{f_3,f_4\}\right) = Tame_P \left\{ \frac{-1}{\alpha y}, y^{-1} \right\} = \alpha$$

have no relationship whatsoever.

To compare $\mathcal{Q}_{C,P}$ with the 4-function local symbol, invariance under blow-up again allows us to restrict to the strict normal crossing divisor setting. As above we write $f_i = x^{a_i} y^{b_i} g_i$, and compute $\mathcal{Q}_{C,P}\left(\{f_1, f_2\} \otimes \{f_3, f_4\}\right) =$

$$\begin{aligned}
&= \mathrm{Tame}_P\{\mathrm{Tame}_C\{f_1, f_2\}, \mathrm{Tame}_C\{f_3, f_4\}\} \\
&= \mathrm{Tame}_P\{\mathrm{Tame}_C\{x^{a_1}y^{b_1}g_1, x^{a_2}y^{b_2}g_2\}, \mathrm{Tame}_C\{x^{a_3}y^{b_3}g_3, x^{a_4}y^{b_4}g_4\}\} \\
&= \mathrm{Tame}_P\left\{(-1)^{a_1a_2}x^{a_1b_2-a_2b_1}\frac{(g_1|_C)^{a_2}}{(g_2|_C)^{a_1}}, (-1)^{a_3a_4}x^{a_3b_4-a_4b_3}\frac{(g_3|_C)^{a_4}}{(g_4|_C)^{a_3}}\right\} \\
&= (-1)^{(a_1b_2-a_2b_1)(a_3b_4-a_4b_3)}\frac{\left\{(-1)^{a_1a_2}\left(\frac{(g_1(P))^{a_2}}{(g_2(P))^{a_1}}\right)\right\}^{(a_3b_4-a_4b_3)}}{\left\{(-1)^{a_3a_4}\left(\frac{(g_3(P))^{a_4}}{(g_4(P))^{a_3}}\right)\right\}^{(a_1b_2-a_2b_1)}} \\
&= (-1)^{b_1a_2a_3a_4+a_1b_2a_3a_4+a_1a_2b_3a_4+a_1a_2a_3b_4}\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P}.
\end{aligned}$$

This motivates the following

Definition A.4. *For any $f_1, f_2, f_3, f_4 \in \mathbb{C}(\mathrm{X})^*$, we set*

$$^K[f_1, f_2, f_3, f_4]^{(1)}_{C,P} := \mathcal{Q}_{C,P}\left(\{f_1, f_2\} \otimes \{f_3, f_4\}\right)$$

$$(f_1, f_2, f_3, f_4)^{(1)}_{C,P} := {}^K[f_1, f_2, f_3, f_4]^{(1)}_{C,P}\Big/\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P}.$$

In light of the above computation and the properties of $^K[\ ,\ ,\ ,\ ]^{(1)}_{C,P}$ and $\{\ ,\ ,\ ,\ \}^{(1)}_{C,P}$, we have

Proposition A.5. *The symbol $(\ ,\ ,\ ,\ )^{(1)}_{C,P}$ is invariant under blow-up, takes values in $\{1, -1\}$, and satisfies the first reciprocity law. It is given by*

$$(f_1, f_2, f_3, f_4)^{(1)}_{C,P} = (-1)^{b_1a_2a_3a_4+a_1b_2a_3a_4+a_1a_2b_3a_4+a_1a_2a_3b_4}, \tag{A.8}$$

where $a_k = \mathrm{ord}_C(f_k)$ and $b_k = \mathrm{ord}_P((x^{-a_k}f_k)|_C)$. Here x is a rational function representing an uniformizer at C such that P is not an intersection point of the irreducible components of the support of the divisor (x).

Proof. Validity of (A.8) in general follows from the strict normal crossing divisor case (the computation above) and invariance under blow-up. □

A direct proof of $\prod_{P\in C(\mathbb{C})}(f_1, f_2, f_3, f_4)^{(1)}_{C,P} = 1$ may be obtained from the following observation. For given f_k and C, a_k is constant on C. Setting $\omega_k = \left(-a_k\frac{dx}{x} + \frac{df_k}{f_k}\right)\Big|_C$ gives $b_k(P) = Res_P(\omega_k)$, so that (using (A.8))

$$\begin{aligned}
(f_1, f_2, f_3, f_4)^{(1)}_{C,P} = \quad & \exp\left(\tfrac{1}{2}\left(a_1a_2a_3Res_P(\omega_4) + a_2a_3a_4Res_P(\omega_1) + \right.\right. \\
& \left.\left. + a_3a_4a_1Res_P(\omega_2) + a_4a_1a_2Res_P(\omega_3)\right)\right)
\end{aligned}$$

The first reciprocity law then follows from the fact that the sum of residues of each ω_k on C is equal to zero.

This leads to an independent proof of Theorem 2.13, using the reciprocity laws for $^K[\ ,\ ,\ ,\ ]^{(1)}_{C,P}$ and $(\ ,\ ,\ ,\ )^{(1)}_{C,P}$. In our view, *both* proofs are of conceptual importance. In particular, the proof in §2 seems promising for the general prospect of building symbols satisfying reciprocity laws in higher dimension from bilocal symbols.

Analogues of the second 4-function local symbol

Now we proceed toward an alternative proof of the second type of reciprocity laws for the new 4-function local symbol.

Let E be the exceptional curve for the blowup of X at the point P. Let $\tilde{C}$ be the irreducible component sitting above the curve C in the blow-up. We define $\tilde{P} = \tilde{C} \cap E$. Similarly to the Proof of Proposition 3.5, we obtain

$$\{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = \left(\{f_1, f_2, f_3, f_4\}^{(1)}_{E,\tilde{P}}\right)^{-1}$$

for the 4-function local symbols. Similarly we define

$$(f_1, f_2, f_3, f_4)^{(2)}_{C,P} = \left((f_1, f_2, f_3, f_4)^{(1)}_{E,\tilde{P}}\right)^{-1} \tag{A.9}$$

for the sign and

$${}^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = \left({}^K[f_1, f_2, f_3, f_4]^{(1)}_{E,\tilde{P}}\right)^{-1} \tag{A.10}$$

for the K-theoretic symbol.

Proposition A.6. *For the sign and the K-theoretic symbol we have a second type of reciprocity laws.*

$$\prod_C (f_1, f_2, f_3, f_4)^{(2)}_{C,P} = 1$$

and

$$\prod_C {}^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = 1,$$

where the product is taken over all curves C, passing through the point P. Here we assume that the union of the support of the divisors $\bigcup_{i=1}^4 |div(f_i)|$ in $\tilde{X}$ have normal crossings and no two components have a common point with the exceptional curve E in $\tilde{X}$ above the point P. We denote by $\tilde{X}$ the blow-up of X at the point P.

Proof. For the K-theoretic symbol we have

$$\prod_C {}^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = \left(\prod_{\tilde{P}} {}^K[f_1, f_2, f_3, f_4]^{(1)}_{E,\tilde{P}}\right)^{-1} = 1.$$

The first equality follows from the definition of ${}^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P}$ and the second equality from Definition A.2 and Weil reciprocity on E. □

Proof. (an alternative proof of Theorem 3.11) We have the following equalities

$$\prod_C \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = \prod_C {}^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} \prod_C (f_1, f_2, f_3, f_4)^{(2)}_{C,P} = 1.$$

The first equality follows from Definition A.4 and Equations (A.9) and (A.10). The second equality follows from Proposition A.6. □

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