Table of Content of Algebra I

1. RATIONAL NUMBERS

1.1. Introducing the negative numbers.

Suppose that on a cold winter evening, the temperature at 6pm is $5°F$. If the temperature at midnight is $3°$ colder than at 6pm, what is the temperature at midnight? We know how to solve this using subtraction.

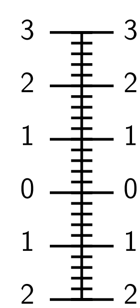
$$5 - 3 = 2,$$

so the midnight temperature is $2°F$.

Now let's change the problem slightly. Suppose the temperature at 6pm is again $5°F$, but now the temperature at midnight is $8°$ colder than at 6pm. What is the temperature at midnight? Just as before, we subtract the drop (8 this time) from 5, to get

$$5 - 8.$$

But what does this mean? How can we subtract a bigger number from a smaller one? Well, let's look at a thermometer.



Notice there are numbers on both sides of zero. There is a 1 above it, and a 1 below it. To distinguish them, we call the first one "one degree above zero" and the second one "one degree below zero".

One degree below zero is even colder than zero. Similarly, it is normal to measure altitude in feet (or meters) above or below sea level. Mount Whitney in California is $14,495$ feet above sea level, and Death Valley is 282 feet below sea level.

We don't want to have to say (and write) "above zero" or "below zero" all the time, so we introduce some convenient notation. Numbers that are below zero we write with a minus sign. So $-1°F$, read as "minus one degree Fahrenheit", means one degree Fahrenheit below zero. The altitude of Death Valley can then be written as -282 feet. If we aren't specifying the unit, we write them like -5 (read "minus five" or "negative five"), -53, -768, and so on. We call these numbers *negative* numbers.

Numbers that are above zero we either write with no sign, or, to emphasize that they are positive, with a plus sign. So $+5$ (read "plus five") is the same as the number 5.

Let us return to our temperature example. If the temperature dropped $8°$ from $5°$, we want to say that the temperature at midnight is $5 - 8$ degrees. How do we evaluate this?

$$\begin{aligned} 5 - 8 &= 5 - (5 + 3) \\ &= 5 - 5 - 3 \\ &= (5 - 5) - 3 \\ &= 0 - 3 \\ &= -3. \end{aligned}$$

So the temperature at midnight is -3 degrees (or, the same thing, 3 degrees below zero).

Let's consider another example. Interstate Highway 70 runs from Denver to Kansas City, a distance of 610 miles, and from Kansas City to Saint Louis, a distance of 260 miles. Suppose you start in Kansas City, and drive 100 miles on I-70. How far are you from Denver?

It depends, of course, on whether you drove west, towards Denver, or east, towards St. Louis. In situations like this, it is convenient to pick one direction as positive, and the other as negative. Let us take east as the positive direction, and measure miles from Denver.

Then you start out at $+610$ miles. If you go east 100 miles, then you are now

$$610 + (100) \;=\; 710$$

miles from Denver. If instead you drove west, you would be

$$610 + (-100) \;=\; 510$$

miles from Denver.

Exercises 1.1.

1. Use positive or negative numbers to express the following temperatures.

(a) $12°F$ above zero.

(b) $6°F$ below zero.

(b) $6°F$ below zero.

(d) $296°F$ below zero.

2. On a north-south highway, if we take north as the positive direction, what does traveling $+100$ miles mean? What does traveling -150 miles mean? How would you express traveling 50 miles north? Traveling 215 miles south?

3. In calculating the budget of a city, if we write $+2,000,000$ dollars for the income from tax, how would we write the mayor's salary of $55,000$ dollars?

4. If we use $+3$ hours to express 3 hours after 11 am, what does $+5$ hours mean? What does -6 hours mean?

5. In measuring the change of water volume in a lake, if $+300$ cubic yard/min. means water flowing in at a rate of 300 cubic yards per minute, what does -400 cubic yards/min. mean?

1.2. Integers and Rational Numbers.

Whole numbers, the numbers 0, 1, 2, 3, . . . and -1, -2, -3, . . . are called *integers*. The numbers 1, 2, 3, 4, ... are called *positive integers*, and the numbers -1, -2, -3, ... are called *negative integers*. The number 0 is neither positive (which means bigger than zero) or negative (which means less than zero).

We can also put a minus sign in front of fractions, like $-\frac{1}{2}$ or $-2\frac{1}{3}$. We interpret these numbers the same way: $-\frac{1}{2}$ means half a unit below (or to the left of) zero, $-2\frac{1}{3}$ means $2\frac{1}{3}$ units below zero. When we want to talk about all the fractions and integers together we call them the *rational numbers*, because they can be written as a *ratio* of two whole numbers:

$$\begin{aligned}
2\tfrac{1}{3} &= \frac{8}{3} \\
-2\tfrac{1}{3} &= \frac{-8}{3} \\
1.5 &= \frac{15}{10} = \frac{3}{2} \\
-1.5 &= \frac{-15}{10} = \frac{-3}{2} \\
7 &= \frac{7}{1} \\
0 &= \frac{0}{1}.
\end{aligned}$$

The relationships between these sets of numbers are summarized below:

$$\text{rational number}\begin{cases} \text{positive} & \begin{cases}\text{positive fraction}\\ \text{positive integer}\end{cases} \\ \text{zero} & \\ \text{negative} & \begin{cases}\text{negative fraction}\\ \text{negative integer}\end{cases}\end{cases}$$

NOTE. A *non-negative* number means a number that it is either zero or positive. Non-negative and positive are different, because zero is non-negative but not positive.

EXERCISES 1.2.

1. Is 0 a positve number? a negative number?

2. Is 254 a positive number? an integer? a rational number?

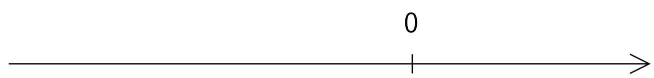
3. Write down 5 rational numbers, with 3 non-negative numbers and three non-positive numbers.

1.3. The Number Axis.

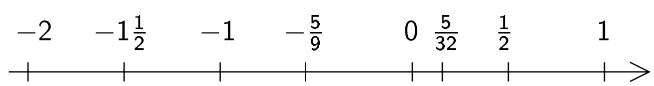
If you wanted to measure your height, one way to do it would be to stand a ruler against a wall, and stand up straight beside it. On the ruler are many marks with numbers beside them. If the zero end of the ruler is next to your feet, then the mark next to the top of your head gives your height.

Of course, any real ruler only has certain subdivisions marked: it may be marked in eighths of an inch, or tenths of an inch. We can imagine that there could be an ideal ruler that had every rational number, positive, negative, and zero, on it. How would we make it?

Draw a horizontal line. Imagine it is infinite in both directions (no ending in either direction). Pick a point on the line, and mark it. This point is called the *origin,* and represents the number zero. We always choose the positive direction to be to the right, so we put an arrow at the right to show that that is the direction in which numbers increase.



Now we need to decide the length of a unit. It doesn't matter what we choose — it can be a centimeter, an inch, or 3 inches. Let's suppose we decide on an inch. Then we mark the point one inch to the right of the origin with the number 1, the point two inches to the right with the number 2, the point one inch to the left of the origin with the number -1, the point two inches to the left with the number -2, and so on. The number $\frac{1}{2}$ is midway between 0 and 1, the number $-1\frac{1}{2}$ is midway between -2 and -1, and so on. Obviously, we can't mark *every* number on the line, because there are an infinite number of them. But given any *particular* rational number, like $\frac{5}{32}$ or $-\frac{5}{9}$, we can mark that. Every rational number corresponds to a unique point on the line.



Summarizing:

> *A number axis is a line that represents numbers. It has a positive direction, an*

origin, and a unit length.

From the above figue, it is easy to see that all numbers to the right of the origin are positive, and to the left are negative.

Exercises 1.3.

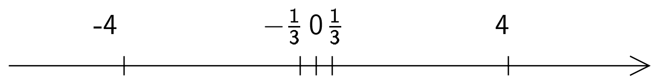
1. Draw a number axis with an origin, unit length, and positive direction. Mark $+2, -2.5, 0, 1\frac{1}{2}, -1\frac{1}{2}$ on the number axis. Arrange these five numbers from left to right according to their positions on the axis.

2. On a number axis, can two points represent the same number? Can one point represent two different numbers?

3. On a number axis, what are the points to the right of the origin? What are the points to the left of the origin?

1.4. Opposite Numbers and Absolute Values.

The *opposite number* of any number is the number on the number axis that is the same distance from the origin, but in the opposite direction. Thus, the opposite number of 4 is -4, and the opposite number of -4 is 4. Likewise $\frac{1}{3}$ and $-\frac{1}{3}$ are opposite numbers. Zero is special: the opposite number of 0 is 0.



If we start with a positive number, like 4, and want to get its opposite, we just put a minus sign in front to get -4. We extend this rule to say that a minus sign put in front of any number means we take the opposite of that number. Thus, $-(-4)$ means the opposite of -4, which is 4.

$$-(-4) \;=\; +4 \;=\; 4.$$

So two negative signs together are the same as a positive sign. This is the mathematical equivalent of the rule in English grammar that two negatives make a positive: "I don't want to have no pie" means "I want to have some pie".

EXAMPLE 1. Find the opposite numbers of the following.

(1) $+3.1$ (2) $-5\frac{1}{3}$.

SOLUTION. (1) The opposite number of $+3.1$ is

$$-(+3.1) = -3.1.$$

(2) The opposite number of $-5\frac{1}{3}$ is

$$-(-5\tfrac{1}{3}) = 5\tfrac{1}{3}.$$

The *absolute value* of a number is its distance from the origin — we ignore whether it is to the left or the right. For instance, if we are interested in the gasoline consumption of a car, we would like to know how far it has traveled; whether it drove west or east doesn't matter. The absolute value of a positive number is the number itself, and the absolute value of a negative number is its opposite number. The absolute value of 0 is 0.

We draw a vertical line on each side of the number to indicate the absolute value. For example, the absolute value of $+15$ is 15, written as

$$|+15| = 15\,,$$

read as "the absolute value of plus 15 is 15". Similarly,

$$|-15| = 15\,,$$

read as "the absolute value of minus 15 is 15"; and

$$|0| = 0\,,$$

read as "the absolute value of 0 is 0".

Notice that the absolute values of opposite numbers are the same.

Exercises 1.4.

1. Find the opposite numbers of $+3$,-4, $-1\frac{1}{2}$, 0. Mark these numbers together with their opposite numbers on a number axis. What are the absolute values of these numbers?

2. What is the opposite number of $+4$? What is the opposite number of the opposite number of $+4$? In general, what is the opposite number of the opposite number of a number?
3. Can the absolute value of a number be negative? Is the absolute value of a number always positive?
4. Write down the absolute values of the following numbers:

$$+5,\ -7,\ 3,\ -6\frac{5}{7}.$$

5. Simplify the following:

(1)	$+(+3)$	(2)	$+(-10.2)$
(3)	$-(-2.5)$	(4)	$-(+0.4)$
(5)	$-[+(-5)]$	(6)	$-[-(+3)]$
(7)	$-\|-3\|$	(8)	$-\|+3\|$

6. Calculate the following:

(1) $|-10|+|12|+|-5|$ (2) $|-1|\times|-2\frac{1}{2}|$

(3) $|-4.1|-|2.3|$

(4) $|-10|+|12|-|-15|$

1.5. Comparing the sizes of rational numbers.

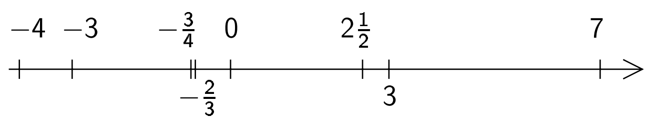
The sign $>$ means greater than (or bigger than); the sign $<$ means less than (or smaller than). So we can say

$7 > 3$, read as "7 is greater than 3"

or

$$3 < 7, \qquad \text{read as "3 is less than 7".}$$

To extend the comparisons $>$ and $<$ to all rational numbers, we say that given two different points on the number line, the one on the right is greater than the one on the left. So $3 > -3$, $-3 > -4$, $2\frac{1}{2} > 0$, $-\frac{3}{4} < -\frac{2}{3}$, and so on.



It follows that:

> *All positive numbers are greater than all negative numbers.*
> *Given two positive numbers, the one with larger absolute value is bigger.*
> *Given two negative numbers, the one with larger absolute value is smaller.*

EXAMPLE 1. Which is larger, 1.23 or -4.56?

SOLUTION. Since positive numbers are greater than negative numbers, we have $1.23 > -4.56$.

EXAMPLE 2. Which is smaller, -9.1 or -8.5?

SOLUTION. Since both numbers are negative, the one with larger absolute value is smaller. That is $-9.1 < -8.5$.

EXAMPLE 3. Arrange the numbers 1, -3.1, and 2 from smallest to largest.

SOLUTION. Since -3.1 is smaller than 1 and 1 is smaller than 2, we have $-3.1 < 1 < 2$.

NOTE. When comparing three or more numbers, we will use either '$<$' or '$>$', but will not mix them together. In the above example, we would not write '$1 > -3.1 < 2$', because that formulation does not say how 1 compares to 2.

EXERCISES 1.5.

1. Write out all the positive integers that are smaller than 6. Write out all the negative integers that are greater than -7.

2. Write out all the integers that are greater than -3 and less than 5, and mark them on a number axis.

3. Compare each pair of numbers and link them by '$<$'.

(1) 4, 6 (2) $6\frac{1}{2}$, $6\frac{3}{4}$
(3) 7, -4 (4) -3, -8

4. Compare each pair of numbers and link them by '$>$'.

(1) $3\frac{1}{3}$, 3.4 (2) 102, -68
(3) -6, -5 (4) -2.1, -3.9

5. Compare the numbers $-4, 2, -3, 5$. (Use '$<$')

6. Compare the numbers $-5, -2.1, |3.2|, |-4.3|$. (Use '$>$')

1.6. Addition of Integers.

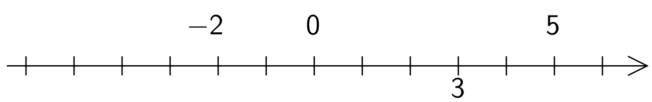
How do we add two integers? If they are both positive, then there's no problem:

$$(+5) + (+2) \;=\; 5 + 2 \;=\; 7.$$

If they aren't both positive, then we think in terms of the number line.

Problem 1. Calculate (5) + (-2).

Solution.



The number 5, which is the same as +5, represents going 5 units to the right of the origin. The number -2 represents going 2 units to the left. If you go 5 units right and 2 units left, the net effect is to go 3 units right. Therefore

$$(5) + (-2) \;=\; 5 - 2 \;=\; 3.$$

Problem 2. Calculate (-7) + 3.

Solution. Now we go 7 units to the left, and 3 to the right. This is the same as 4 units to the left, so

$$(-7) + 3 \;=\; -(7 - 3) \;=\; -4.$$

Problem 3. Calculate (-4) + (-5).

SOLUTION. Now we go 4 units left, then another 5 units left, for a total of 9 units left.

$$(-4) + (-5) \;=\; -(4 + 5) \;=\; -9.$$

Problems 1, 2 and 3 could have been done by carefully counting. We want a procedure that allows us to easily transform them into arithmetic problems. To phrase it neatly, we will use the terms *summand* and *sum*. The summands are the numbers being added, and the sum is their total; so in Problem 3, the summands are -4 and -5, and the sum is -9. This is what we do:

I. To add two numbers with the same sign, the absolute value of the sum is the sum of the absolute values of the summands, and the sign of the sum is the same as the sign of the summands.
II. To add two numbers with opposite signs, the absolute value of the sum is the difference between the two absolute values, and the sign of the sum is the same as the sign of the summand that has the larger absolute value.
III. When 0 is added to any number, the number doesn't change.

These rules may be a little hard to understand at first. Reread Problems 1-3 and see how the rules apply in those cases. Then read Examples 1-4 below. Think of moving left or right along the number line, and the rules will start to make sense.

Example 1. Calculate the following:
(1) $(+21) + (+23)$;
(2) $(-5) + (-23)$.

Solution.
(1) $(+21) + (+23) = +44$;
(2) $(-5) + (-23) = -28$.

Explanation. Here we are adding two numbers with the same sign. We just need to add their absolute values and then put in the appropriate sign.

Note. When adding positive numbers, we normally omit the positive sign. For example $(+21) + (+23) = +44$ would be written as $21 + 23 = 44$.

Example 2. Calculating the following:
(1) $(-14) + (+20)$;
(2) $(+19) + (-21)$;
(3) $(-17) + (+17)$.

Solution.
(1) $(-14) + (+20) = +6$;
(2) $(+19) + (-21) = -2$;
(3) $(-17) + (+17) = 0$.

Explanation. The two summands have different signs. We subtract the smaller absolute value from the larger one, then put in the sign of the number that has the larger absolute value. In number (3) we are adding opposite numbers and they cancel each other out.

NOTE. The sign of the positive number can be omitted. For example, $(-14)+(+20)$ can be written as $(-14)+20$ or just $-14+20$.

EXAMPLE 3. Calculate the following:
(1) $(214)+0$;
(2) $0+(-214)$;
(3) $0+0$.

SOLUTION.
(1) $(214)+0=214$;
(2) $0+(-214)=-214$;
(3) $0+0=0$.

Explanation. Here one of the summands is zero, so the sum is simply the other summand.

EXAMPLE 4. Calculating the following:
(1) $(+21)+(-16)+(-4)+(+15)$;
(2) $(-16)+(+5)+(-4)+(+5)$.

SOLUTION. We will carry out the calculations from left to right:
(1)

$$\begin{aligned}&(+21)+(-16)+(-4)+(+15)\\&\qquad=(+5)+(-4)+(+15)\\&\qquad=(+1)+(+15)\\&\qquad=+16.\end{aligned}$$

(2)

$$\begin{aligned}(-16)+(+5)+(-4)+(+5)\\ &=(-11)+(-4)+(+5)\\ &=(-15)+(+5)\\ &=-10.\end{aligned}$$

EXERCISES 1.6.

1. Calculate the following:

(1) $(-10)+(+21)+(+8)$

(2) $(-10)+(-12)+(-8)$

(3) $(+123)+(-20)+(-50)+(+7)$

(4) $123+(-20)+(-50)+7$

(5) $-11+6+5+(-3)$

2. A repair man needs to visit several clients living along an east-west road. However he cannot simply visit his clients starting from the east and moving to the west. His schedule depends on when his clients are available. He starts the day by visiting his first client who lives 6 *miles* east of his office, then his second client who lives 2 *miles* west of his first client, then his third client who lives 4 *miles* east of the second client, and finally, he visits his last client who lives 5 *miles* east of the third client. At the end of the day, how far was he from his office, and was he east or west of it?

1.7. Subtraction of Integers.

Subtraction is the inverse of addition. If you add 5 and then subtract 5, you move 5 units to the right and then 5 units to the left, and end up where you started. So subtracting a number is the same as adding its opposite number; and that is how we do it:

> *Subtracting a number is the same as adding its opposite number.*

Example 1. Calculate:
(1) $(+16)-(+13)$;
(2) $(-11)-(+12)$;
(3) $(-14)-(-15)$;
(4) $0-(-6)$;
(5) $(+13)-(+16)$;
(6) $(+12)-(-11)$;
(7) $(-5)-(-5)$;
(8) $(-6)-0$.

Solution.
(1) $(+16)-(+13) \;=\; (+16)+(-13) \;=\; 3$;
(2) $(-11)-(+12) \;=\; (-11)+(-12) \;=\; -23$;
(3) $(-14)-(-15) \;=\; (-14)+(+15) \;=\; 1$;
(4) $0-(-6) \;=\; 0+(+6) \;=\; 6$;
(5) $(+13)-(+16) \;=\; (+13)+(-16) \;=\; -3$;
(6) $(+12)-(-11) \;=\; (+12)+(11) \;=\; 23$;
(7) $(-5)-(-5) \;=\; (-5)+(+5) \;=\; 0$;
(8) $(-6)-0 \;=\; (-6)+0 \;=\; -6$.

EXAMPLE 2. Calculate

(1) $(+6) - (-3) - (-5)$;

(2) $(-7) - (-5) - (+12)$;

(3) $(-14) - (-15) + (-3)$;

(4) $(-1) + (-3) - (-5) + (-7)$.

SOLUTION.

(1) $(+6)-(-3)-(-5) = 6+3+5 = 9+5 = 14$;

(2) $(-7) - (-5) - (+12) = -7 + 5 + (-12) = -2 + (-12) = -14$;

(3) $(-14) - (-15) + (-3) = -14 + 15 + (-3) = 1 + (-3) = -2$;

(4) $(-1)+(-3)-(-5)+(-7) = -4+5+(-7) = 1 + (-7) = -6$.

EXERCISES 1.7.

Calculate the following:

1. $(+5) - (+12)$
2. $(+5) - (-12)$
3. $(-7) - (-5)$
4. $(+12) - (5) - (-8)$
5. $(-5) - (12) - (-8)$
6. $(-8) - (-12) + (-5)$
7. $(+9) - (+9) - (-3)$
8. $(-21) - (-12) + (+9) - (+8) - (-3)$

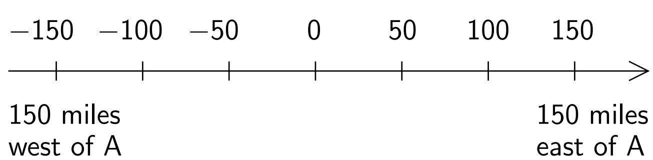
1.8. Multiplication of Integers.

Problem 1. Consider a car traveling on an east-west road at 50 miles per hour. Suppose that it passed point A at noon. Where was the car three hours from noon?

Solution. In 3 hours, the car travels $3 \times 50 = 150$ miles. So the car was 150 miles from point A.

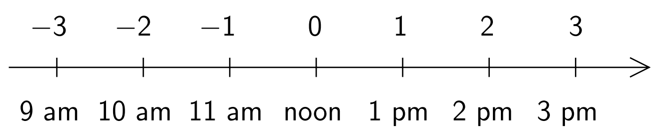
In this problem, we don't know if the car is traveling east or west, nor do we know if "three hours from noon" means three hours before noon or three hours after noon. So, though we do know the car was 150 miles from A, we don't know if it was 150 miles west or east of A.

There are four cases. To discuss them, we need to decide what is the positive direction. Let us assume that east is the positive direction. Then points east of A will be labeled with positive numbers, corresponding to their distance from A in miles, and points west of A will be labeled with negative numbers, the negative of their distance from A. So we think of the road as a number line, with A as the origin.



If the car is traveling east, we will say its velocity is $+50$ miles per hour, because its position on the number line increases by 50 miles every hour; and if the car is traveling west, we will say its velocity is -50 miles per hour, because its position on the number line decreases by 50 miles every hour.

For time, we let noon be the origin, and the positive direction corresponds to times after noon. Thus 3 hours before noon becomes -3, and 3 hours after noon becomes $+3$.



There are 4 possible cases in Problem 1.

PROBLEM 2.

(i) Suppose the car is traveling east. Where is the car three hours after it passes A?

(ii) Suppose the car is traveling east. Where is the car three hours before it passes A?

(iii) Suppose the car is traveling west. Where is the car three hours after it passes A?

(iv) Suppose the car is traveling west. Where is the car three hours before it passes A?

SOLUTION.

(i) We multiply the velocity, $+50$, by the time, $+3$, to get $+150$. So the car is 150 miles east of A.

(ii) We multiply the velocity, $+50$, by the time, -3, to get -150. So the car is 150 miles west of A.

(iii) We multiply the velocity, -50, by the time, $+3$, to get -150. So the car is 150 miles west of A.

(iv) We multiply the velocity, -50, by the time, -3, to get $+150$. So the car is 150 miles east of A.

In each of the four cases (i) - (iv), it is clear from looking at a picture whether the car is east or west of A. The principle, however, is very important. If we multiply a negative number by a positive number, as in (ii) or (iii), we get a negative number; and if we multiply two numbers of the same sign, as in (i) (both positive) or (iv) (both negative) we get a positive number.

Another observation: if in the problem either the velocity is zero (the car is not moving) or the time is zero (we want to know where the car is at noon), the answer will be 0: the car is at A.

Summarizing:

The product of two positive or two negative numbers is always positive, and the product is equal to the product of the absolute values.

The product of a positive and a negative number is always negative, and the absolute value of the product is equal to the product of the absolute values.

The product of zero and any other number is always zero.

EXAMPLE 1. Calculate:

(1) $(-12) \times (+13)$;

(2) $(11) \times (-12)$;

(3) $(-11) \times (14)$;

(4) $(-12) \times (0)$;

(5) $(12) \times (12)$.

SOLUTION.

(1) $(-12) \times (+13) = -156$

(2) $(11) \times (-12) = -132$;

(3) $(-11) \times (-14) = 154$;

(4) $(-12) \times (0) = 0$;

(5) $(12) \times (12) = 144$.

To multiply more than two numbers, we can do the multiplication by starting from the left and moving to the right.

EXAMPLE 2. Calculate:

(1) $(-3) \times (-4) \times (13)$;

(2) $(-1) \times (2) \times (-5) \times (-3) \times (4)$;

(3) $(-1) \times (-1) \times (-1) \times (-1) \times (-1)$.

SOLUTION.

(1) $(-3) \times (-4) \times (13) \;=\; 12 \times 13 \;=\; 156$;

(2)

$$\begin{aligned}
&(-1)\times(2)\times(-5)\times(-3)\times(4)\\
&\qquad= -2\times(-5)\times(-3)\times(4)\\
&\qquad= 10\times(-3)\times(4)\\
&\qquad= -30\times 4\\
&\qquad= -120;
\end{aligned}$$

(3)

$$\begin{aligned}
&(-1)\times(-1)\times(-1)\times(-1)\times(-1)\\
&\qquad= 1\times(-1)\times(-1)\times(-1)\\
&\qquad= -1\times(-1)\times(-1)\\
&\qquad= 1\times(-1)\\
&\qquad= -1.
\end{aligned}$$

NOTE. In (1) there are two negative numbers, and the product is positive. In (2), there are three negative numbers, and the product is negative. In (3), there are five negative numbers, and the product is negative.

From the above example, we see that the sign of the product depends on how many negative numbers there are among the factors. If there are an even number of negative factors, then the product is positive; if there are an odd number of negative factors, then the product is negative.

EXAMPLE 3. Calculate $(+3)\times(-2)\times(-3)\times(+2)\times(-1)$.

SOLUTION. Since there are three minus signs, that is an odd number of minus signs, and the product will be negative.

$$
\begin{aligned}
&(+3) \times (-2) \times (-3) \times (+2) \times (-1) \\
&\qquad = -(3 \times 2 \times 3 \times 2 \times 1) = -36.
\end{aligned}
$$

NOTE. We glossed over one point in Problems 1 and 2. It is important to use consistent units in physical problems. So if the velocity is given in miles per hour (mi/hr), then time should be measured in hours, and the product will give the position in miles. Likewise, if the velocity is given in meters per second (m/s), then time should be measured in seconds, and the product of the velocity and the time will be the position in meters.

EXERCISES 1.8.
Calculate the following:
1. $(+5) \times (-12)$
2. $(-5) \times (-12)$
3. $(-7) \times (0)$
4. $(+12) \times (-5) \times (2)$
5. $(-2) \times (12) \times (-8)$
6. $(-11) \times (-12) \times 0$
7. $(+2) \times (-3) \times (-3) \times (+5) \times (-4)$

1.9. Division of Integers.

Division is the inverse of multiplication. For example, from $4 \times 5 = 20$, we have $20 \div 4 = 5$ (and $20 \div 5 = 4$). In the equation $20 \div 4 = 5$, 20 is called the *dividend*, 4 is called the *divisor*, and 5 is called the *quotient*.

Consider the following examples:

From multiplication $(+3) \times (+5) = (+15)$,
we get for division $(+15) \div (+3) = (+5)$.

From multiplication $(-3) \times (+5) = (-15)$,
we get for division $(-15) \div (-3) = (+5)$.

From multiplication $(+3) \times (-5) = (-15)$,
we get for division $(-15) \div (+3) = (-5)$.

From multiplication $(-3) \times (-5) = (+15)$,
we get for division $(+15) \div (-3) = (-5)$.

From multiplication $0 \times (-3) = 0 \times (+3) = 0$,
we get for division $0 \div (+3) = 0 \div (-3) = 0$.

However, $(-3) \div 0$ is not defined. That is because if it were a number, then that number multiplied by 0 would be -3, and we know that is impossible.

We conclude the following:

When dividend and divisor are non-zero:
When the signs of the dividend and divisor are the same, the quotient is positive.
When the signs of the dividend and divisor are different, the quotient is negative.

The absolute value of the quotient is the quotient of the absolute values.

For zero:

You can never divide by 0.
Zero divided by any nonzero number is 0.

NOTE. To help remember which is the divisor and which is the dividend, it is useful to know that in Latin the ending "and" or "end" on a verb means "that which is required to be (whatever the verb is)". So if Fiona has to zap Alice, Fiona is the *zapper* and Alice is the *zappend*. Unfortunately, long usage has resulted in much inconsistency in the labeling of the numbers in arithmetic operations. The full list is:

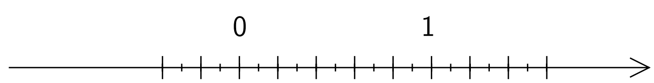
$12+4=16$ *12 and 4 are summands, 16 is the sum.*
$12-4=8$ *12 is the minuend (that which is to be made smaller), 4 is the subtrahend (that which is to be subtracted) and 8 is the difference.*
$12\times 4=48$ *12 and 4 are the factors, 48 is the product.*
$12\div 4=3$ *12 is the dividend, 4 is the divisor, and 3 is the quotient.*

Exercises 1.9.

1. $(-20) \div (+5)$
2. $(-30) \div (-5)$
3. $(+24) \div (-3)$
4. $(144) \div (-6)$
5. $0 \div (-102)$

1.10. Multiplication and Division of Rational Numbers.

What is the meaning of $\frac{3}{5}$ on the number axis? Divide the interval from 0 to 1 into 5 parts, then $\frac{3}{5}$ corresponds to the third point to the right of the origin.



Of course, if we divide the interval from 0 to 1 into 10 parts, then $\frac{3}{5}$ corresponds to the sixth point to the right to the origin. This is because $\frac{6}{10} = \frac{3}{5}$. Indeed, in any fraction, if we multiply the numerator (the number on the top) and the denominator (the number on the bottom) by the same nonzero number, the value of the fraction is unchanged. For example, $\frac{(-30)}{(-12)} = \frac{5\times(-6)}{2\times(-6)} = \frac{5}{2}$.

We know that

$$(-5) \div 3 = 5 \div (-3) = -(5 \div 3) = -\frac{5}{3}.$$

So
$$-\frac{5}{3} = \frac{(-5)}{3} = \frac{5}{(-3)}.$$
This is a useful property of fractions: we can move the minus sign from in front of the fraction to the numerator, or to the denominator, and the value of the fraction remains the same.

MULTIPLICATION OF RATIONAL NUMBERS. Of all the operations on fractions, multiplication is the simplest. To multiply two fractions, just multiply the numerators and multiply the denominators. For example: $\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$.

When multiplying two rational numbers (positive or negative fractions), the rules on the signs are the same as when multiplying two integers. If the two numbers have the same sign their product is positive; if the two numbers have different signs their product is negative. So we have:

$$\begin{aligned}
(-\frac{3}{5}) \times \frac{2}{7} &= -\frac{6}{35}, \\
\frac{3}{5} \times (-\frac{2}{7}) &= -\frac{6}{35} \\
(-\frac{3}{5}) \times (-\frac{2}{7}) &= \frac{6}{35} \\
\frac{3}{5} \times \frac{2}{7} &= \frac{6}{35}.
\end{aligned}$$

If we want to multiply mixed fractions, we must first convert them into pure fractions. For example,

$$2\frac{3}{5} \times \frac{3}{4} = \frac{13}{5} \times \frac{3}{4} = \frac{39}{20}.$$

EXAMPLE 1. Calculate $(-\frac{1}{3}) \times (-1\frac{2}{7}) \times (-\frac{2}{5})$.

SOLUTION. Perform the multiplication from left to right:

$$\begin{aligned}(-\frac{1}{3}) \times (-1\frac{2}{7}) \times (-\frac{2}{5}) &= (-\frac{1}{3}) \times (-\frac{9}{7}) \times (-\frac{2}{5}) \\ &= \frac{3}{7} \times (-\frac{2}{5}) = -\frac{6}{35}.\end{aligned}$$

Reciprocals. The reciprocal of a number is 1 divided by that number. The product of two reciprocal numbers is 1. The number 0 has no reciprocal. To find the reciprocal of a rational number, first write it as a pure fraction, then swap the numerator and denominator. For example, the reciprocal of -3 is $\frac{-1}{3}$, and the reciprocal of $-1\frac{2}{3}$ is $\frac{1}{-\frac{5}{3}} = -\frac{3}{5}$.

DIVISION OF RATIONAL NUMBERS. Since the reciprocal of a number is 1 divided by that number, so to divide a number is the same as to multiply by its reciprocal.

EXAMPLE 2. Calculate the following and simplify the answers.

(1) $\left(\frac{-3}{10}\right) \div \left(\frac{2}{5}\right)$

(2) $\left(-3\frac{2}{5}\right) \div \left(2\frac{1}{3}\right)$

(3) $\left(-3\frac{2}{5}\right) \div \left(-3\frac{1}{3}\right)$

SOLUTION.

(1) $\left(\frac{-3}{10}\right) \div \left(\frac{2}{5}\right) = -\frac{3}{10} \times \frac{5}{2} = -\frac{15}{20} = -\frac{3}{4}$

(2) $\left(-3\frac{2}{5}\right) \div \left(2\frac{1}{3}\right) = -\frac{17}{5} \times \frac{3}{7} = \frac{51}{35}$

(3) $\left(-3\frac{2}{5}\right) \div \left(-3\frac{1}{3}\right) = \frac{17}{5} \div \frac{10}{3} = \frac{17}{5} \times \frac{3}{10} = \frac{51}{50}$

EXAMPLE 3. Calculate the following and simplify the answers.

(1) $\left(-2\frac{1}{3}\right) \div (-14) \times \left(-3\frac{1}{3}\right)$

(2) $\left(-3\frac{1}{3}\right) \div \left(2\frac{1}{3}\right) \div \left(1\frac{1}{4}\right)$

SOLUTION. We will do the calculations from left to right.

(1)

$$\begin{aligned}
&\left(-2\tfrac{1}{3}\right) \div (-14) \times \left(-3\tfrac{1}{3}\right) \\
&\qquad = \left(-\tfrac{7}{3}\right) \times \left(-\tfrac{1}{14}\right) \times \left(-\tfrac{10}{3}\right) \\
&\qquad = \tfrac{1}{6} \times \left(-\tfrac{1}{14}\right) = -\tfrac{10}{18} = -\tfrac{5}{9}
\end{aligned}$$

(2)

$$\begin{aligned}
&\left(-3\tfrac{1}{3}\right) \div \left(2\tfrac{1}{3}\right) \div \left(1\tfrac{1}{4}\right) \\
&\qquad = \left(-\tfrac{10}{3}\right) \div \left(\tfrac{7}{3}\right) \div \left(\tfrac{5}{4}\right) \\
&\qquad = \left(-\tfrac{10}{3} \times \tfrac{3}{7}\right) \times \tfrac{4}{5} \\
&\qquad = -\tfrac{10}{7} \times \tfrac{4}{5} = -\tfrac{40}{35} = -\tfrac{8}{7}
\end{aligned}$$

NOTE. It is always a good habit to simplify your answer as we did in the above example, canceling any common factors in the numerator and denominator.

EXAMPLE 4. Simplify the following fractions:

(1) $-\frac{-8}{-24}$ (2) $\frac{-2}{-\frac{2}{3}}$

SOLUTION.

(1) $-\frac{-8}{-24} = -\frac{1}{3}$

(2) $\frac{-2}{-\frac{2}{3}} = 2 \times \frac{3}{2} = 3$

EXERCISES 1.10.

Calculate the following and simplify the answers.

(1) $\left(-1\frac{1}{2}\right) \times \left(-1\frac{1}{3}\right) \times \left(-1\frac{1}{4}\right)$.

(2) $(-8) \times \left(-\frac{3}{2}\right) \times \left(-1\frac{1}{4}\right)$.

(3) $(-16) \times \left(1\frac{1}{2}\right) \times \left(1\frac{1}{3}\right) \times \left(-1\frac{1}{4}\right)$.

(4) $3.3 \div \left(-3\frac{1}{3}\right)$

(5) $\left(-1\frac{1}{2}\right) \div \left(2\frac{1}{3}\right) \times \left(-\frac{1}{4}\right)$.

(6) $\left(-\frac{1}{4}\right) \div \left(-\frac{1}{0.2}\right) \div \left(-\frac{-4}{7}\right)$.

2. Simplify the following:

(1) $-\frac{-6}{-8}$ (2) $\frac{-\frac{1}{3}}{-2}$

(3) $\frac{1}{-\frac{1}{2}}$ (4) $\frac{-0.3}{3\frac{2}{3}}$

1.11. Addition and subtraction of rational numbers.

To add two rational numbers that have the same denominator just add the numerators.

EXAMPLE 1. Calculate the following:

(1) $\frac{3}{7} + (-\frac{4}{7})$;

(2) $(-\frac{4}{9}) + (-\frac{3}{9})$.

SOLUTION.
(1) $\frac{3}{7} + (-\frac{4}{7}) = \frac{3}{7} + \frac{-4}{7} = \frac{3+(-4)}{7} = \frac{-1}{7} = -\frac{1}{7}$
(2) $(-\frac{4}{9}) + (-\frac{3}{9}) = (\frac{-4}{9}) + (\frac{-3}{9}) = \frac{(-4)+(-3)}{9} = \frac{-7}{9} = -\frac{7}{9}$.

When the denominators are not the same, convert all the fractions so they have the same denominator. Its easiest if the denominator is always positive and we keep track of signs in the numerator.

EXAMPLE 2. Calculate the following:
(1) $\frac{3}{5} + (-\frac{4}{7})$;
(2) $(-\frac{4}{5}) + (-\frac{3}{6})$;
(3) $-\frac{2}{5} + (2\frac{1}{4})$.

SOLUTION.
(1) $\frac{3}{5} + (-\frac{4}{7}) = \frac{21}{35} + \frac{-20}{35} = \frac{21+(-20)}{35} = \frac{1}{35}$;
(2) $(-\frac{4}{5}) + (-\frac{3}{6}) = \frac{(-24)}{30} + (\frac{-15}{30}) = \frac{(-24)+(-15)}{30} = \frac{-39}{30} = -\frac{13}{10} = -1\frac{3}{10}$;
(3) $-\frac{2}{5} + (2\frac{1}{4}) = \frac{-8}{20} + \frac{45}{20} = \frac{-8+45}{20} = \frac{37}{20} = 1\frac{17}{20}$.

NOTE. It is always a good habit to simplify your answer as we did in (2), canceling any common factors in the numerator and denominator. Whether you prefer $-\frac{13}{10}$ or $-1\frac{3}{10}$ is a matter of taste.

Subtracting a number is the same as adding its opposite number, so to subtract a number you change the sign and add.

EXAMPLE 3. Calculate the following:

(1) $\frac{2}{5} - (-\frac{3}{2})$;
(2) $(-\frac{4}{5}) - (-\frac{3}{6})$;
(3) $-\frac{1}{5} - (2\frac{1}{4}) - (-1\frac{1}{3})$.

Solution.
(1) $\frac{2}{5} - (-\frac{3}{2}) = \frac{2}{5} + \frac{3}{2} = \frac{4}{10} + \frac{15}{10} = \frac{25}{10} = \frac{5}{2} = 2\frac{1}{2}$;
(2) $(-\frac{4}{5}) - (-\frac{3}{6}) = \frac{(-24)}{30} + \frac{15}{30} = \frac{(-24)+15}{30} = \frac{-9}{30} = -\frac{3}{10}$;
(3) $-\frac{1}{5} - (2\frac{1}{4}) - (-1\frac{1}{3}) = \frac{(-12)}{60} + \frac{(-135)}{60} + \frac{80}{60} = \frac{(-12)+(-135)+(80)}{60} = \frac{-67}{60} = -1\frac{7}{60}$.

Exercises 1.11.
Calculate the following:
1. $\big(-1\frac{1}{3}\big) - \big(-1\frac{1}{2}\big)$
3. $\big(3\frac{5}{6}\big) + \big(-\frac{3}{4}\big) + \big(-\frac{-11}{12}\big)$
4. $\big(7\frac{1}{3}\big) - \big(-\frac{4}{5}\big) + \big(1\frac{1}{2}\big)$
5. $\big(2\frac{1}{5}\big) - \big(-\frac{5}{6}\big) + \big(\frac{3}{-2}\big)$
6. $\big(-1\frac{1}{2}\big) + \big(-2\frac{3}{4}\big) - \big(-3\frac{1}{2}\big)$

1.12. Properties of Addition and Subtraction.

Commutativity of Addition

We know that $3+8 = 8+3$. This property of addition, that you can change the order of the summands, is called *commutativity*. It holds not just for adding positive integers, but for adding any numbers. To see this, imagine that you are on an east-west road. If you went 30 miles east then 50 miles west, you would be in the same position as if you first went 50 miles west and then went 30 miles east. Symbolically, this says that

$$30 + (-50) = (-50) + 30.$$

If you replace 30 and -50 by any other rational numbers, and think of them as representing distances to the left or right on a number line, you can see that it doesn't matter in what order you add them.

Example 1. $(-\frac{4}{5}) + \frac{5}{6} = \frac{-24}{30} + \frac{25}{30} = \frac{1}{30}$.
$\frac{5}{6} + (-\frac{4}{5}) = \frac{25}{30} + \frac{-24}{30} = \frac{1}{30}$.

Associativity of Addition

When adding three or more numbers together, it does not matter how we group them. For example,

$$\begin{aligned}(3+5)+7 &= 8+7 = 15\\ 3+(5+7) &= 3+12 = 15.\end{aligned}$$

This property is called *associativity*. Again, thinking about moving backwards and forwards on a number line, you can see that associativity applies to adding rational numbers as well.

Example 2.

$$\begin{aligned}(\frac{5}{2} + \frac{-7}{2}) + \frac{9}{2} &= \frac{-2}{2} + \frac{9}{2} = \frac{7}{2}\\ \frac{5}{2} + (\frac{-7}{2} + \frac{9}{2}) &= \frac{5}{2} + \frac{2}{2} = \frac{7}{2}.\end{aligned}$$

We can combine associativity and commutativity to simplify calculations.

EXAMPLE 3. Calculate $6\frac{3}{5} + \left(-5\frac{2}{3}\right) + 4\frac{2}{5} + 2\frac{1}{7} + \left(-1\frac{1}{3}\right) + \left(-1\frac{1}{7}\right)$.

SOLUTION. First we use commutativity and associativity to group fractions with the same denominators before adding them up.

$$\begin{aligned}
&+6\tfrac{3}{5} + \left(-5\tfrac{2}{3}\right) + 4\tfrac{2}{5} + 2\tfrac{1}{7} + \left(-1\tfrac{1}{3}\right) + \left(-1\tfrac{1}{7}\right)\\
=&6\tfrac{3}{5} + 4\tfrac{2}{5} + \left(-5\frac{2}{3}\right) + \left(-1\frac{1}{3}\right) + 2\tfrac{1}{7} + \left(-1\tfrac{1}{7}\right)\\
=&\left(6\tfrac{3}{5} + 4\tfrac{2}{5}\right) + \left[-5\tfrac{2}{3} + \left(-1\tfrac{1}{3}\right)\right] + \left[2\tfrac{1}{7} + \left(-1\tfrac{1}{7}\right)\right]\\
=&11 + (-7) + 1 = 4 + 1 = 5.
\end{aligned}$$

EXAMPLE 4. Calculate $(+32) + (-18) + (+164) + (-32) + (-164)$.

SOLUTION. Group the opposite numbers together:

$$\begin{aligned}
&(+32) + (-18) + (+164) + (-32) + (-164)\\
=&32 + (-32) + (-18) + 164 + (-164)\\
=&0 + (-18) + 0 = -18.
\end{aligned}$$

EXAMPLE 5. Calculate $32 + (-18) + 157 + (-243) + 24 + (-9)$.

SOLUTION. First group all the positive and negative terms together.

$$\begin{aligned}
&32 + (-18) + (+157) + (-243) + 24 + (-9) \\
&\quad = 32 + 157 + 24 + (-17) + (-243) + (-9) \\
&\quad = (+213) + (-269) = -56.
\end{aligned}$$

From the above examples, we see that we can sometimes simplify calculations by using the commutative and associative properties of addition. In general,

(1) add any opposite numbers first to get zero;

(2) group numbers with the same denominator, and add those;

(3) when there are many positive and negative numbers, first add all the positive terms, then add all the negative terms, finally add the sum of the positive numbers to the sum of the negative numbers.

PROPERTIES OF SUBTRACTION

Unlike addition, subtraction is neither commutative, nor associative. (That is to say $3 - 4 \neq 4 - 3$ and $(1 - 2) - 3 \neq 1 - (2 - 3)$.) Subtraction does have one property, however, that can be useful. Consider the following example.

EXAMPLE 6. Mike wants to read 30 pages of his math book this week. On Monday he reads 5 pages, on Tuesday he reads 7 pages. How many pages does he have to read during the rest of the week?

SOLUTION. There are two ways we could solve this.

We could calculate the total number of pages he read on Monday and Tuesday, $5 + 7 = 12$, and subtract this from 30 to get the answer. Or, we could first subtract from 30 the 5 pages he read Monday, and then the 7 pages he read Tuesday. Mathematically this corresponds to the two calculations:

$$\begin{aligned} 30 - (5 + 7) &= 30 - 12 = 18 \\ (30 - 5) - 7 &= 25 - 7 = 18. \end{aligned}$$

This is called the *distributive property* of subtraction.

> *Subtracting the sum of two numbers is the same as first subtracting one and then subtracting the other.*

EXAMPLE 7. Calculate $(-12) - [(-24) + 36]$.

SOLUTION 1.

$$\begin{aligned} (-12) - [(-24) + 36] &= (-12) - (12) \\ &= (-12) + (-12) = -24 \end{aligned}.$$

SOLUTION 2.

$$\begin{aligned} (-12) - [(-24) + (36)] &= (-12) - (-24) - (36) \\ &= [(-12) + (24)] + (-36) \\ &= 12 + (-36) = -24. \end{aligned}$$

EXERCISES 1.12.

Calculate the following

1. $132 + (-124) + (-16) + 0 + (-132) + (+16)$.
2. $127 + 13 + (-307) + (-140) + (-189) + 307$.

3. $127 + (-373) + 233 + (-125) + (-12) + 540.$
4. $6 + (-12) + 8.3 + (-7.4) + 9.1 + (-2.5).$
5. $+3\frac{5}{6} + 5\frac{1}{7} + \left(-2\frac{1}{6}\right) + 32\frac{6}{7}.$
6. $7\frac{3}{4} + \left(-5\frac{4}{11}\right) + \left(-3\frac{1}{4}\right) + \left(-6\frac{5}{11}\right) + 17\frac{1}{4} + 1\frac{1}{4}.$
7. $3.543 + (-0.543) + 6.457 + (-0.417)$
8. $3\frac{2}{5} + \left(-2\frac{7}{8}\right) + \left(-3\frac{5}{12}\right) + 5\frac{3}{5} + \left(-1\frac{1}{8}\right) + 5\frac{5}{12}$
9. $2\frac{1}{3} - \left(-2\frac{1}{4} + 3\frac{2}{5}\right)$
10. $153 - [53 + (-231) + (-69)]$
11. $-11\frac{1}{3} - \left[9\frac{2}{3} + \left(-8\frac{1}{4}\right) + 3\frac{3}{4}\right]$
12. $2.3 - \left[(-2.3) + (-2.4) + 5\frac{1}{3}\right]$
13. $-16 - \left[\left(-5\frac{2}{3}\right) + \left(-2\frac{1}{7}\right) + \left(-3\frac{1}{3}\right) + 3\frac{1}{7}\right]$
14. $-1\frac{1}{3} - \left[\left(-12\frac{2}{3}\right) + \left(-11\frac{2}{15}\right) + 14\right]$

1.13. Properties of Multiplication and Division.

Properties of Multiplication

Multiplication is *commutative* — interchanging the order of the factors doesn't matter:

$$\begin{aligned} 5 \times 3 &= 3 \times 5 \\ -7\tfrac{1}{3} \times 2\tfrac{2}{5} &= 2\tfrac{2}{5} \times -7\tfrac{1}{3}. \end{aligned}$$

It is also *associative*. When multiplying three or more numbers together, it doesn't matter how you group the factors:

$$\begin{aligned} (5 \times 3) \times 11 &= 5 \times (3 \times 11) \\ ((6\tfrac{1}{4} \times 3) \times -4\tfrac{1}{5}) \times \tfrac{1}{2} &= (6\tfrac{1}{4} \times 3) \times (-4\tfrac{1}{5} \times \tfrac{1}{2}). \end{aligned}$$

There is a third property of multiplication, illustrated by the following equation:

$$5 \times (7+3) \;=\; (5 \times 7) + (5 \times 3).$$

This is called the *distributive property of multiplication over addition.* It says that if you multiply one number by the sum of two others, it's the same as if you multiply each of the summands separately and add the results. Here are two more examples:

$$\begin{aligned} -2 \times (3\tfrac{1}{2} + 7\tfrac{1}{2}) \;&=\; -2 \times (3\tfrac{1}{2}) + (-2 \times 7\tfrac{1}{2}) \\ 3\tfrac{1}{3} \times (9-12) \;&=\; (3\tfrac{1}{3} \times 9) - (3\tfrac{1}{3} \times 12). \end{aligned}$$

Notice that in the last example we had a difference, not a sum. But that is okay because

$$\begin{aligned} 3\tfrac{1}{3} \times (9-12) \;&=\; 3\tfrac{1}{3} \times (9 + (-12)) \\ &=\; (3\tfrac{1}{3} \times 9) + (3\tfrac{1}{3} \times -12) \\ &=\; (3\tfrac{1}{3} \times 9) - (3\tfrac{1}{3} \times 12). \end{aligned}$$

So multiplication is also distributive over subtraction.

Example 1. Calculate $(+7)\times(-123)\times(\frac{1}{7})\times(-\frac{2}{41})$.

Solution. Since $7 \times \frac{1}{7} = 1$ and $123 = 3 \times 41$, we will change the order of the multiplications, using both the commutative and associative rules.

$$\begin{aligned} &(+7) \times (-123) \times (\tfrac{1}{7}) \times (-\tfrac{2}{41}) \\ &\qquad =\Big((+7) \times (\tfrac{1}{7})\Big) \times (-123) \times (-\tfrac{2}{41}) \\ &\qquad =1 \times \tfrac{123}{41} \times 2 \\ &\qquad =3 \times 2 = 6. \end{aligned}$$

EXAMPLE 2. Calculate $(-60) \times (-\frac{1}{4} + \frac{2}{5} - \frac{2}{3})$.

SOLUTION. Here we have three different denominators. Instead of finding a common denominator and adding all the fractions, we can do all the multiplications first:

$$
\begin{aligned}
&(-60) \times (-\tfrac{1}{4} + \tfrac{2}{5} - \tfrac{2}{3}) \\
=&(-60) \times (-\tfrac{1}{4}) + (-60) \times \tfrac{2}{5} + (-60) \times (-\tfrac{2}{3})\,. \\
=&15 + (-24) + (40) = 31
\end{aligned}
$$

PROPERTIES OF DIVISION Let us consider the following problem.

PROBLEM 1. Wing bought 900 bottles of wine in France. To ship the wine back to the US, she asked Philippe to carry the wine whenever he made a trip from France to the US. The maximum amount of wine that he can carry back is 3 cases, and each case contains 12 bottles of wine. How many trips must Philippe take to carry all the wine back for Wing?

SOLUTION 1. Since each cases contains 12 bottles, and on each trip Philippe can take 3 cases, on each trip he can take 12×3 bottles. So the number of trips that is required is:

$$900 \div (12 \times 3) = 900 \div 36 = 25\,.$$

ANSWER. It will take Philippe 25 trips.

SOLUTION 2. We can formulate the problem differently. Each case contains 12 bottles, so in total there are $900 \div 12$ cases. On each trip he can take three cases, so the number of trips that is required is:

$$(900 \div 12) \div 3 = 75 \div 3 = 25 \ .$$

ANSWER. It will take Philippe 25 trips.

The answers of course are the same. The point is that if you want to divide one number by the product of two others, it's the same as if you divide by the first number and then by the second. This fact could be deduced from properties we already know: dividing is the same as multiplying by the reciprocal, and the associativity of multiplication. For example,

$$\begin{aligned} 900 \div (12 \times 3) = 900 \times \frac{1}{12 \times 3} &= (900 \times \frac{1}{12}) \times \frac{1}{3} \\ &= (900 \div 12) \div 3 \ . \end{aligned}$$

This property is also valid for division of rational numbers. For example,

$$(-56) \div \big((-2) \times (3\tfrac{1}{2})\big) = (-56) \div (-7) = 8 \ ,$$

$$(-56 \div -2) \div (3\tfrac{1}{2}) = (+28) \div (3\tfrac{1}{2}) = 8 \ .$$

To summarize, we get the first property of division:

> *The quotient of a number divided by the product of several factors is the same as the number divided by each factor in turn.*

Let's consider another problem.

PROBLEM 2. John planted four blueberry bushes. The first year he got 24 blueberries, the second year he got 106 blueberries, and the third year he got 66 blueberries. On average, how many blueberries did he get from each bush over the three year period?

SOLUTION 1. In total, John got $24 + 106 + 66$ berries from 4 bushes. Dividing by 4, we get the following expression:

$$(24 + 106 + 66) \div 4 = 196 \div 4 = 49 \ .$$

ANSWER. On average, each bush produced 49 berries.

SOLUTION 2. We can also calculate the average for each bush each year, then add them up to get the average yield for the whole period. That is

$$24 \div 4 + 106 \div 4 + 66 \div 4 = 6 + 26.5 + 16.5 = 49 \ .$$

ANSWER. On average, each bush produced 49 berries.

Problem 2 illustrates the following rule:

A sum of numbers divided by another number is the same as the sum of the quotients of each number separately.

This rule also follows from thinking of division as multiplication by the reciprocal, and using the distributivity of multiplication over addition.

Here is another example.

$$\Big((-8)+4+(-1)\Big)\div 3=(-5)\div 3=-\frac{5}{3}\,,$$

$$\begin{aligned}\Big((-8)+4+(-1)\Big)\div 3 & \\ &= -8\div 3+4\div 3+-1\div 3\\ &=\frac{-8}{3}+\frac{4}{3}+\frac{-1}{3}\\ &=-\frac{5}{3}\,.\end{aligned}$$

Caution: We cannot reverse the distributive rule. For example, $4\div(1+1)\neq 4\div 1+4\div 1$.

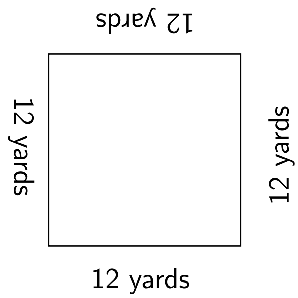
Exercises 1.13.

Calculate the following:

1. $-7\times\left(\frac{4}{3}\right)\times\left(\frac{2}{7}\right)$
2. $\frac{3}{4}\times(-6)+3\times\frac{3}{4}$
3. $\left(-\frac{2}{5}\right)\times(-3)-\left(\frac{3}{5}\right)\times(-3)$
4. $24\times\left(\frac{2}{3}-\frac{1}{2}+\frac{1}{6}\right)$
5. $\left(\frac{5}{8}-\frac{4}{7}\right)\times 56$
6. $123\times(-24)+(-123)\times(23)-123\times(-25)$
7. $2277\div(11\times 9)$
8. $1052\div(16\times 9)$
9. $\Big(123\div\left(-\frac{5}{6}\right)\Big)\div\frac{3}{5}$
10. $\left(2\frac{1}{3}\right)\div 5+\left(-3\frac{1}{3}\right)\div 5-2\frac{2}{7}\div 5-3\frac{3}{7}\div 5$

1.14. Powers of Rational Numbers.

Problem . Suppose a garden has the shape of a square, with each side 12 yards long. What is its area?



Solution. Its area is $12 \times 12 = 144$ square yards.

There is a special notation for writing this; we write 12^2 to mean 2 copies of 12 multiplied together. So by definition,

$$12^2 \quad \text{means} \quad 12 \times 12.$$

We read 12^2 as "twelve to the power of 2" or "12 squared".

What do you think 5^4 means? We use this notation to mean four copies of 5 multiplied together:

$$\begin{aligned} 5^4 &= 5 \times 5 \times 5 \times 5 \\ (-3)^5 &= (-3) \times (-3) \times (-3) \times (-3) \times (-3) \\ (\tfrac{1}{2})^3 &= (\tfrac{1}{2}) \times (\tfrac{1}{2}) \times (\tfrac{1}{2}). \end{aligned}$$

We read these as "five to the power of four" (or "five to the fourth power"), "minus three to the power of five" and "one half to the power of three" respectively.

The geometric fact that the area of a square with side-length $7\frac{1}{2}$ is $(7\frac{1}{2})^2$ leads to calling $7\frac{1}{2}$ to the power of 2 "$7\frac{1}{2}$ *squared*" (or the square of $7\frac{1}{2}$). Similarly, a cube with side length 5 has volume 5^3, so we often say "five cubed" for "five to the power of three". For powers above three, we just use the expression "to the power of".

Since the product of an even number of negative numbers is positive, and the product of an odd number of negative numbers is negative, when we exponentiate a negative number, we can decide the sign immediately.

EXAMPLE 1. Calculate the following:
(1) $(-2)^2$;
(2) $(-2)^3$;
(3) $(-2)^4$;
(4) $(-2)^5$.

SOLUTION.
(1) $(-2)^2 = (-2) \times (-2) = 4$;
(2) $(-2)^3 = (-2) \times (-2) \times (-2) = -8$;
(3) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$;
(4) $$\begin{aligned}(-2)^5 &= (-2) \times (-2) \times (-2) \times (-2) \times (-2)\\ &= -32\,.\end{aligned}$$

When we write 8^3, the number 8 is called *the base*, and the number 3 is called *the exponent*.

EXAMPLE 2. Read the following exponentiations and find the base and the exponent in each one.

$$(-4)^2, \quad (+3)^4, \quad (\frac{3}{5})^4, \quad (-1)^{31}, \quad (-0.3)^3.$$

SOLUTION. In $(-4)^2$, read as "minus four to the power of two", the base is -4, and the exponent is 2.

In $(3)^4$, read as "three to the power of four", the base is 3, and the exponent is 4.

In $(\frac{3}{5})^2$, read as "three over five to the power of two", the base is $\frac{3}{5}$, and the exponent is 2.

In $(-1)^{31}$, read as "minus one to the power of thirty one", the base is -1, and the exponent is 31.

In $(-0.3)^3$, read as "minus zero point three to the power of three", the base is -0.3, and the exponent is 3.

What is 5^1? This is just 5 multiplied by itself one time, *i.e.* 5. What about 5^0? What does 5 multiplied by itself 0 times mean? When confronted with a notation whose meaning is unclear, mathematicians assign it a meaning. The convention is that any number to the power of 0 is 1. We will see later why this turns out to be convenient.

Any nonzero number to the power of 0 *is defined to be* 1*.*

Example 3. Calculate the following:

(1)	3^4	(2)	2^5	(3)	2^0
(4)	$(-1)^5$	(5)	$(-2)^3$	(6)	$(-3)^2$
(7)	$(0.1)^3$	(8)	$(-\frac{3}{4})^2$	(9)	$(-\frac{1}{2})^3$

Solution.

(1)	$3^4 = 81$	(2)	$2^5 = 32$
(3)	$2^0 = 1$	(4)	$(-1)^5 = -1$
(5)	$(-2)^3 = -8$	(6)	$(-3)^2 = 9$
(7)	$(0.1)^3 = 0.001$	(8)	$(-\frac{3}{4})^2 = \frac{9}{16}$
(9)	$(-\frac{1}{2})^3 = -\frac{1}{8}$		

Example 4. What is $2^3 - 3^2$?

Solution. 2^3 is 2 multiplied by itself 3 times, *i.e.* $2 \times 2 \times 2 = 8$. Whereas 3^2 is 3 multiplied by itself twice, *i.e.* $3 \times 3 = 9$. So

$$2^3 - 3^2 = 8 - 9 = -1.$$

Exercises 1.14.

1. Read the following exponentiations and find the base and the exponent in each one.

(1)	6^5	(2)	$(-3)^8$
(3)	$\left(\frac{1}{4}\right)^4$	(4)	$\left(-\frac{5}{2}\right)^{10}$

2. Calculate the following:

(1)	2^3	(2)	3^2
(3)	$(-4)^3$	(4)	$(-2)^6$

(5) $(-8)^2$ (6) 0^4

3. Calculate the following:

(1) $\left(\frac{1}{2}\right)^3$ (2) $\left(\frac{1}{3}\right)^2$

(3) $\left(-\frac{1}{2}\right)^5$ (4) $\left(-\frac{1}{3}\right)^3$

(5) $\left(-\frac{1}{5}\right)^3$ (6) $\left(-1\frac{1}{3}\right)^4$

4. Calculate the following:

(1) $(-1)^2 7$ (2) $(-0.1)^4$

(3) $(-0.04)^2$ (4) $(0.2)^4$

1.15. The order of performing calculations.

We have seen five operations on rational numbers: addition, subtraction, multiplication, division, and exponentiation. Keep in mind that subtraction and addition are inverse operations to each other, and multiplication and division are also inverse operations to each other, and exponentiation is a long multiplication. So in some sense we are really only dealing with two distinct operations.

When a mathematical expression has more than one operation, we must decide in what order to perform the operations. We call this idea *precedence*. An operation with higher precedence gets performed before any operation with lower precedence.

The ordering of precedence among our five operations is:

(1) Exponentiation has the highest precedence, then

multiplication and division, and then addition and subtraction last.

(2) If two operations have the same precedence, we go from left to right.

(3) If there are parentheses, we always perform the operations inside parentheses first, over-riding the ordering in (1).

If we can use commutative, associative, or distributive rules to make our calculations easier, we are always encouraged to do so.

EXAMPLE 1. Calculate $6 + 3 \times 5$.

SOLUTION. Because multiplication has higher precedence than addition, we do the multiplication first. We can make things less confusing by introducing extra parentheses to make obvious the order of operations.

$$\begin{aligned} 6 + 3 \times 5 &= 6 + (3 \times 5) \\ &= 6 + 15 \\ &= 21. \end{aligned}$$

EXAMPLE 2. Calculate $6 \div 2 \times 3$.

SOLUTION. Both division and multiplication have the same precedence, so we calculate from left to right.

$$\begin{aligned} 6 \div 2 \times 3 &= (6 \div 2) \times 3 \\ &= 3 \times 3 \\ &= 9. \end{aligned}$$

Notice that $6 \div (2 \times 3)$ is different. To prevent accidents, it is good to put extra parentheses in an expression. So though $6 \div 2 \times 3 = (6 \div 2) \times 3$, the latter expression is less likely to lead to confusion.

EXAMPLE 3. Calculate $(-\frac{5}{27}) \times (-3)^2 - 0.5 \times (-1.1) \times (-2)^3$.

SOLUTION.

$$\begin{aligned}
&\Big(-\frac{5}{27}\Big) \times (-3)^2 - 0.5 \times (-1.1) \times (-2)^3 \\
&= \Big(-\frac{5}{27}\Big) \times 9 - \big(0.5 \times (-1.1) \times (-8)\big) \\
&= -\frac{5}{3} - \big((1.1) \times (-4)\big) \\
&= -\frac{5}{3} - (-4.4) \\
&= -\frac{5}{3} + \frac{44}{10} \\
&= -\frac{25}{15} + \frac{66}{15} = \frac{41}{15} = 2\,\frac{11}{15}
\end{aligned}$$

NOTE. Here we first did all the exponentiation, then the multiplication. Since it is easier to multiply 0.5×8 first, we used the commutative rule to interchange the ordering of -1.1 and 8 in the expression $0.5 \times (-1.1) \times (-8)$. We dealt with the subtraction in the last step.

EXAMPLE 4. Calculate

$$-6\left(\frac{7}{8}\right) - \left(\frac{3}{2} \times (-\frac{4}{5}) + 0.2 + 1\frac{3}{8} \div \frac{8}{7}\right).$$

SOLUTION.

$$
\begin{aligned}
&-6\left(\frac{7}{8}\right)-\left(\frac{3}{2}\times(-\frac{4}{5})+0.2+1\frac{3}{5}\div\frac{8}{7}\right)\\
=&-6\left(\frac{7}{8}\right)-\left(-\frac{12}{10}+0.2+\frac{8}{5}\times\frac{7}{8}\right)\\
=&-6\left(\frac{7}{8}\right)-\left(-\frac{6}{5}+0.2+\frac{7}{5}\right)\\
=&-6\left(\frac{7}{8}\right)-\left(0.2+\frac{-6+7}{5}\right)\\
=&-6\left(\frac{7}{8}\right)-(0.2+0.2)\\
=&-6\left(\frac{7}{8}\right)-0.4=-6.875+0.4=-6.475
\end{aligned}
$$

EXAMPLE 5. Calculate $\Big(12\div((-5)+(+14))\div 3\Big)^3$.

SOLUTION.

$$
\begin{aligned}
&\Big(12\div((-5)+(+14))\div 3\Big)^3\\
&\qquad=\big(12\div(9)\div 3\big)^3\\
&\qquad=\left(\frac{4}{3}\div 3\right)^3\\
&\qquad=\left(\frac{4}{3}\times\frac{1}{3}\right)^3=\left(\frac{4}{9}\right)^3=\frac{64}{729}.
\end{aligned}
$$

EXAMPLE 6. Calculate $(-4)\times\left(-4\frac{8}{9}\right)+(-5)\times\left(-4\frac{8}{9}\right)-(-8)\times\left(-4\frac{8}{9}\right)$.

SOLUTION. We will take advantage of the distributive rule for multiplication:

$$\begin{aligned}
(-4)\times\left(-4\frac{8}{9}\right)+(-5)\times\left(-4\frac{8}{9}\right)-(-8)\times\left(-4\frac{8}{9}\right)&\\
&=\big((-4)+(-5)-(-8)\big)\times\left(-4\,\frac{8}{9}\right)\\
&=(-9+8)\times\left(-4\,\frac{8}{9}\right)\\
&=(-1)\times\left(-4\,\frac{8}{9}\right)=4\,\frac{8}{9}.
\end{aligned}$$

Exercises 1.15.

Calculate the following:

1. $13-2\times(-3)$
2. $21\div 3+4\times(-2)$
3. $2\frac{1}{3}-\big(-\frac{2}{3}\big)\div 2$
4. $\big(-2\frac{1}{3}\big)\div\frac{1}{4}\times 3$
5. $12\div\big(1\frac{1}{2}\big)-\big(1\frac{1}{2}\big)\times 3$
6. $(-4)-\big(\frac{2}{3}\times\frac{4}{5}-2\frac{1}{3}\times\frac{8}{7}\big)$
7. $2\times\big([(-2)^2-(-1)^3]\div 3\big)$
8. $(-\frac{1}{3})\div\big(2\times[3-(-4+6)^2]\big)$

2. ALGEBRAIC FORMULAS

2.1. Using letters to represent numbers.

In the last chapter, we learned various rules for arithmetic with rational numbers. Sometimes the rules were hard to describe. For example, in Section 1.13, we had the distributive property of multiplication over addition. It said that "if you multiply one number by the sum of two others, it's the same as if you multiply each of the summands separately and add the results." How could we say this so it is easier to understand?

The key idea is to give the numbers names. Then, instead of having to talk about "the first number" and "the second number", which are hard to keep straight, one just uses their names. The names could be John and Wing, but to keep things concise it is normal to use a single letter to denote a number. So, in the distributive rule above, let a be the first number, b the second, and c the third. Then the rule becomes

$$a \times (b + c) \;=\; (a \times b) + (a \times c).$$

This is much better! Why? It is much easier to understand (and to remember). It is short and precise — exactly how a rule should be written. We call something like this, where letters are used to represent arbitrary numbers, an *algebraic formula.*

Similarly, the associative rule for addition becomes:

$$(a+b)+c = a+(b+c),$$

where a, b, c are any rational numbers.

When multiplying two fractions, the rule is "the denominator of the product of two fractions is the product of the denominators from each fraction, and the numerator of the product is the product of the numerators." How can we express this algebraically? The first fraction can be written as $\frac{a}{b}$, where a is the numerator and b the denominator. For the second fraction we need two more letters, say c and d, and so we can write it as $\frac{c}{d}$. Now the multiplication rule for fractions becomes:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times b}{c \times d} .$$

The rules for addition, subtraction and division of fractions can be written algebraically as:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d + b \times c}{b \times d}, \quad \frac{a}{b} - \frac{c}{d} = \frac{a \times d - b \times c}{b \times d},$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c} .$$

It is often easier to use symbols in a formula instead of words. For example, the area of a rectangle equals the product of its length and its width.

Area of a rectangle=length $\times$ width.

This is not bad, but we can make it better. Let A represent the area, l the length, and w the width. Then we have the formula for the area of a rectangle:

$$A = l \times w.$$

In Chapter 1 we learned several properties of arithmetic. We summarize them here algebraically:

Let a, b, c, and d be rational numbers.

commutative rule for addition:

$$a + b = b + a;$$

associative rule for addition:

$$(a + b) + c = a + (b + c);$$

commutative rule for multiplication:

$$a \times b = b \times a;$$

associative rule for multiplication:

$$(a \times b) \times c = a \times (b \times c);$$

distributive rule for multiplication over addition:

$$a \times (b + c) = a \times b + a \times c;$$

a rule for subtraction:

$$a - (b + c + d) = a - b - c - d;$$

two rules for division:

$$a \div (b \times c \times d) = ((a \div b) \div c) \div d;$$

$$(a + b + c) \div d = a \div d + b \div d + c \div d.$$

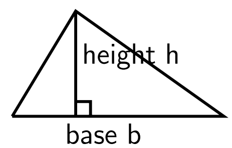
(In the rules for division, the numbers appearing as divisors cannot be zero.)

Exercises 2.1.

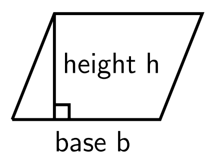
1. Use letters to express the following rule about addition: the opposite number (or the negative) of the sum of two numbers is the sum of the opposites of each summand.

2. Use letters to express the following property about fractions: in a fraction, if we divide both the numerator and the denominator by a nonzero number, the resulting number is unchanged. (Hint: use a and b to represent the numerator and the denominator of the fraction, and use m to represent the nonzero number.)

3. The area of a triangle is equal to half of the product of its height and the length of the base. Use letters to rewrite the formula for the area of triangle.



4. The area of a parallelogram is equal to the height of the parallelogram multiplied by the length of its base. Use letters to express the formula for the area of the parallelogram.



5. The average miles-per-gallon of a car is calculated by dividing the number of miles the car traveled by the number of gallons of gasoline it used. Use letters to express the formula for miles-per-gallon of a car. Try to calculate the miles-per-gallon of your family car. Is it the same as the manufacture claimed?

2.2. Constructing algebraic formulas.

One reason to use an algebraic formula is to say something *precisely*; another is to say it *concisely*. This is not just to save ink; it is easier to grasp something that is short than something that is long. The most common arithmetic operation in a formula is multiplication, which we have up to now represented by $\times$. In the interest of conciseness, the symbol $\times$ was first replaced by $\cdot$, so

$$\begin{aligned} a \cdot b &= a \times b; \\ (-6) \cdot (-28) &= (-6) \times (-28); \\ x \cdot (x - y) &= x \times (x - y). \end{aligned}$$

Do not confuse the symbol $\cdot$ with the decimal point! Frequently the multiplication sign is dropped entirely: the convention is that two symbols side-by-side in a formula are supposed to be multiplied together.

$$\begin{aligned} ab &= a \cdot b = a \times b; \\ (-6) \cdot (-28) &= (-6) \cdot (-28) = (-6) \times (-28); \\ x(x - y) &= x \cdot (x - y) = x \times (x - y). \end{aligned}$$

Note. 34 still means thirty-four; if we want to multiply 3 by 4 we must write $(3)(4)$ or $3 \cdot 4$.

Let us look at how to convert a phrase that has mathematical meaning into an algebraic formula.

Example 1. Let x be a number. Find the algebraic formula that represent the number which is 1 greater then x, and the algebraic formula that represent the number which is 5 less than x.

ANSWER. The number that is 1 greater then x is $x+1$, and the number that is 5 less then x is $x-5$.

EXAMPLE 2. Let x be a number. Find the number that is 7 larger then 5 times x, and the number that is 6 less then the reciprocal of x.

ANSWER. The number that is 7 larger then 5 times x is $5x+7$, and the number that is 6 less than the reciprocal of x is $\frac{1}{x}-6$.

EXAMPLE 3. Let x be the numerator of a fraction. The denominator of this fraction is 3 plus 4 times x. Find the reciprocal of this fraction.

SOLUTION. This fraction is $\frac{x}{3+4x}$. So the reciprocal is $\frac{3+4x}{x}$.

ANSWER. The reciprocal of this fraction is $\frac{3+4x}{x}$.

EXAMPLE 4. Use algebraic formulas to express the following:
(1) a number that is three times a;
(2) the cube of b;
(3) the sum of a and b;
(4) 8 less than two times a.

ANSWER.
(1) $3a$;
(2) b^3;
(3) ab;

(4) $2a - 8$.

NOTE. When we have a letter times a number, such as $a \times 3$, it is conventional to always write the number in front of the letter, so we write $3a$, not $a3$.

EXAMPLE 5. Use algebraic formulas to express the following:
(1) three times the sum of a and b;
(2) the sum of 4 times a and 5 times b;
(3) the sum of the square of a and the square of b;
(4) the square of the sum of A and B.

ANSWER.
(1) $3(a+b)$;
(2) $4a+5b$;
(3) a^2+b^2;
(4) $(A+B)^2$.

NOTE. The order of operations in an algebraic formula is the same as in a numerical formula (see Section 1.15). Anything in parentheses is done first; then exponentiation; then multiplication and division; finally addition and subtraction. You can always put extra parentheses in to make things clearer.

EXAMPLE 6. Use algebraic formulas to express the following:
(1) 5 times the square of a;
(2) the square of 5 times a;
(3) the opposite number of the square of a;
(4) the square of the opposite number of a.

Answer.
(1) $5a^2$;
(2) $(5a)^2$;
(3) $-a^2$;
(4) $(-a)^2$.

Example 7. Suppose the average speed of a car is p miles/hour.
(1) How far does the car travel in 3 hours?
(2) How far does the car travel in t hours?
(3) How long does it take to travel 200 miles?
(4) How long does it take to travel x miles?

Analysis. the relation between speed, time and distance is: $speed \times time = distance$, or $time = \frac{distance}{speed}$.

Answer.
(1) The car travels $3p$ miles in 3 hours.
(2) The car travels pt miles in t hours.
(3) It takes $\frac{200}{p}$ hours to travel 200 miles.
(4) It takes $\frac{x}{p}$ hours to travel x miles.

Example 8. A piece of rectangular land has one side bordering a river. Suppose the length of the side bordering the river is a miles, and the length of the other side of the land is b yards. (1) Find the area of the land. (2) If we want to surround the land with a fence, what is the length of the fence?

Analysis.
(1) Since a and b have different units, we must first convert them to the same unit. As 1760 yards =

1 mile, we need to write either the side of the land along the river as $1760a$ yards or the other side as $\frac{b}{1760}$ miles.

(2) To surround the land with a fence, we only need to put the fence along three sides of the land and skip the side bordering the river.

ANSWER.

(1) The area of the land is $\frac{ab}{1760}$ square miles. (Or $1760ab$ square yards.)

(2) The length of the fence is $1760a + 2b$ yards.

EXERCISES 2.2.

1. Let a be a number. Use algebraic formulas to express the following:

(a) a plus 5. (Ans. $a + 5$)

(b) 10 times a.

(c) One 10th of a.

(d) a divided by 5.

(e) a subtracted from 5.

(f) 5 subtracted from a.

(g) m times a.

2. Let a and b be numbers. Use algebraic formulas to express the following:

(a) the sum of a and b. (Ans. $a + b$)

(b) 13 times a minus 4 times b.

(c) the product of a and b.

(d) the sum of 2 times a and a times b.

(e) the sum of a squared and b squared.

(f) the square of the sum of a and b.

(g) the product of a and b divided by the sum of a and b.

3. Let a and b be numbers. Use algebraic formulas to express the following:

(a) minus a (or the opposite number of a, or the negative of a). (Ans. $-a$)

(b) minus the sum of a and b.

(c) minus the product of a and b.

(d) the absolute value of a.

(e) the sum of the absolute value of a and the absolute value of b.

(f) the absolute value of the sum of a and b.

(g) the absolute value of the difference of a and b.

(h) the absolute value of the sum of the reciprocal of a and the reciprocal of b.

4. Write down a fraction where the denominator is x and the numerator is 5 times the denominator plus 3.

5. Write down a fraction where the numerator is x and the denominator is the square of the sum of x and 3.

6. Write down a fraction such that the numerator is x and the denominator is 5 minus the reciprocal of the numerator.

7. Suppose the average miles-per-gallon of a passenger car is $36mi/gal$, and the average miles-per-gallon

of an SUV is $12 mi/gal$.

(a) How many gallons of gasoline are needed for the passenger car to travel x miles? for the SUV to travel x miles?

(b) For y gallons of gasoline, how many miles can the passenger car travel? the SUV travel?

(c) If the car and the SUV both travel x miles, how much more gasoline does the SUV consume then the car?

(d) If John drives the passenger car on weekdays and the SUV on weekends, and on average he drives x miles every weekday and y miles every weekend day, how many gallons of gasoline does John need every week?

8. Suppose the length of the side of a small square is x inches and the length of the side of a big square is y feet. (All answers should have units except (d).)

(a) What is the area of the small square? the big square?

(b) What is the total area of the two squares?

(c) What is the perimeter of the two squares?

(d) What is the ratio of the areas of the two squares?

(e) If you want to paint the back and the front of the small square and just the front of the big square with a paint that covers 5 square feet per ounce, how many ounces of paint do you need?

2.3. Values of algebraic formulas.

As we have seen in the discussion of the last two sections, algebraic formulas represent numbers. If we assign a value to each letter in an algebraic formula, then the algebraic formula becomes a number.

Example 1. Calculate the value of $-2ab^2$ when
(1) $a = -2,\ b = 3$;
(2) $a = -4,\ b = -0.1$
(3) $a = \frac{2}{5},\ b = 1\frac{2}{3}$.

Solution.
(1) $-2ab^2 = -2(-2)(3)^2 = -2\cdot(-2)\cdot 9 = 4\cdot 9 = 36$;
(2) $-2ab^2 = -2(-4)(-0.1)^2 = -2\cdot(-4)\cdot(0.01) = 8\cdot(0.01) = 0.08$;
(3) $-2ab^2 = -2(\frac{2}{5})(1\frac{2}{3})^2 = -2\cdot\frac{2}{5}\cdot\left(\frac{5}{3}\right)^2 = \frac{-4}{5}\cdot\frac{25}{9} = -\frac{20}{15}$.

Note. When we substitute numbers into letters, sometimes we need to insert parentheses to keep track of the order of arithmetic operations.

Example 2. Fill in the following table for $x^2 - 3x$.

x	-2	-1	0	1	2	$1/2$
x^2	4	1	0	1	2	$1/4$
$-3x$	6	3	0	-3	-6	$-3/2$
$x^2 - 3x$	10	4	0	-2	-4	$-5/4$

Example 3. Calculate the value of $\frac{a+2b}{2a-b}$ if
(a) $a = -2,\ b = -1$;

(b) $a = 2,\ b = -1$;

(c) $a = 1,\ b = -2$;

(d) $a = 1,\ b = 2$.

Solution.

(a) $\frac{a+2b}{2a-b} = \frac{-2+2(-1)}{2(-2)-(-1)} = \frac{-4}{-3} = \frac{4}{3}$.

(b) $\frac{a+2b}{2a-b} = \frac{2+2(-1)}{2\cdot 2-(-1)} = \frac{0}{3} = 0$.

(c) $\frac{a+2b}{2a-b} = \frac{1+2(-2)}{2\cdot 1-(-2)} = \frac{-3}{4} = -\frac{3}{4}$.

(d) When $a = 1,\ b = 2$, the denominator $2a - b = 2 \cdot 1 - 2 = 0$, thus the value of the fraction $\frac{a+2b}{2a-b}$ is undefined.

Example 4. Calculate the values of $-x^2$ and $(-x)^2$ for the following values of x.

(a) $x = 1.1$;

(b) $x = -1.1$;

(c) $x = 0.2$;

(d) $x = -0.2$.

Solution.

(a) $-x^2 = -(1.1)^2 = -1.21$,
$(-x)^2 = (-1.1)^2 = 1.21$.

(b) $-x^2 = -(-1.1)^2 = -1.21$,
$(-x)^2 = (-(-1.1))^2 = 1.21$.

(c) $-x^2 = -(0.2)^2 = -0.04$,
$(-x)^2 = (-0.2)^2 = 0.04$.

(d) $-x^2 = -(-0.2)^2 = -0.04$,
$(-x)^2 = (-(-0.2))^2 = 0.04$.

Exercises 2.3.

1. Fill in the following table:

a	-2.7	-2	1.7	$\frac{1}{3}$	0
$\lvert a\rvert$					
$-\lvert a\rvert$					
$a-\lvert a\rvert$					
$a+\lvert a\rvert$					

2. Fill in the following table:

a	-2	2	1	-1	1	1
b	1	1	2	2	-2	0
a^2						
ab						
a/b						

3. Calculate the values of $(x+1)^2$ and $(x-1)^2$ for the following values of x:
(a) $x = 1.5$
(b) $x = -1.5$
(c) $x = 0$.

4. Calculate the value of $\frac{a^2+b^2}{2ab}$ for the following values of a and b:
(a) $a = 2,\ b = 3$
(b) $a = -1,\ b = 3$
(c) $a = 3,\ b = -1$
(d) $a = 0,\ b = 0$.