Math 131, Exam 3, November 14th

This exam should have 16 multiple choice questions, and two hand-written questions. Each multiple-choice question is worth 5 points; each hand-written problem is worth 10 points.

Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card.

Write your name and student ID number on each of the hand-graded sheets.

Choose the answer that is *closest* to the solution — the exact solution may not be on the list. Mark your answer card with a **PENCIL** by shading in the correct box.

You may use a calculator, but not one that has a graphing function, or does symbolic differentiation. Remember that all angles are assumed to be in **radians**.

You may not have any written aids.

1. Let $f(x) = x \sin x$. What is $f'''(1)$

A. -3.00
B. -3.01
C. -3.02
D. -3.03
E. -3.04
F. -3.05
G. -3.06
H. -3.07
I. -3.08
J. -3.09

$f'(x) = x \cos x + \sin x$

$f''(x) = -x \sin x + 2 \cos x$

$f'''(x) = -x \cos x - 3 \sin x$

$f'''(1) = -3.0647$

2. If $y = A\sin x + B\cos x$, and $y'' - 2.1y' = \cos x$, what is A?

A. -.31
B. -.32
C. -.33
D. -.34
E. -.35
F. -.36
G. -.37
H. -.38
I. -.39
J. -.40

$y' = A \cos x - B \sin x$

$y'' = -A \sin x - B \cos x$

$y'' - 2.1y' =$

$(-A + 2.1B) \sin x + (-B - 2.1A) \cos x$

$\therefore \quad -A + 2.1B = 0$
$-B - 2.1A = 1$

$B = \frac{1}{2.1} A$

$\therefore 1 = (-\frac{1}{2.1} - 2.1) A$

$\therefore A = -.38817$

3. If $e^{xy} = 2y^2 - 1$, calculate $\frac{dy}{dx}$ at $P = (0, -1)$.

A. .21
B. .22
C. .23
D. .24
E. .25
F. -.21
G. -.22
H. -.23
I. -.24
J. -.25

$\frac{d}{dx}$: $e^{xy}\left(y + x\frac{dy}{dx}\right) = 4y\frac{dy}{dx}$

At P : $(-1) = -4\frac{dy}{dx}$

$\frac{dy}{dx} = 1/4 = .25$

4. If $\sin x + \cos y = 1$, calculate $\frac{d^2y}{dx^2}$ at $P = (\frac{\pi}{6}, \frac{\pi}{3})$

A. -1.10
B. -1.11
C. -1.12
D. -1.13
E. -1.14
F. -1.15
G. -1.16
H. -1.17
I. -1.18
J. -1.19

$\frac{d}{dx}$: $\cos x - \sin y \frac{dy}{dx} = 0$ (1)

$\frac{d}{dx}$: $-\sin x - \cos y \frac{dy}{dx}\frac{dy}{dx} - \sin y \frac{d^2y}{dx^2} = 0$ (2)

At P: (2) $\Rightarrow$ $-\frac{1}{2} - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - \frac{\sqrt{3}}{2}\frac{d^2y}{dx^2} = 0$

(1) $\Rightarrow$ $\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\left.\frac{dy}{dx}\right|_P = 0 \Rightarrow \left.\frac{dy}{dx}\right|_P = 1$

$\therefore \frac{d^2y}{dx^2} = -\frac{2}{\sqrt{3}} = -1.1547$

5. What equation represents the best linear approximation of $f(x) = \sin(2\pi x)$ for values of x close to zero?

A. $y = 0$
B. $y = \cos(x)$
C. $y = x$
D. $y = 2\pi x + 1$
E. $y = 1$
F. $y = 2\pi x$
G. $y = 2\pi(x + 1)$
H. $y = -x$

$f(0) = 0$

$f'(0) = 2\pi \cos(2\pi x)\big|_0$

$= 2\pi$

$\therefore y = 2\pi x$

6. Simplify $x \cosh(\ln x)$.

A. $\dfrac{x^2 + 1}{2}$

B. $x^2 - 1$

C. 0

D. $\dfrac{x^2 - 1}{2}$

E. $\dfrac{x^2}{2}$

F. $x + x^{-1}$

G. $\dfrac{e^x}{2}$

H. $\dfrac{e^x - e^{-x}}{2}$

I. $\dfrac{e^{-x}}{2}$

$x \dfrac{e^{\ln x} + e^{-\ln x}}{2}$

$= x \dfrac{x + \frac{1}{x}}{2}$

$= \dfrac{x^2 + 1}{2}$

7. Suppose $F'(x) = 4x^2$ and $F(0) = .9$. What is $F(2)$?

A. 11.0
B. 11.1
C. 11.2
D. 11.3
E. 11.4
F. 11.5
G. 11.6
H. 11.7
I. 11.8
J. 11.9

$$\frac{d}{dx} x^3 = 3x^2$$

$$\therefore \frac{d}{dx} \frac{4}{3} x^3 = 4x^2$$

$$\therefore F(x) = \frac{4}{3}x^3 + C$$

$$F(0) = C = .9$$

$$\therefore F(x) = \frac{4}{3}x^3 + .9$$

$$F(2) = \frac{4}{3} \cdot 2^3 + .9$$

$$= 11.5667$$

8. Evaluate $\frac{d}{dy}\arctan(y^{1.5})$ at $y = 9$.

A. .001
B. .002
C. .003
D. .004
E. .005
F. .006
G. .007
H. .008
I. .009
J. .100

1. What is $\frac{d}{dx}$ Arctan x?

$$t = \text{Arctan } x$$

$$\text{Tan } t = x$$

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\text{Sec}^2 t} = \frac{1}{1 + \text{Tan}^2 t} = \frac{1}{1+x^2}$$

$$\therefore \frac{d}{dx} \text{Arctan } x = \frac{1}{1+x^2}$$

$$\frac{d}{dy} \text{Arctan}(y^{1.5}) = \frac{1}{1+(y^{1.5})^2} \cdot (1.5) y^{.5}$$

$$= \frac{1.5\sqrt{y}}{1+y^3}$$

At 9: $\frac{4.5}{730} = .00616$

9. Let $g(x) = 3x + \sin(x) + \cos(x)$. How many critical points does g have on $\mathbb{R}$?

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5
G. 6
H. 7
I. 8
J. infinitely many

$g'(x) = 3 + \cos x - \sin x$

As Cos & Sin vary between -1 & 1,

$g'(x) \geq 1$ for all x.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f(0) = 1$ and $f(2) = 5$.

Consider the assertions:

(I) f must have a global maximum on $[0, 2]$. True (Extreme Value Thm)
(II) f must have a critical point in $(0, 2)$. False (eg.)
(III) There must exist some $c \in (0, 2)$ with $f'(c) = 2$. True (Mean Value Theorem)

A. All 3 are true.
B. (I) and (II) are true, (III) is false.
C. (I) and (III) are true, (II) is false.
D. (II) and (III) are true, (I) is false.
E. (I) is true, (II) and (III) are false.
F. (II) is true, (I) and (III) are false.
G. (III) is true, (I) and (II) are false.
H. All three are false.

Pi G

11. A spherical balloon is losing air at a rate of 10 cubic centimeters per minute. At what rate is its radius r decreasing when r is 2 centimeters?

A. .14 cm/min
B. .15 cm/min
C. .16 cm/min
D. .17 cm/min
E. .18 cm/min
F. .19 cm/min
G. .20 cm/min
H. .21 cm/min
I. .22 cm/min
J. .23 cm/min

$\frac{dV}{dt} = -10$

Wa: $\frac{dr}{dt}$

R: $V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\left.\frac{dr}{dt}\right|_{r=2} = \frac{1}{16\pi} \cdot (-10)$

$= -\frac{5}{8\pi}$

$= -.1989$

12. Suppose y and x are functions of the time variable t and that $y = x^3$ for all t. Also, at time t_0, $\frac{dx}{dt} = 4.75$ and $x = .33$. What is $\frac{dy}{dt}$ at time t_0?

A. 1.54
B. 1.55
C. 1.56
D. 1.57
E. 1.58
F. 1.59
G. 1.60
H. 1.61
I. ∞
J. $-\infty$

$y = x^3$

$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$

$\left.\frac{dy}{dt}\right|_{t_0} = 3(.33)^2(4.75) = 1.5518$

13. Find the point of inflection of $f(x) = 2x^3 - 7x^2 - 5x + 2$.

A. 1.0
B. 1.1
C. 1.2
D. 1.3
E. 1.4
F. 1.5
G. 1.6
H. 1.7
I. 1.8
J. 1.9

$f'(x) = 6x^2 - 14x - 5$

$f''(x) = 12x - 14$

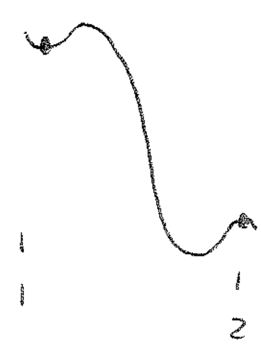
This is zero only at $14/12 \approx 1.167$, where it changes from -ve to +ve.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that f has a local minimum at 1 and a local maximum at 2. Consider which of the following three statements must be true:

(I) $f''(2) \leq 0$. T
(II) $f(1) \leq f(2)$. F
(III) $f'(1) = 0$. T (Fermat)

A. All 3 are true.
B. (I) and (II) are true, (III) is false.
C. (I) and (III) are true, (II) is false.
D. (II) and (III) are true, (I) is false.
E. (I) is true, (II) and (III) are false.
F. (II) is true, (I) and (III) are false.
G. (III) is true, (I) and (II) are false.
H. All three are false.

Ad II: Could have



15. Let the population of a country be 14.1 million in 2005 and 15.8 million in 2010.

Assuming the population is growing linearly, what was the population in 2008 in millions?

A. 15.10
B. 15.11
C. 15.12
D. 15.13
E. 15.14
F. 15.15
G. 15.16
H. 15.17
I. 15.18
J. 15.19

$$P(2008) = 14.1 + 3\,\frac{15.8 - 14.1}{5}$$

$$= 15.12$$

16. Let the population of a country be 14.1 million in 2005 and 15.8 million in 2010.

Now assume the population is growing exponentially. What was the population in 2008 in millions?

A. 15.10
B. 15.11
C. 15.12
D. 15.13
E. 15.14
F. 15.15
G. 15.16
H. 15.17
I. 15.18
J. 15.19

Now Log P is linear

$$\text{Log } P(2005) = 7.1492$$

$$\text{Log } P(2010) = 7.1987$$

$$\therefore \text{Log } P(2008) = 7.1492 + 3\,\frac{7.1987 - 7.1492}{5}$$

$$= 7.1789$$

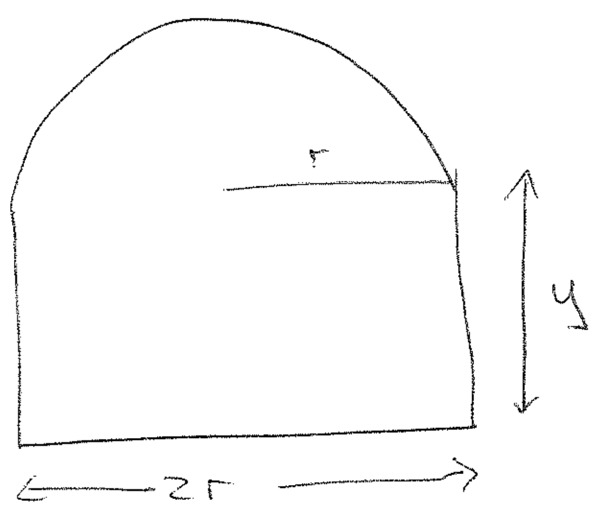
$$\therefore P = 10^{7.1789} = 15.0967 \times 10^6$$

Math 131 Hand-graded Problems

Student Name:

Student ID:

Problem A: A window is in the shape of a rectangle surmounted by a semi-circle. The area of the window is 10 square meters. What should the radius of the semi-circle be to minimize the perimeter?



$$A = \frac{1}{2}\pi r^2 + 2ry = 10 \qquad (1)$$

$$P = 2r + 2y + \pi r$$

From (1): $2ry = 10 - \frac{1}{2}\pi r^2 \implies y = \frac{5}{r} - \frac{\pi}{4} r$

$$P = (2+\pi) r + \frac{10}{r} - \frac{\pi}{2} r = \left(2+\frac{\pi}{2}\right) r + \frac{10}{r}$$

$\boxed{\text{Note } P(r) \to \infty \text{ as } r \to 0 \text{ or } r \to \infty.}$

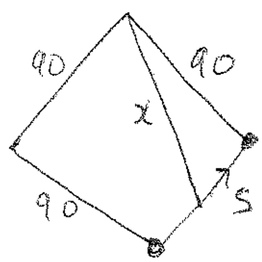
$$\frac{dP}{dr} = 2 + \frac{\pi}{2} - \frac{10}{r^2}$$

This has critical point only when

$\left(2 + \frac{\pi}{2}\right) r^2 = 10 \qquad r = \sqrt{\frac{10}{2+\pi/2}}$.

Problem B: A baseball diamond is a square with a side of 90 ft. A player runs from home plate to first base at a speed of 24 ft/s. At what rate is his distance from second base decreasing when he is halfway to first base?

Pi:



G: $\frac{ds}{dt} = -24$

Wa: $\frac{dx}{dt}$

R: $s^2 + 90^2 = x^2$

$\therefore \quad 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$

When $s = 45$, $\quad x = \sqrt{45^2+90^2}$

$$\therefore \quad \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{45}{\sqrt{45^2+90^2}}(-24)$$

$$= \frac{1}{\sqrt{5}}(-24)$$

Answer: At $\frac{24}{\sqrt{5}}$ ft/s.