Math 131 , Spring 2004
Exam 1 Solutions

Name: _____ **ID#** _____

Only scientific calculators are allowed. Be sure your calculator is set for "radians", not "degrees", if you do any calculus computations with trig functions. For Parts I and II, please mark your answer on the answer card. For Part III, please solve the problems in the space provided.

Part I, Multiple Choice, 5 points/problem:

1. Suppose $f(x) = x^2$, $g(x) = \sqrt{1 + \ln x}$, and $h(x) = e^{4x}$. What is the value of $(f \circ g \circ h)(1)$?

A) 0 B) 1 C) 2 D) e^4 E) $\sqrt{5}$

F) $\sqrt{6}$ G) $2 + e^4$ H) $1 + e$ **I) 5** J) 6

Solution:

$$(f \circ g \circ h)(x) = f(g(h(x))) = \left(\sqrt{1 + \ln(e^{4x})}\right)^2 = \left(\sqrt{1 + 4x}\right)^2 = 1 + 4x$$

Then $(f \circ g \circ h)(1) = 1 + 4(1) = 5$.

2. Find the vertical asymptotes of the function $f(x) = \dfrac{x^2 - 3x - 4}{2x^2 - 2x - 24}$.

A) $x = -1$ and $x = 4$ **B) $x = -3$** C) $x = \frac{1}{2}$ D) $x = -3$ and $x = 4$

E) $x = \frac{1}{3}$ and $x = 4$ F) $x = 4$ G) $x = 3$ H) $x = \frac{1}{3}$

I) $x = -1$ and $x = -3$ J) no vertical asymptotes

Solution: Intuitively, vertical asymptotes occur when the "y" values can be made arbitrarily large. (See the definintion on page 132 of your text for a more precise statement.) We first simplify $f(x)$.

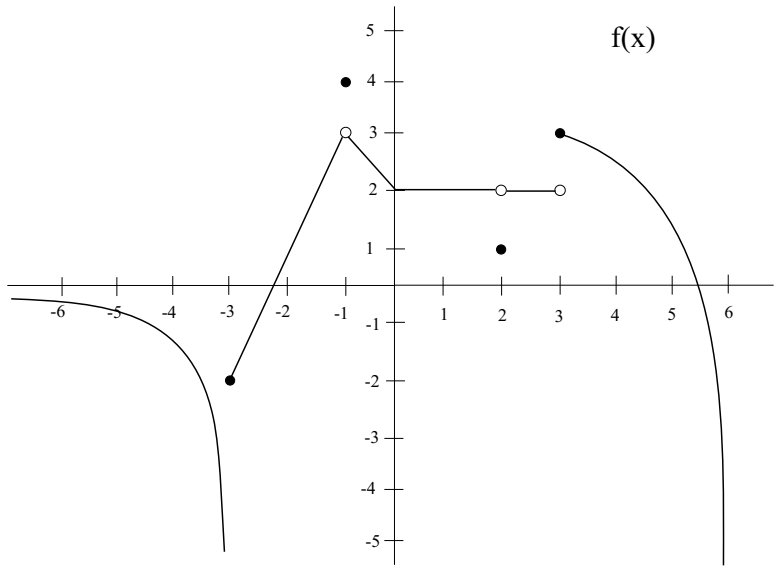
$$f(x) = \frac{x^2 - 3x - 4}{2x^2 - 2x - 24} = \frac{(x-4)(x+1)}{2(x-4)(x+3)}.$$

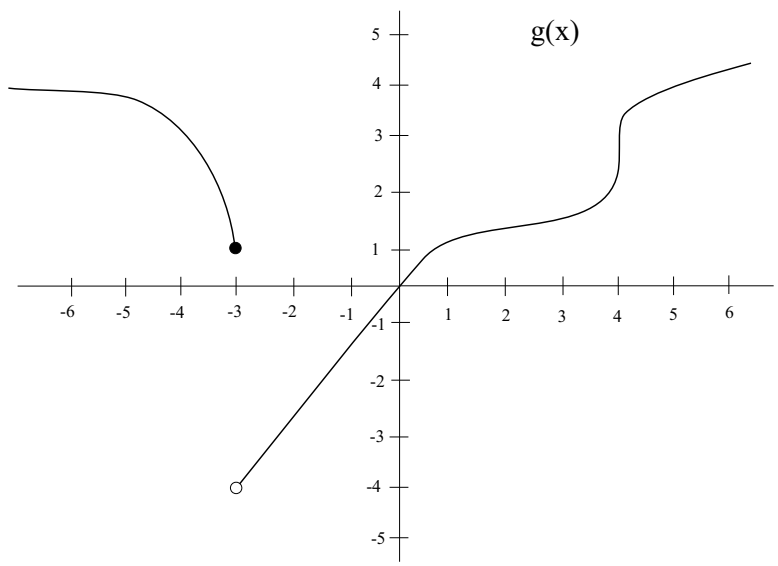
Since

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{(x-4)(x+1)}{2(x-4)(x+3)} = \lim_{x \to -3} \frac{(x+1)}{2(x+3)} = \infty,$$

the line $x = -3$ is a vertical asymptote. For no other value $x = a$ is it true that $\lim_{x \to a} f(x) = \pm\infty$, so $x = -3$ is the only vertical asymptote.

For questions 3 - 12, please refer to the graphs of $f(x)$ and $g(x)$ given below. Questions 3 - 7 (worth 2 points each) are "fill in the blank", where you will select your answers from those listed below the graphs. Questions 8 - 12 (worth 2 points each) are true/false questions.





The "fill in the blank" answer choices are:

A) -4	B) -3	C) -2	D) -1	E) 0
F) 1	G) 2	H) 3	I) 4	J) ∞

3. $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$ $\boxed{\text{(i.e. J)}}$

4. $\lim_{x\to -\infty} g(x) = 4.$ $\boxed{\text{(i.e. I)}}$

5. $\lim_{x\to -3^-} g(x) = 1.$ $\boxed{\text{(i.e. F)}}$

6. $\lim_{x\to 0} f(x)g(x) = 2(0) = 0.$ $\boxed{\text{(i.e. E)}}$

7. $\lim_{x\to -1} f(x) = 3.$ $\boxed{\text{(i.e. H)}}$

The following 5 questions are true/false and still refer to the functions $f(x)$ and $g(x)$, whose graphs are shown on the previous page.

8. f is discontinuous at exactly five points.

A) True B) False **Solution:** f is discontinuous at $x = -3, -1, 2, 3, 6$.

9. $\lim_{x\to 2} f(x)$ does not exist.

A) True **B) False** **Solution:** $\lim_{x\to 2} f(x) = 2$

10. g is not differentiable at $x = 4$.

A) True B) False **Solution:** $g(x)$ has a vertical tangent line at $x = 4$.

11. f is continuous from the right at $x = 3$.

A) True B) False **Solution:** By definition of continuous from the right,
$$\lim_{x\to 3^+} f(x) = 3 = f(3).$$

12. $\lim_{x\to -3} g(x) = 1.$

A) True **B) False** **Solution:** $\lim_{x\to -3} g(x)$ DNE because $\lim_{x\to -3^-} g(x) = 1$ and $\lim_{x\to -3^+} g(x) = -4$

13. Solve the following equation for x: $\dfrac{10}{e^{-5x}+2} = 1$.

A) $x = -5\dfrac{\ln 2}{8}$ B) $x = \dfrac{8\ln 5}{2}$ C) $x = \dfrac{1}{5}$ D) $x = -10\dfrac{\ln 2}{5}$ E) $x = e^{-\frac{1}{5}}$

F) $x = \dfrac{\ln 5}{10}$ **G)** $x = -\dfrac{\ln 8}{5}$ H) $x = -10\dfrac{\ln 8}{2}$ I) $x = \dfrac{\ln(-2)}{10}$ J) $x = 0$

Solution: We proceed as follows:

$$\begin{aligned} 10 &= e^{-5x} + 2 \\ 8 &= e^{-5x} \\ \ln 8 &= \ln\left(e^{-5x}\right) = -5x \\ x &= -\frac{\ln 8}{5} \end{aligned}$$

14. Suppose

$$1 + \ln(x+1) \leq f(x) \leq 2x + e^x.$$

Evaluate $\lim_{x\to 0} f(x)$.

A) $1 + \ln 2$ B) $2 + e$ C) 0 D) 2 E) $\ln 2$
F) e G) $e + \ln 2$ **H)** 1 I) 3 J) does not exist

Solution: We use the Squeeze Theorem. First note that

$$\lim_{x\to 0}\left(1 + \ln(x+1)\right) = 1 + \ln 1 = 1 + 0 = 1 \quad \text{and} \quad \lim_{x\to 0}\left(2x + e^x\right) = 0 + e^0 = 1.$$

Then by the Squeeze Theorem,

$$\lim_{x\to 0}\left(1 + \ln(x+1)\right) \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0}\left(2x + e^x\right),$$

so

$$1 \leq \lim_{x\to 0} f(x) \leq 1.$$

Therefore, $\lim_{x\to 0} f(x) = 1$.

15. Let $f(x) = x^2 + x + a$, where a is a constant. For which value of a will the Intermediate Value Theorem guarantee that the equation $f(x) = 0$ has a root in the interval $(0, 2)$.

A) -20	B) -8	$\boxed{\textbf{C)}\ -2}$	D) 1	E) 2
F) 4	G) 6	H) 8	I) 10	J) 12

Solution: Note that $f(x)$ is continuous for all x since f is a polynomial. Then, since

$$f(0) = a \quad \text{and} \quad f(2) = 6 + a,$$

by the Intermediate Value Theorem, f assumes all values between a and $6 + a$. We want 0 to be one such value, so we must choose a so that

$$a < 0 < 6 + a.$$

Then $a = -2$ is the only value that works. (For this value, $f(0) = -2$ and $f(2) = 4$, so by IVT, there is a number c in $(0, 2)$ such that $f(c) = 0$.)

16. What is the value of $\left(\log_2 2^6\right)\left(\ln\sqrt{e^5}\right)\left(\log_2 \frac{1}{2}\right)$?

A) -30	B) $\frac{5}{2}$	C) $\frac{1}{6}$	$\boxed{\textbf{D)}\ -15}$	E) 24
F) $-\frac{12}{5}$	G) 6	H) $-\frac{5}{4}$	I) $-\frac{e}{5}$	J) 0

Solution: First note that

$$\ln\sqrt{e^5} = \ln\left(e^5\right)^{\frac{1}{2}} = \ln(e)^{\frac{5}{2}} = \frac{5}{2}\ln e = \frac{5}{2}$$

and

$$\log_2 \frac{1}{2} = \log_2\left(2^{-1}\right) = -\log_2(2) = -1.$$

Then

$$\begin{aligned}\left(\log_2 2^6\right)\left(\ln\sqrt{e^5}\right)\left(\log_2 \frac{1}{2}\right) &= (6\log_2 2)\left(\frac{5}{2}\right)(-1)\\ &= 6\left(\frac{5}{2}\right)(-1)\\ &= -15.\end{aligned}$$

17. Evaluate $\lim_{x \to 0^-} \frac{4x + |x|}{x}$.

A) ∞ B) -4 C) -3 D) -1 E) 4

F) 2 G) 0 H) 5 **I) 3** J) 1

Solution: When $x \to 0^-$, the values of x are less than zero, so $|x| = -x$. Then

$$\lim_{x \to 0^-} \frac{4x + |x|}{x} = \lim_{x \to 0^-} \frac{4x - x}{x} = \lim_{x \to 0^-} \frac{3x}{x} = \lim_{x \to 0^-} 3 = 3.$$

18. The table below shows the average age of marriage for women $A(t)$ in the last half of the 20th century.

t	1950	1960	1970	1980	1990
A(t)	23.0	25.0	24.2	26.5	28.6

What was the average rate of change of average marriage age over the time interval $[1960, 1980]$?

A) $-\frac{6}{50}$ years/year B) $-\frac{22}{50}$ years/year C) -1.2 years/year D) $\frac{23}{100}$ years/year

E) $\frac{11}{100}$ years/year **F) $\frac{3}{40}$ years/year** G) $\frac{1}{50}$ years/year H) $\frac{14}{25}$ years/year

I) $\frac{3}{25}$ years/year J) 1.5 years/year

Solution:

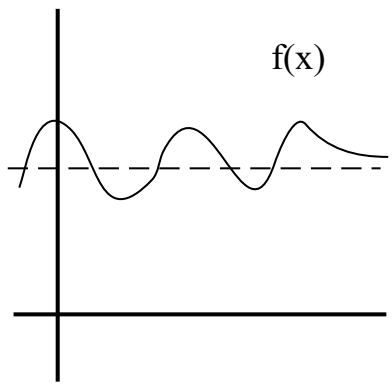
$$\underset{\text{over } [1960,1980]}{\text{average rate of change}} = \frac{26.5 - 25.0}{1980 - 1960} = \frac{1.5}{20} = \frac{3/2}{20} = \frac{3}{40} \text{ years/year.}$$

Part II: True/False (2 points each)

19. The graph of a function $f(x)$ can intersect a horizontal asymptote.

A) True B) False

Solution: For example:



In fact, a function can intersect a horizontal asymptote infinitely many times.

20. If $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, then $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 1$.

A) True **B) False**

Solution: Let $f(x) = 2x$ and $g(x) = x$. Then $\lim_{x\to 0} \dfrac{2x}{x} = \lim_{x\to 0} 2 = 2$ even though $\lim_{x\to 0} 2x = 0 = \lim_{x\to 0} x$.

21. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

A) True **B) False**

Solution: Let $y = f(x)$. Then $\dfrac{d^2y}{dx^2}$ is the second derivative of f (i.e. $f''(x)$). The notation $\dfrac{dy}{dx}$ represents the first derivative of f (i.e. $f'(x)$), so $\left(\dfrac{dy}{dx}\right)^2 = \left(f'(x)\right)^2$ represents the square of the first derivative. The two, in general, are not equal. For example, if $y = f(x) = x^2$, then

$$\frac{dy}{dx} = 2x \quad \text{and} \quad \left(\frac{dy}{dx}\right)^2 = 4x^2.$$

However, $\dfrac{d^2y}{dx^2} = 2$, and $4x^2 \neq 2$.

22. If $f(x)$ is differentiable at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

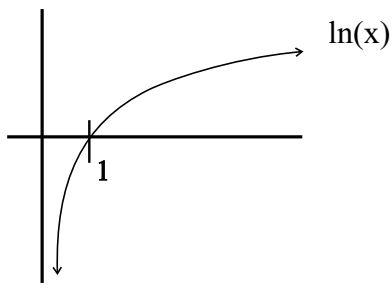
A) True B) False

Solution: First note that $\lim_{x \to a} f(x) = f(a)$ says that f is continuous at $x = a$. Since differentiability implies continuity, this statement is true.

23. $\lim_{x \to 0^+} \ln x = 1$

A) True **B) False**

Solution: From the graph, we see that $\lim_{x \to 0^+} \ln x = -\infty$.



Part III: These are the "free response" problems worth a total of 30 points. Write your answers on the test pages. <u>Show your work</u> neatly and cross out irrelevant scratch-work, false starts, etc.

Please put your NAME on each of the following pages, since they may be separated during grading. Also, please add your Discussion Section Letter (available on your exam front cover) on each page so that we can return these pages in your discussion section.

Name: _____ **Discussion Section:** _____

24. The $\lim_{x\to 1} \dfrac{\sqrt{x+2}-\sqrt{3}}{x-1}$ represents the derivative of some function $f(x)$ at some point a.

a) (2 points) What are f and a?

Solution:

$$f'(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

so $f(x) = \sqrt{x+2}$ and $a = 1$.

b) (4 points) Find the exact value of $f'(a)$ by evaluating the limit.

Solution:

$$\begin{aligned} f'(1) = \lim_{x\to 1} \frac{\sqrt{x+2}-\sqrt{3}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{x+2}-\sqrt{3}}{x-1}\left(\frac{\sqrt{x+2}+\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right) \\ &= \lim_{x\to 1} \frac{(x+2)-3}{(x-1)(\sqrt{x+2}+\sqrt{3})} \\ &= \lim_{x\to 1} \frac{(x-1)}{(x-1)(\sqrt{x+2}+\sqrt{3})} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{x+2}+\sqrt{3}} \\ &= \frac{1}{2\sqrt{3}}. \end{aligned}$$

c) (2 points) Find the equation of the tangent line to $f(x)$ at the point $x = a$ (using the value for a you found in part (a)).

Solution: The slope of the tangent line at $x = 1$ is $f'(1) = \frac{1}{2\sqrt{3}}$, and the point on the tangent line is $(1, f(1)) = (1, \sqrt{3})$. Therefore, the equation of the tangent line is

$$y - \sqrt{3} = \frac{1}{2\sqrt{3}(x-1)} \quad \text{which is} \quad y = \frac{1}{2\sqrt{3}}x - \frac{1}{2\sqrt{3}} + \sqrt{3}.$$

Name: _____ **Discussion Section:** _____

25. (12 points) Solve the following problems.

a) Evaluate $\lim_{x\to 2} \dfrac{2x^3 - 5x^2 + 2x}{x^2 + 2x - 8}$.

Solution:

$$\begin{aligned}\lim_{x\to 2} \frac{2x^3 - 5x^2 + 2x}{x^2 + 2x - 8} = \lim_{x\to 2} \frac{x(2x^2 - 5x + 2)}{x^2 + 2x - 8} &= \lim_{x\to 2} \frac{x(2x-1)(x-2)}{(x+4)(x-2)} \\ &= \lim_{x\to 2} \frac{x(2x-1)}{(x+4)} \\ &= \frac{2(4-1)}{6} \\ &= 1.\end{aligned}$$

b) Suppose

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ c(x-1) & \text{if } x < 1. \end{cases}$$

For what value of c will f have a derivative at $x = 1$ (i.e. will $f'(1)$ exist)?

Solution: Since

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \qquad (*)$$

and f is a piecewise defined function, we need to make sure that the above limit exists. First,

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{(x^2 - 1) - 0}{x - 1} = \lim_{x\to 1^+} \frac{(x-1)(x+1)}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2.$$

Next,

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{c(x-1) - 0}{x - 1} = \lim_{x\to 1^-} c = c.$$

In order for the limit in (*) to exist, the left and right hand limits must agree. Therefore,

$$c = 2$$

will guarantee that f will have a derivative at $x = 1$.

c) Find the horizontal asymptotes of the function $f(x) = \sin\left(\dfrac{4x^4 - 2x + 1}{x^6 + 2x^3 - 2}\right)$.

Solution: To find the horizontal asymptotes, we need to evaluate

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x).$$

Then

$$\begin{aligned}
\lim_{x\to\infty} \sin\left(\frac{4x^4 - 2x + 1}{x^6 + 2x^3 - 2}\right) &= \sin\left(\lim_{x\to\infty} \frac{4x^4 - 2x + 1}{x^6 + 2x^3 - 2}\right) \\
&= \sin\left[\lim_{x\to\infty} \frac{4x^4 - 2x + 1}{x^6 + 2x^3 - 2}\left(\frac{1/x^6}{1/x^6}\right)\right] \\
&= \sin\left(\lim_{x\to\infty} \frac{\frac{4}{x^2} - \frac{2}{x^5} + \frac{1}{x^6}}{1 + \frac{2}{x^3} - \frac{2}{x^6}}\right) \\
&= \sin(0) \\
&= 0
\end{aligned}$$

In the above calculation, we used the fact that $\lim\limits_{x\to\infty} \dfrac{1}{x^n} = 0$.

This calculation does not change when we evaluate $\lim\limits_{x\to-\infty} \sin\left(\dfrac{4x^4 - 2x + 1}{x^6 + 2x^3 - 2}\right)$, so we conclude that the horizontal asymptote for $f(x)$ is $y = 0$.

Name: _____ **Discussion Section:** _____

26. (10 points) The following page gives graphs of five functions (labeled $f_1(x) - f_5(x)$) and graphs of their derivatives (labeled (i) - (v)). Please match the correct function with its derivative, and please explain your choice.

a) $f_1(x)$ (iii) because the slope of the tangent line is always 0.

b) $f_2(x)$ (v) because the slope of the tangent line is constant and negative.

c) $f_3(x)$ (i) because the slope of the tangent line is constant and positive.

d) $f_4(x)$ (ii) because the slope of the tangent line at 0 is negative. (Also, the slopes of the tangent lines start positive, become negative, and then become positive again.)

e) $f_5(x)$ (iv) because the slope of the tangent line at 0 is positive. (Also, the slopes of the tangent lines start negative, become positive, and end negative.)

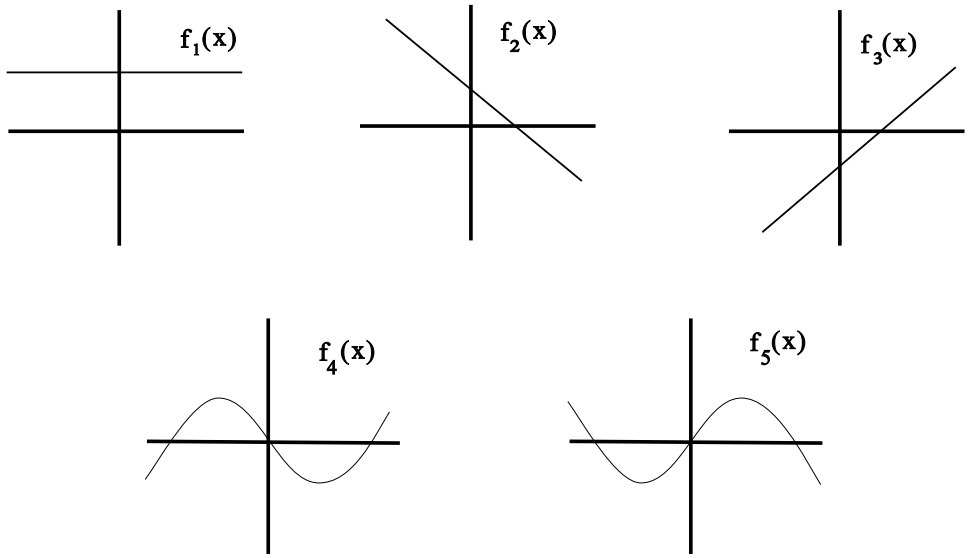


Figure 1: $f_1(x) - f_5(x)$ are functions.

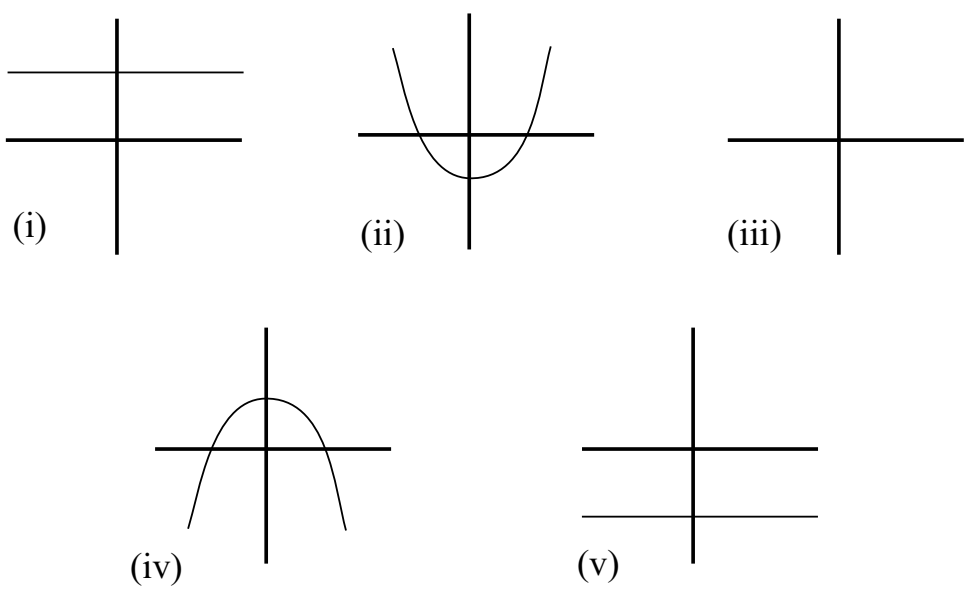


Figure 2: Graphs (i) - (v) are graphs of the derivatives of the functions $f_1(x) - f_5(x)$.