Math 131 , Spring 2004
Exam 3 Solutions

Name: _____ ID# _____

No graphing calculators or calculators with a CAS are allowed. You may use a scientific calculator. Be sure your calculator is set for "radians", not "degrees", if you do any calculus computations with trig functions. For Parts I and II, please mark your answer on the answer card. For Part III, please solve the problems in the space provided.

Part I, Multiple Choice, 3 points/problem:

1. Find the local maxima (if any) of the function $f(x) = \dfrac{2x^2}{x^3+1}$.

A) $x = 0$

B) $x = -1,\ 0,\ \sqrt[3]{2}$

C) $x = -1,\ \sqrt[3]{-\dfrac{2}{5}}$

D) $x = -1$

E) $x = 0,\ \sqrt[3]{2}$

F) $x = \sqrt[3]{-\dfrac{2}{5}},\ 0$

G) $x = \sqrt[3]{-\dfrac{2}{5}}$

H) $x = -1,\ \sqrt[3]{-\dfrac{2}{5}},\ 0$

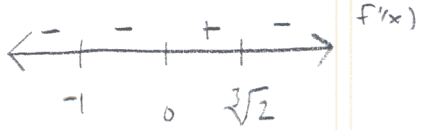
I) $x = \sqrt[3]{2}$

J) no local maxima

Solution: We first compute the derivative of $f(x)$:

$$f'(x) = \frac{(x^3+1)(4x) - 2x^2(3x^2)}{(x^3+1)^2} = \frac{4x^4 + 4x - 6x^4}{(x^3+1)^2} = \frac{-2x(x^3-2)}{(x^3+1)^2}.$$

The critical values are where $f'(x) = 0$ or does not exist, so $f'(x) = 0$ when $x = 0, \sqrt[3]{2}$ and $f'(x)$ does not exist when $x = -1$. From the sign chart, we see that $x = \sqrt[3]{2}$ is a local max since the derivative changes sign from positive to negative.



2. Find the local minima (if any) for the function $f(x) = \dfrac{2x^2}{x^3+1}$, which was given in problem 1.

A) $x = 0$

B) $x = -1,\ 0,\ \sqrt[3]{2}$

C) $x = -1,\ \sqrt[3]{-\dfrac{2}{5}}$

D) $x = -1$

E) $x = 0,\ \sqrt[3]{2}$

F) $x = \sqrt[3]{-\dfrac{2}{5}},\ 0$

G) $x = \sqrt[3]{-\dfrac{2}{5}}$

H) $x = -1,\ \sqrt[3]{-\dfrac{2}{5}},\ 0$

I) $x = \sqrt[3]{2}$

J) no local minima

Solution: From the sign chart on the previous page, we see that $x = 0$ is a local min since the derivative changes sign from negative to positive.

3. Let a be a positive constant, and consider the function $f(x) = x^3 - 3a^2x$ defined on the closed interval $[0, 2a]$. For what value of a will the absolute minimum value of $f(x)$ be -16?

A) $a = 0$ B) $a = \frac{1}{3}$ C) $a = \frac{2}{3}$ D) $a = 1$ **E) $a = 2$**

F) $a = 3$ G) $a = 4$ H) $a = 6$ I) $a = 8$ J) $a = 16$

Solution: We will use the Closed Interval Method. First,

$$f'(x) = 3x^2 - 3a^2 = 3(x - a)(x + a),$$

so

$$f'(x) = 0 \text{ when } x = a, -a.$$

We exclude $x = -a$ since we are on $[0, 2a]$. The three candidates for absolute minimums are the critical value $x = a$ and the endpoints of the interval $x = 0, 2a$. Then

$$f(0) = 0, \quad f(2a) = 8a^3 - 6a^3 = 2a^3, \quad \text{and} \quad f(a) = a^3 - 3a^3 = -2a^3.$$

The smallest value occurs when $x = a$ since $-2a^3 < 0$ (because $a > 0$). Then we choose a so that $-2a^3 = -16$, which means that $a^3 = 8 \Longrightarrow a = 2$.

4. A baseball field may be thought of as $\frac{1}{4}$ of a circle of radius 350 feet with a possible error in radius of ± 1 foot. Estimate the maximum error in calculating the area of the baseball field.

A) $\frac{349\pi}{2}$ ft^2 B) 87.5π ft^2 C) 250π ft^2 D) $\frac{350^2\pi}{2}$ ft^2 E) 500π ft^2

F) 350π ft^2 G) 625π ft^2 **H) 175π ft^2** I) 700π ft^2 J) $\frac{351\pi}{2}$ ft^2

Solution: We use differentials to estimate the error. The area of $\frac{1}{4}$ of a circle is given by $A(r) = \frac{1}{4}\pi r^2$. Then

$$dA = A'(r)dr = \left(\frac{1}{2}\pi r\right) dr = \frac{1}{2}\pi(350)(1) = 175\pi \text{ feet}^2.$$

5. Evaluate $\lim_{x\to 1} \dfrac{x-1}{\ln x}$.

A) 0 B) $\frac{1}{2}$ **C) 1** D) 2 E) e

F) $-\frac{1}{2}$ G) -1 H) -2 I) $-e$ J) does not exist

Solution:

$$\lim_{x\to 1} \frac{x-1}{\ln x} \overset{\text{"}\frac{0}{0}\text{"}}{\underset{\text{L'H}}{=}} \lim_{x\to 1} \frac{1}{\frac{1}{x}}$$
$$= \lim_{x\to 1} x$$
$$= 1.$$

6. Find the equation of the tangent line to the curve $h(x) = \ln(x^4)$ at the point $(1, 0)$.

A) $y = x$ B) $y = -x + 1$ C) $y = -4x + 4$ D) $y = x - 1$ E) $y = 2(x + 1)$

F) $y = 4x - 3$ G) $y = \frac{1}{4}x - 1$ H) $y = 3(x - 1)$ I) $y = 0$ **J) $y = 4(x - 1)$**

Solution:

$$h'(x) = \frac{1}{x^4}\left(4x^3\right) = \frac{4}{x} \Longrightarrow h'(1) = 4.$$

Then the equation of the tangent line is

$$y = 4(x - 1).$$

Part II: True/False (2 points each)

7. $\frac{d}{dx}(\ln 10) = \frac{1}{10}$

A) True B) False

Solution: $\ln 10$ is a number, so $\frac{d}{dx}(\ln 10) = 0$.

8. Suppose $f(t)$ is differentiable, $f(0) = 0$, and that $f'(t) \leq 20$ for every t. It is possible for $f(2) = 70$.

A) True B) False

Solution: By the Mean Value Theorem,

$$f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{70 - 0}{2} = 35,$$

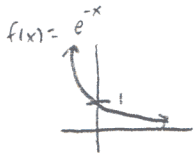
for some c in $(0, 2)$. The hypothesis says that $f'(t) \leq 20$ for all t, which is a contradiction. Thinking about this in terms of physics, with $f(t)$ equal to your position at time t moving in a straight line, this statement says that it is possible to travel 70 miles in 2 hours with a velocity that is always ≤ 20 miles per hour.

9. If $f''(2) = 0$, then $x = 2$ is an inflection point for the function $f(x)$.

A) True B) False

Solution: Inflection points occur when there is a change in concavity. Points where $f''(x) = 0$ are just candidates for inflection points. For example, consider $f(x) = 1$. Then $f''(x) = 0$, for all x values, but $f(x)$ does not change concavity. (In fact, since it is a constant function, it has no concavity.)

10. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.



A) True B) False

Solution: $f'(x) < 0$ says that f is decreasing and $f''(x) > 0$ says that f is concave up, so we are looking for a decreasing, concave up function that is always positive. The function $f(x) = e^{-x}$ fits this description.

11. If $f'(c) = 0$, then $f(x)$ must have either a local maximum or a local minimum at c.

A) True B) False

Solution: Consider $f(x) = x^3$. Then $f'(x) = 3x^2$, so $f'(0) = 0$. However, $x = 0$ is neither a local maximum or a local minimum. (The sign of the first derivative does not change.)

Part III: These are the "free response" problems worth a total of 72 points. Write your answers on the test pages. Show your work neatly and cross out irrelevant scratch-work, false starts, etc.

Please put your NAME on each of the following pages, since they may be separated during grading. Also, please add your Discussion Section Letter (available on your exam front cover) on each page so that we can return these pages in your discussion section.

Name: _____ **Discussion Section:** _____

12. (32 points) Your ultimate goal is to sketch the graph of $f(x) = \dfrac{x^2}{x^2-4}$. To save time,

$$f'(x) = \frac{-8x}{(x^2-4)^2} \qquad \text{and} \qquad f''(x) = \frac{8(3x^2+4)}{(x-2)^3(x+2)^3}.$$

a) Find all vertical asymptotes (if any) of $f(x)$.

Solution: Since

$$\lim_{x\to 2} \frac{x^2}{x^2-4} = \lim_{x\to 2} \frac{x^2}{(x+2)(x-2)} = \infty,$$

so $x = 2$ is a vertical asymptote.

Similarly,

$$\lim_{x\to -2} \frac{x^2}{x^2-4} = \lim_{x\to -2} \frac{x^2}{(x+2)(x-2)} = \infty,$$

so $x = -2$ is a vertical asymptote.

b) Find all horizontal asymptotes (if any) of $f(x)$.

Solution: Since

$$\lim_{x\to \infty} \frac{x^2}{x^2-4} = \lim_{x\to \infty} \frac{1}{(1-\frac{4}{x^2})} = 1,$$

so $y = 1$ is a horizontal asymptote. Similarly,

$$\lim_{x\to -\infty} \frac{x^2}{x^2-4} = \lim_{x\to -\infty} \frac{1}{(1-\frac{4}{x^2})} = 1,$$

so $y = 1$ is the only asymptote.

c) Define critical numbers, and find the critical numbers for $f(x)$?

Solution: Critical numbers are values where $f'(x) = 0$ or $f'(x)$ does not exist. For this particular function,

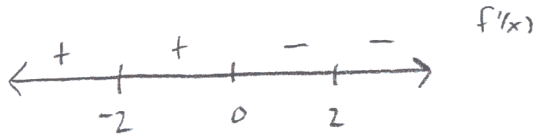
$$f'(x) = \frac{-8x}{(x^2-4)^2} = 0 \qquad \text{when} \qquad -8x = 0 \iff x = 0,$$

and $f'(x)$ does not exist when

$$(x^2-4)^2 = 0 \iff x = \pm 2.$$

d) Where is $f(x)$ increasing and decreasing? Please explain your reasoning.

Solution:



$f(x)$ is increasing when $f'(x) > 0$, which occurs on $(-\infty, -2)$ and $(-2, 0)$
$f(x)$ is decreasing when $f'(x) < 0$, which occurs on $(0, 2)$ and $(2, \infty)$

e) Find all local extrema (if any) of $f(x)$. Please explain your reasoning.

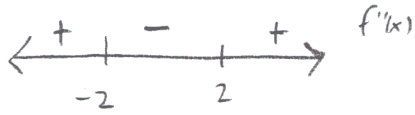
Solution:

$x = 0$ is a local maximum because $f'(x)$ changes from positive to negative. Notice that $f(0) = 0$. f has no local minimum because $f'(x)$ does not change from negative to positive at any value of x.

f) Where is $f(x)$ concave up and concave down? Please explain your reasoning.

Solution:

$f''(x) = \dfrac{8(3x^2+4)}{(x-2)^3(x+2)^3}$ is never zero since $3x^2 + 4 > 0$ for all x, but $f''(x)$ does not exist when its denominator $(x-2)^3(x+2)^3 = 0$, which occurs when $x = -2, 2$. Then



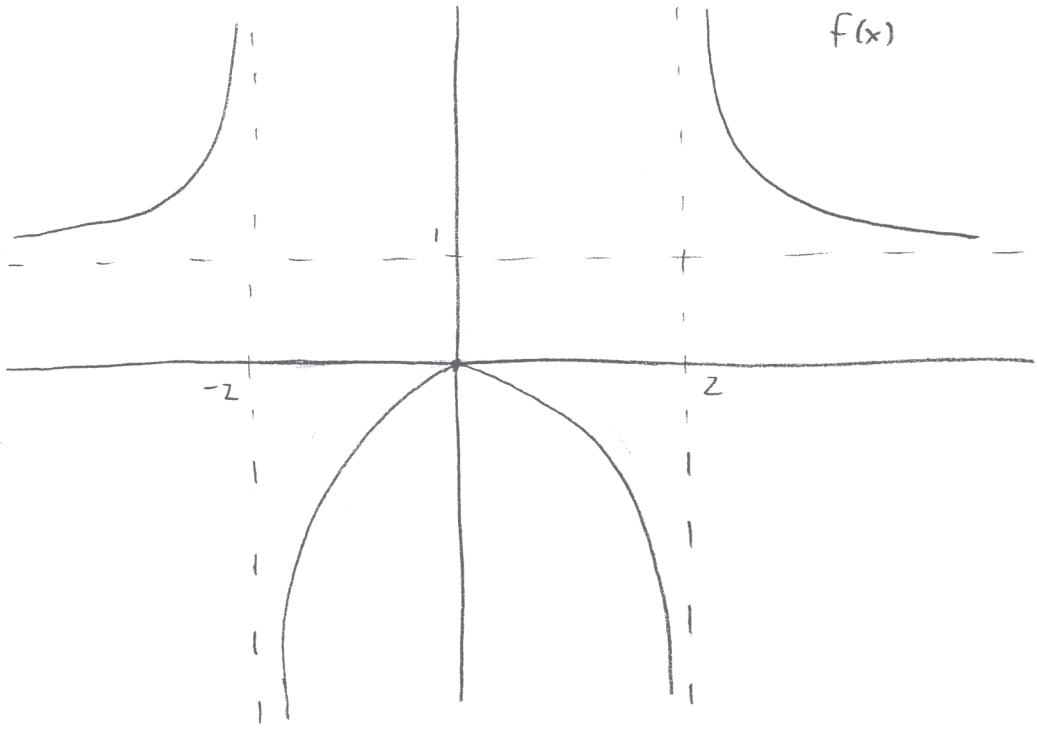
f is concave up when $f''(x) > 0$, which occurs on $(-\infty, -2)$ and $(2, \infty)$ and f is concave down when $f''(x) < 0$, which occurs on $(-2, 2)$.

g) Find all inflection points (if any), and explain your reasoning.

Solution:

Inflection points are points in the domain of the function where the concavity changes. For the given function, there are no such points because $x = \pm 2$ are not in the domain of $f(x)$.

h) Sketch the graph of $f(x)$.



Name: _____ **Discussion Section:** _____

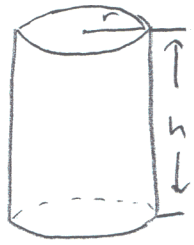
13. (10 points) Please work through the following problems. Remember, I am particularly interested in your thought process (how you tackle problems, how you think them through, and the methods you use to solve problems), so please give me a sense of your thinking even if you are unsure of your solution.

a) The radius of a cylinder is increasing at a rate of 2 cm/s, while the volume of the cylinder is increasing at a rate of 4π cm^3/s. At what rate is the height changing when the radius is 10 cm and the volume is 20π cm^3? Note that the volume of a cylinder is given by $V = \pi r^2 h$.

Solution: Let r be the radius of the cylinder, h its height, and V its volume. We are given that $\frac{dr}{dt} = 2$ cm/s and $\frac{dV}{dt} = 4\pi$ cm^3/s. We are asked to find $\frac{dh}{dt}$ when $V = 20\pi$ cm^3 and $r = 10$ cm.

Since $V = \pi r^2 h$, differentiating (using the product rule) yields

$$\frac{dV}{dt} = \pi \left[r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt} \right].$$



Solving for $\frac{dh}{dt}$, we see that

$$\frac{dh}{dt} = \frac{\frac{1}{\pi}\frac{dV}{dt} - 2rh\frac{dr}{dt}}{r^2}.$$

Before we can plug in the specific information, we need to find the height h at the instant $V = 20\pi$ cm^3 and $r = 10$ cm. From the formula for volume,

$$h = \frac{V}{\pi r^2},$$

so

$$h = \frac{20\pi}{\pi(10^2)} = \frac{1}{5}.$$

Plugging in the specific information, we see that

$$\frac{dh}{dt} = \frac{\frac{1}{\pi}(4\pi) - 2(10)(\frac{1}{5})(2)}{10^2} = -\frac{1}{25}.$$

b) Suppose the sum of two numbers x and y is π. What is the largest possible value of their product xy?

Solution: Let x and y be the two numbers. Then we want to find x and y such that

$$x + y = \pi \quad \text{and} \quad xy \quad \text{is a maximum.}$$

We will first turn xy into a function of one variable. Since $x + y = \pi$, then $y = \pi - x$ so

$$xy = x(\pi - x) = \pi x - x^2 = f(x),$$

where $0 \leq x \leq \pi$. We now maximize $f(x)$.

$$f'(x) = \pi - 2x = 0 \quad \text{when} \quad x = \frac{\pi}{2}.$$

We will verify that $x = \frac{\pi}{2}$ is a maximum by using the Second Derivative Test:

$$f''(x) = -2 < 0 \implies f \text{ is concave down } \implies x = \frac{\pi}{2} \text{ is a maximum.}$$

Then

$$y = \pi - x = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

and the maximum product is

$$xy = \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}.$$

14. (20 points) The following problem contains two limits and two derivatives for you to evaluate. Please justify your work.

a) Evaluate $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

Solution: This limit is an indeterminate product of the form "$\infty \cdot 0$". Notice that $\frac{1}{x} \to 0$ as $x \to \infty$, so $\sin\left(\frac{1}{x}\right) \to 0$ as $x \to \infty$. Then

$$\begin{aligned}
\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \\
&= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \\
&= \cos(0) \\
&= 1.
\end{aligned}$$

b) Evaluate $\lim_{x\to 1} x^{1/(x-1)}$.

Solution: This limit is an indeterminate form of the type "1^0". We work with it as follows: Let

$$L = \lim_{x\to 1} x^{1/(x-1)}.$$

Then

$$\begin{aligned}
\ln L &= \ln\left(\lim_{x\to 1} x^{1/(x-1)}\right) \\
&= \lim_{x\to 1} \ln\left(x^{1/(x-1)}\right) \\
&= \lim_{x\to 1} \frac{1}{x-1}\ln x \\
&= \lim_{x\to 1} \frac{\ln x}{x-1} \\
&\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} \qquad \text{L'H} \\
&= 1.
\end{aligned}$$

Thus $\ln L = 1$, so exponentiating gives us that

$$L = \lim_{x\to 1} x^{1/(x-1)} = e.$$

c) Differentiate $y = x^{e^x}$.

Solution: We need to use logarithmic differentiation. The first step is to take the natural log of both sides and to simplify (if possible) using log properties.

$$\ln y = \ln x^{e^x} = e^x \ln x.$$

Next we implicitly differentiate (and use the product rule):

$$\frac{1}{y}\frac{dy}{dx} = e^x\left(\frac{1}{x}\right) + \ln x(e^x) = e^x\left(\frac{1}{x} + \ln x\right).$$

Lastly, we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left[e^x\left(\frac{1}{x} + \ln x\right)\right] = x^{e^x}\left(e^x\left(\frac{1}{x} + \ln x\right)\right) = e^x x^{e^x}\left(\frac{1}{x} + \ln x\right).$$

d) Differentiate $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$. You should not have to multiply out the terms of this function to take its derivative.

Solution: We have three options for solving this one: 1) We could expand everything first and then take the derivative, but . . . that would be long and messy, 2) We could use the product rule, which is not too bad, but is still a lot of work, or 3) We could use logarithmic differentiation and simplify the expression using log properties, which . . . is the best solution (cleanest and least amount of work). We will choose option (3). Then,

$$\begin{aligned}\ln(f(x)) &= \ln[(x-1)(x-2)(x-3)(x-4)(x-5)] \\ &= \ln(x-1) + \ln(x-2) + \ln(x-3) + \ln(x-4) + \ln(x-5).\end{aligned}$$

Then we differentiate implicitly and solve for $f'(x)$:

$$\frac{1}{f(x)}f'(x) = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5},$$

so

$$\begin{aligned}f'(x) &= f(x)\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5}\right] \\ &= (x-1)(x-2)(x-3)(x-4)(x-5)\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x-4} + \frac{1}{x-5}\right].\end{aligned}$$

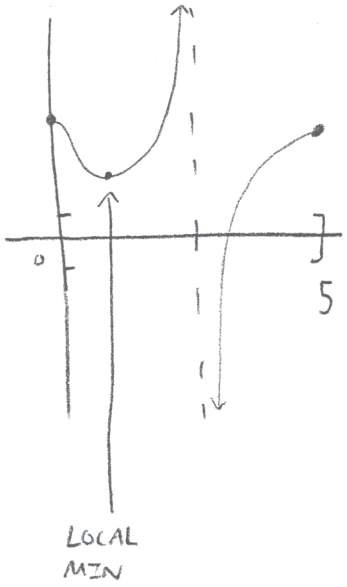
15. (10 points) Please complete the following problems.

a) Sketch the graph of a function $f(x)$ on $[0, 5]$ that has a local minimum but no absolute minimum. What must be true about $f(x)$ in order for it to have the requisite properties? How do you know?

Solution: Since $f(x)$ is defined on a closed interval, the only way f can not have an absolute minimum is for $f(x)$ to NOT be continuous. (Note that the Extreme Value Theorem says that a continuous function on a closed interval attains an absolute maximum and an absolute minimum. Our function is defined on a closed interval but does not have an absolute minimum. If we make $f(x)$ continuous, then we will contradict the Extreme Value Theorem, so $f(x)$ is not continuous.)

We have two options:

I.



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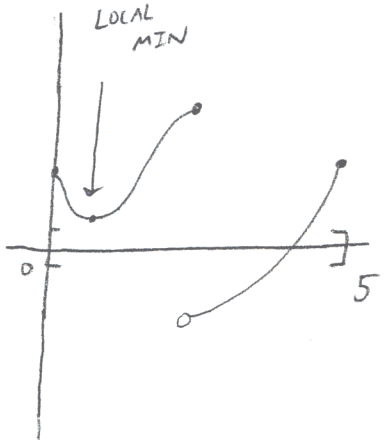
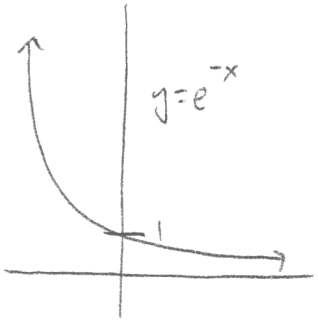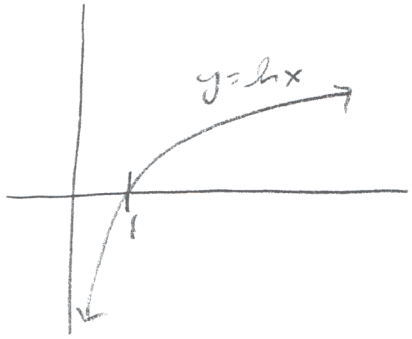
b) Sketch the graphs of $f(x) = e^{-x}$ and $g(x) = \ln x$, and label any important points. (You may simply sketch the graphs from memory if you know them. You do not have to take derivatives or look for increasing/decreasing, concavity, or any of the other properties you found in question 12.)