Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points. Mark the correct answer on the answer card. On Part I, only the answer on the card will be graded.

1. Find all horizontal and vertical asymptotes for the function \( f(x) = \frac{25x^2 + 24x - 1}{x^2 - 1} \)

   (A) horizontal \( y = 25 \); vertical \( x = -1, x = 1 \)

   (B) horizontal \( y = 25 \); vertical \( x = 1 \)

   (C) horizontal \( y = 25 \); vertical \( x = -1 \)

   (D) horizontal \( y = 0 \); vertical \( x = 1 \)

   (E) horizontal \( y = 1 \); vertical \( x = 1, x = -1 \)

   (F) horizontal \( y = 1 \); vertical \( x = 1 \)

   (G) no horizontal asymptotes; vertical \( x = -1 \)

   (H) no horizontal asymptotes; vertical \( x = 1 \)

   (I) no horizontal asymptotes; vertical \( x = -1, x = 1 \)

   (J) horizontal \( y = 1 \); vertical \( x = -1, x = 1 \)

\[
\lim_{x \to \infty} \frac{25x^2 + 24x - 1}{x^2 - 1} = \lim_{x \to \infty} \frac{\frac{1}{x^2} \left(25x^2 + 24x - 1\right)}{1 - \frac{1}{x^2}} = \lim_{x \to \infty} \frac{25 + 24x - \frac{1}{x^2}}{1 - \frac{1}{x^2}}
\]

\[
\lim_{x \to -\infty} \frac{25x^2 + 24x - 1}{x^2 - 1} = 25 \quad \text{(similarly)}
\]

So horizontal asymptote: \( y = 25 \)

\[
\lim_{x \to -1} \frac{25x^2 + 24x - 1}{x^2 - 1} = \lim_{x \to -1} \frac{(25x - 1)(x + 1)}{(x-1)(x+1)} = \lim_{x \to -1} \frac{25x-1}{x-1} = \frac{-20}{-2} = 10
\]

\[
\lim_{x \to 1} \frac{25x^2 + 24x - 1}{x^2 - 1} = \pm \infty \quad \text{so vertical asymptote:} \quad x = 1
\]
2. Find \( \lim_{x \to \infty} \frac{\sqrt{cx^2 + 4}}{x - b} \). (Assume \( c > 0 \) and \( b > 0 \)).

(A) \(-\frac{\sqrt{c}}{b}\)  (B) \(\frac{\sqrt{c}}{b}\)  (C) \(\sqrt{c}\)  (D) \(-\sqrt{c}\)  (E) 0

(F) \(-\frac{2}{b}\)  (G) \(\frac{2}{b}\)  (H) \(-\frac{c}{b}\)  (I) \(\infty\)  (J) \(-\infty\)

\(\text{indeterminate} : \frac{\infty}{\infty}\)

\[
\lim_{x \to -\infty} \frac{\sqrt{cx^2 + y}}{x - b} = \lim_{x \to -\infty} \frac{-\frac{1}{x^2} \sqrt{cx^2 + y}}{1 - \frac{b}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{c}}{1} = -\sqrt{c}
\]
3. Find \( \lim_{{h \to 0}} \frac{e^{c+h} - e^c}{h} \).

\[
\begin{array}{cccccc}
(A) & \frac{c}{h} & (B) & \frac{e^c}{h} & (C) & c \\
(D) & e^h & (E) & 0 \\
(F) & 1 & (G) & \frac{1}{h} & (H) & e^c \\
(I) & \infty & (J) & -\infty
\end{array}
\]

\[
\lim_{{h \to 0}} \frac{e^{c+h} - e^c}{h}
\]

is the definition of the derivative of \( f(x) = e^x \) at \( x = c \),

so \( \lim_{{h \to 0}} \frac{e^{c+h} - e^c}{h} = e^c \)

(Or \( \lim_{{h \to 0}} \frac{e^{c+h} - e^c}{h} = \lim_{{h \to 0}} \frac{e^c e^h - e^c}{h} \))

\[
= \lim_{{h \to 0}} \frac{e^c(e^h - 1)}{h} = e^c \cdot \lim_{{h \to 0}} \frac{e^h - 1}{h}
\]

\[
= e^c \cdot 1 = e^c
\]
4. The linear approximation (tangent line approximation) of $\sqrt[3]{x}$ at $x = 8$ can be used to approximate the value of $\sqrt[3]{9}$.

What is the approximation and is it too large or too small?

(A) $\frac{49}{24}$, too small (B) $\frac{49}{24}$, too large

(C) $\frac{21}{10}$, too small (D) $\frac{21}{10}$, too large

(E) $\frac{25}{12}$, too small (F) $\frac{25}{12}$, too large

(G) $\frac{51}{24}$, too small (H) $\frac{51}{24}$, too large

(I) $\frac{23}{12}$, too small (J) $\frac{23}{12}$, too large

\[ f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \]

\[ f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3}x^{\frac{1}{3}} = \frac{1}{3(\sqrt[3]{x})^2} \]

\[ f'(8) = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{3(2)^2} = \frac{1}{12} \]

So equation of tangent line at $x = 8$

\[ y - \sqrt[3]{8} = \frac{1}{12}(x - 8) \]

or \[ y = 2 + \frac{1}{12}(x - 8) = L(x) \] (linear approximation)

\[ \sqrt[3]{9} \approx L(9) = 2 + \frac{1}{12}(9 - 8) = 2 + \frac{1}{12}(1) = \frac{25}{12} \]

The graph of $\sqrt[3]{x}$ is above the tangent line so the approximation is too large.
5. Given the graph of the function $y = f(x)$.

Which of the following statements are true?

a: $f$ is differentiable on the interval $-5 < x < 7$  
no

b: $f$ is continuous on the interval $-5 < x < 7$  
yes

c: $f''(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$ exists  
no (not differentiable at $x=1$)

d: $f'(7) = \lim_{x\to 7} \frac{f(x) - f(7)}{x - 7}$ exists  
no (not differentiable at $x=7$)

(A) a, b, c, d

(B) only a and b

(C) only a, c, and d

(D) only b and c

(E) only b

(F) only c

(G) only d

(H) only c and d

(I) only a and c

(J) only a
6. The graph of the derivative of $f$ is given below.

Which of the following are true?

- a: $f$ has a local maximum at $x = 40$  
  - yes (increasing, then decreasing)
- b: $f$ has a local maximum at $x = 30$  
  - no (increase / increase)
- c: $f$ has an inflection point at $x = 10$  
  - yes ($f'' < 0$, then $f'' > 0$)
- d: $f$ has an inflection point at $x = 40$  
  - no ($f'' < 0$, $f'' < 0$)

(A) a, b, c, d  
(B) only a and b  
(C) only a, c, and d  
(D) only b and c  
(E) only a  
(F) only b  
(G) only c  
(H) only d  
(I) only c and d  
(J) only a and c
7. The following is the graph of the function \( y = f(x) \). There is exactly one marked point where \( f'(x) \) and \( f''(x) \) are both zero. And there are 2 marked points where \( f' \) and \( f'' \) both nonzero and have the same sign. These points are:

(A) \( f' = 0 \) and \( f'' = 0 \) at C; \( f' \) and \( f'' \) have the same sign at B and F

(B) \( f' = 0 \) and \( f'' = 0 \) at B; \( f' \) and \( f'' \) have the same sign at D and F

(C) \( f' = 0 \) and \( f'' = 0 \) at B; \( f' \) and \( f'' \) have the same sign at D and E

(D) \( f' = 0 \) and \( f'' = 0 \) at D; \( f' \) and \( f'' \) have the same sign at B and F

(E) \( f' = 0 \) and \( f'' = 0 \) at C; \( f' \) and \( f'' \) have the same sign at D and F

(F) \( f' = 0 \) and \( f'' = 0 \) at C; \( f' \) and \( f'' \) have the same sign at B and E

(G) \( f' = 0 \) and \( f'' = 0 \) at E; \( f' \) and \( f'' \) have the same sign at B and F

(H) \( f' = 0 \) and \( f'' = 0 \) at B; \( f' \) and \( f'' \) have the same sign at A and F

(I) \( f' = 0 \) and \( f'' = 0 \) at E; \( f' \) and \( f'' \) have the same sign at A and F

(J) \( f' = 0 \) and \( f'' = 0 \) at E; \( f' \) and \( f'' \) have the same sign at C and D

C: horizontal tangent and point of inflection
D: decreasing and concave down \((f' < 0, f'' < 0)\)
F: increasing and concave up \((f' > 0, f'' > 0)\)
8. Given the graph of the \( f''(x) \) (the second derivative of \( f \)), which marked points correspond to inflection points of the function \( f \)?

(A) only A, C, and E
(B) only B and E
(C) only A, C, D, and G
(D) only C, D, and F
(E) only C and D
(F) only A, C, and D
(G) only A, E, D, and F
(H) only B, E, and G
(I) only D and F
(J) only F

2nd derivative changes sign at A, C, D.

(Inflection point means change in concavity.)
9. Each of the following four graphs shows the position of a particle moving along a straight path as a function of time. During this time interval $0 < t < 5$ answer the following questions in order:

1. Which has constant velocity?
   - (D)
   - (C)

2. Which has the greatest initial velocity?
   - (B)
   - (A)

3. Which has the greatest average velocity?
   - (A)
   - (D)

4. Which has zero average velocity?
   - (B)
   - (C)

5. Which has zero acceleration throughout?
   - (C)
   - (B)

6. Which has positive acceleration throughout?
   - (B)
   - (C)

7. Which has negative acceleration throughout?
   - (A)
   - (D)

(A) B A B A D C A
(B) D C B A D B C
(C) C D A B D B C
(D) C D A B D C B
(E) D C C D B C B
(F) D A B A C B C
(G) A D C A C A C
(H) D B B C D A C
(I) A D B A B D C
(J) A D C A D B C
10. A point moves along a straight line with so that its position at time $t$ is given by $s = f(t) = 2t^3 - 9t^2 - 24t + 1$ For what values of $t$ is the velocity increasing?

(A) $t < -1$ or $t > 4$  (B) $t < 4$

(C) $t > 0$  (D) $t < \frac{3}{2}$

(E) $-1 < t < 4$  (F) $t > 1$

(G) $t < 2$  (H) $t > -3$

(I) $-3 < t < 2$  (J) $t > \frac{3}{2}$

velocity $= 6t^2 - 18t - 24$

the derivative of velocity is $12t - 18$.

velocity $= 12t - 18 > 0$ for $t > \frac{3}{2}$

so velocity is increasing for $t > \frac{3}{2}$
11. The rate of population growth is given by \( r(t) = 500(1.024)^t \) where \( t \) is the number of years since 1980. Given that the population in 1990 was 20,000, use linear approximation to estimate the population in 1992.

(A) 20,523  (B) 20,872  (C) 21,268  (D) 21,495  (E) 21,563
(F) 21,649  (G) 22,627  (H) 23,021  (I) 23,471  (J) 24,217

\[
P'(t) = \text{rate} = r(t) = 500(1.024)^t
\]

1990 corresponds to \( t = 10 \),

\[
P'(10) = 500(1.024)^{10} \approx 633.8253
\]

Linear Approximation:

\[
L(x) = 20,000 + 633.8253 (t - 10)
\]

(when 1992, \( t = 12 \))

\[
L(12) = 20,000 + 633.8253 (12 - 10) = 21,267.65 \approx 21,268
\]
12. Let \( f(x) = x^3 - 9x^2 - 16x + 1 \) and \( g(x) = x^3 - \frac{1}{2}x^2 + x \). Find the \( x \) value where the tangent line to the graph of \( y = f(x) \) is parallel to the tangent line to the graph of \( y = g(x) \).

(A) -4  (B) -3.5  (C) -2  (D) \(-\frac{3}{2}\)  (E) -1

(F) \(-\frac{2}{3}\)  (G) \(-\frac{1}{2}\)  (H) 0  (I) 1  (J) \(\sqrt{2}\)

\[
\begin{align*}
f'(x) &= 3x^2 - 18x - 16 \\
g'(x) &= 3x^2 - x + 1
\end{align*}
\]

\[f'(x) = g'(x) \quad \text{(parallel tangent lines);}\]

\[
3x^2 - 18x - 16 = 3x^2 - x + 1
\]

\[-17x = 17\]

\[x = -1\]
13. Given \( f(x) = xe^x \) find any local maxima and minima and any inflection points

(A) local maximum at \( x = -2 \); local minimum at \( x = -1 \); inflection point at \( x = 0 \)
(B) \underline{local minimum at \( x = -1 \)}; inflection point at \( x = -2 \)
(C) local minimum at \( x = -2 \); inflection point at \( x = 0 \)
(D) local maximum at \( x = 0 \); inflection point at \( x = -2 \)
(E) local minimum at \( x = 0 \); inflection point at \( x = -2 \)
(F) local maximum at \( x = -1 \); inflection point at \( x = -2 \)
(G) no local maxima or minima; inflection point at \( x = -2 \)
(H) local minimum at \( x = -2 \); no inflection points
(I) local minimum at \( x = -1 \); no inflection points
(J) local minimum at \( x = 0 \); inflection points at \( x = -2 \) and \( x = -1 \)

\[
f'(x) = e^x + xe^x = e^x(1 + x)
\]

\[
f'(x) = 0 \quad \text{when} \quad x = -1; \quad f'(x) < 0 \quad \text{for} \quad x < -1; \quad f'(x) > 0 \quad \text{for} \quad x > -1.
\]

\[
f''(x) = e^x(1 + x) + e^x(1) = e^x(1 + x + 1)
\]

\[
f''(x) = 0 \quad \text{when} \quad x = -2
\]

\[
f''(x) > 0 \quad \text{when} \quad x > -2
\]

\[
f''(x) < 0 \quad \text{when} \quad x < -2
\]

So \underline{local minimum at \( x = -1 \)}

\underline{inflection point at \( x = -2 \)}
14. The quantity of skateboards sold is a function of the price of the skateboards. Assume 
\( Q = f(p) \) where \( Q \) is the quantity of skateboards sold at price \( p \) in dollars. Revenue 
from skateboard sales is equal to the quantity sold times the price. Given \( f(140) = 15,000 \) 
\( f'(140) = -100 \) find the rate that the revenue is changing when the price of the skateboards 
is $140.

(A) 14,000  (B) 10,000  (C) 8000  (D) 1000  (E) 500

(F) 140  (G) 0  (H) -100  (I) -1000  (J) -10,000

\[
\text{Revenue} = f(p) \cdot p
\]

\[
\text{rate} = R'(140) = f'(p)p + f(p).
\]

\[
R'(140) = (f'(140))(140) + (f(140))(1)
\]

\[
= (-100)(140) + 15000 - 1
\]

\[
= -14000 + 15000 = 1000
\]
15. A graph can intersect a horizontal asymptote at most once.
(A) True  (B) False  (Lots of counter examples)

16. If \( \lim_{x \to \infty} f(x) = 50 \) and \( f'(x) > 0 \) then \( \lim_{x \to \infty} f'(x) = 0 \).
(A) True  (B) False

17. Given
\[
f(x) = \begin{cases} 
  x^2, & \text{if } x \geq 0 \\
  0, & \text{if } x < 0 
\end{cases}
\]
\( f'(0) \) does not exist.
(A) True  (B) False

\[
\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}
\]

18. The function \( f(x) = \frac{1}{c + e^x} \) (\( c > 0 \)) has a vertical asymptote at \( x = 0 \).
(A) True  (B) False

\[
\lim_{x \to 0} \frac{1}{c + e^{1/x}} = \infty
\]  
So \( x=0 \) is not a vertical asymptote.

19. Given \( f(x) = x^2 e^x \) then \( \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = 3e \)
(A) True  (B) False

\[
f'(1) = 2(1)e^1 + (1)^2 e^1 = 2e + e = 3e
\]
Part II: (25% of test points) In each problem, clearly show your solution in the space provided. Give any explanation that you think is relevant.

20. Sketch a possible graph of \( y = f(x) \) given the following information.

For \( x < -2 \), \( f'(x) > 0 \) and \( f''(x) > 0 \).

For \( -2 < x < 1 \), \( f'(x) > 0 \) and the derivative of \( f \) is decreasing.

For \( x > 1 \), \( f'(x) < 0 \) and \( f''(x) < 0 \).

\( f'(1) = 0 \) and \( f(0) = 1 \).
21. a) Sketch the graphs of the derivatives of the following functions. Be sure your sketches are consistent with the important features of the original functions.

b) Sketch the graphs of an antiderivative of the following function. You can assume the antiderivative contains the point \((0, 0)\).