Math 131, Exam 3 Solutions
Fall 2004

Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points.

1. If \( \lim_{x \to 0} \frac{\cos(3x) + bx - 1}{7x + \sin bx} = 8 \), what is \( b \)?

A) 0 B) 1 C) 4 D) 7 E) 9
F) \( \) \( \) \( \) \( \) \( \) \( \) I) 8 J) 10

\[
\lim_{x \to 0} \frac{\cos(3x) + bx - 1}{7x + \sin bx} = 0, \quad \text{so we can use L'Hôpital's Rule to write}
\]

\[
\lim_{x \to 0} \frac{\cos(3x) + bx - 1}{7x + \sin bx} = \lim_{x \to 0} \frac{-3 \sin(3x) + b}{7 + b \cos bx} = \frac{b}{7} = 8. \quad \text{Solving for } b, \quad \text{we get}
\]

\[ b = 56 + 8b, \quad \text{so } b = -8. \]

2. A spherical snowball melts in such a way its radius is decreasing at a rate of 3 cm/min at the instant when its radius is 20 cm. At that moment, how fast is its volume changing? (Round your answer to 1 decimal place.)

A) 12478.6 cm\(^3\)/min B) 14682.3 cm\(^3\)/min C) \(-15079.6\) cm\(^3\)/min
D) \(-23156.7\) cm\(^3\)/min E) 16783.8 cm\(^3\)/min F) \(-15983.7\) cm\(^3\)/min
G) \(-14682.3\) cm\(^3\)/min H) 14292.9 cm\(^3\)/min I) \(-12478.6\) cm\(^3\)/min
J) \(-24978.7\) cm\(^3\)/min

Volume \( V = \frac{4}{3} \pi r^3 \), so \( \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} \). When \( r = 20 \), \( \frac{dr}{dt} = -3 \) so

\[
\frac{dV}{dt} = 4 \pi (20)^2 (-3) \approx -15079.6 \text{ cm}^3/\text{min}. \]
3. A certain cone is growing in such a way that its height is always twice its radius. Use differentials to estimate how much the volume changes as the radius grows from 10 m to 10.05 m. \(^{Round your answer to 2 \text{ decimal places.}}\)

A) 31.12 m\(^3\)  B) 31.27 m\(^3\)  C) 31.57 m\(^3\)  D) 31.66 m\(^3\)  E) 31.75 m\(^3\)

**F) 31.42 m\(^3\)**

Since the height \(h = 2r\), we have \(V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3\), so \(dV = 2\pi r^2 dr\). For \(r = 10\) and \(dr = 0.05\), we get \(dV = 2\pi (10)^2 (0.05) \approx 31.42\) m\(^3\).

4. Suppose \(f(x) = 2x^2 + x - 9\). The Mean Value Theorem states that there is a number \(c\) between 0 and 3 with a certain property. What is \(c\)?

A) 0  B) 1  C) 2  D) \(\frac{5}{2}\)  E) \(\frac{3}{2}\)

**F) \(\frac{1}{2}\)**

The Mean Value Theorem says that there is a point \(c\) between 0 and 3 for which \(f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{(18 + 3 - 9) - (-9)}{3 - 0} = \frac{21}{3} = 7\). Since \(f'(c) = 4c + 1\), we get \(4c + 1 = 7\) and therefore \(c = \frac{3}{2}\).
5. What is the slope of the tangent line to the curve \( \begin{align*} x &= \sin t + \cos t \\ y &= \sin t - \cos 2t \end{align*} \) at the point corresponding to \( t = \pi \) (See the figure.)

\[
\begin{align*}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{\cos t + 2 \sin t}{\cos t - \sin t}.
\end{align*}
\]

When \( t = \pi \), \( \frac{dy}{dx} = \frac{-1-0}{-1+0} = 1 \).

6. There are two times \( t \) in \([0, 2\pi]\) for which the curve \( \begin{align*} x &= \sin t + \cos t \\ y &= \sin t - \cos t \end{align*} \) has a vertical tangent line. What are those times?

\[
\begin{align*}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{\cos t + \sin t}{\cos t - \sin t}.
\end{align*}
\]

We have a vertical tangent line where \( \cos t - \sin t = 0 \) (provided \( \cos t + \sin t \neq 0 \)).

\[
\begin{align*}
\cos t - \sin t &= 0 \\
\cos t &= \sin t \\
\tan t &= 1 \\
t &= \frac{\pi}{4}, \frac{5\pi}{4}
\end{align*}
\]
7. The point \( P = (1, 1) \) is on the graph of \( 2x \ln y + y \ln x = 0 \). What is the slope of the tangent line to the graph at \( P \)?

A) \(-2\)  B) \(-\frac{3}{2}\)  C) \(-1\)  D) \(-\frac{1}{2}\)  E) 0

F) \(\frac{1}{2}\)  G) 1  H) \(\frac{3}{2}\)  I) \(e\)  J) \(2e\)

Differentiating implicitly gives \( 2 \ln y + 2x \cdot \frac{1}{y} \frac{dy}{dx} + \frac{dy}{dx} \ln x + \frac{y}{x} = 0 \). At \( (1, 1) \) this equation becomes \( 2 \cdot 0 + 2 \cdot \frac{1}{1} \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot 1 + \frac{1}{1} = 0 \), so \( 2 \frac{dy}{dx} + 1 = 0 \), so \( \frac{dy}{dx} = -\frac{1}{2} \).

8. A rectangular box has a base in which the length is always 3 times the width. What is the largest volume for such a box if its surface area (4 sides + top + bottom) must be 1152 in\(^2\)?

A) 1728 in\(^3\)  B) 512 in\(^3\)  C) 695 in\(^3\)  D) 1048 in\(^3\)  E) 1246 in\(^3\)

F) 1480 in\(^3\)  G) 1624 in\(^3\)  H) 1848 in\(^3\)  I) 2142 in\(^3\)  J) 2304 in\(^3\)

Call the dimensions of the base \( x \) and \( 3x \). Let the height of the box be \( y \). Then the volume \( V = 3x^2 y \). The total surface area is 1152 = \( 6x^2 + 8xy \), so \( y = \frac{1152 - 6x^2}{8x} \).

Then \( V = 3x^2 \left( \frac{1152 - 6x^2}{8x} \right) = \frac{3}{8} x (1152 - 6x^2) = \frac{3}{8} (1152x - 6x^3) \), so \( V' = \frac{3}{8} (1152 - 18x^2) = 0 \) gives \( x^2 = \frac{1152}{18} = 64 \), so (since \( x > 0 \)) we have \( x = 8 \).

Since \( V' > 0 \) for \( x < 8 \) and \( V' < 0 \) for \( x > 8 \), \( V \) has an absolute maximum at \( x = 8 \). For \( x = 8 \), we have \( y = \frac{1152 - 6(64)}{64} = 12 \) and \( V = 3(8)^2(12) = 2304 \text{ in}^3 \).

9. If \( f(x) = \ln \left( \frac{\sqrt{2x+5} \cdot (3x+5)^8}{\sqrt{6x+5}} \right) \), what is \( f'(0) \)?

A) \(\frac{43}{10}\)  B) \(\frac{21}{3}\)  C) \(\frac{5}{2}\)  D) \(\frac{41}{15}\)  E) \(\frac{23}{25}\)

F) \(\frac{14}{17}\)  G) \(\frac{25}{36}\)  H) 0  I) \(\frac{4}{3}\)  J) \(\frac{2}{3}\)

Using properties of logarithms gives \( f(x) = \frac{1}{2} \ln(2x+5) + 8 \ln(3x+5) - \frac{1}{2} \ln(6x+5) \), so \( f'(x) = \frac{1}{4} \frac{2}{2x+5} + 8 \frac{3}{3x+5} - \frac{6}{12x+10} = \frac{1}{4} \left( \frac{1}{2x+5} + 24 \frac{3}{3x+5} - \frac{3}{6x+5} \right) \), so \( f'(0) = \frac{1}{10} + 24 \frac{3}{5} - \frac{3}{5} = \frac{1 + 48 - 6}{10} = \frac{43}{10} \).
10. $f$ is a function defined on the interval $[0, 5]$, and $f(0) = f(5) = 1, \ f(3) = -1$.

Suppose $f'(x) = (x - 1)(x - 2)^2(x - 3)^3(x - 4)^4$.

Then exactly three of the following statements are true. Which three are true?

i) $f$ is increasing on the interval $1 < x < 2$

ii) $f$ has a local min at $x = 1$

iii) $f$ has neither a local max nor a local min at $x = 2$

iv) $f$ has a local min at $x = 3$

v) $f$ has its absolute min at $x = 3$

A) i, ii, iii  B) i, ii, iv  C) i, ii, v  D) i, iii, iv  E) i, iii, v  
F) i, iv, v  G) ii, iii, iv  H) ii, iii, v  I) ii, iv, v  J) iii, iv, v

The following table determines the sign of $f'(x)$ on the important intervals:

<table>
<thead>
<tr>
<th>$(x - 1)$</th>
<th>$(x - 2)^2$</th>
<th>$(x - 3)^3$</th>
<th>$(x - 4)^4$</th>
<th>$f'(x)$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; x &lt; 1$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$1 &lt; x &lt; 2$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>decreasing</td>
</tr>
<tr>
<td>$2 &lt; x &lt; 3$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>decreasing</td>
</tr>
<tr>
<td>$3 &lt; x &lt; 4$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
<tr>
<td>$4 &lt; x &lt; 5$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing</td>
</tr>
</tbody>
</table>

$f'(x)$ is decreasing on $1 < x < 2$ so i) is false

$f'(x)$ is increasing on $0 < x < 1$ and decreasing on $1 < x < 2$ so $f$ has a

local maximum at $x = 1$, so ii) is false

This means that iii), iv) and v) must be true

(You can also verify iii) and iv) directly, from the information in the table. The fact

that $f(0) = 1 > f(3) = -1$ means that the local minimum at $x = 3$ is also an absolute

minimum (why?) )

11. Suppose $f(x) = \ln((\arctan x)^3)$ for $x > 0$. What is $f'(1)$? (Note: $\arctan x$ is the

"inverse tangent function” which the text sometimes also writes as $\tan^{-1}x$.)

A) $0$   B) $\frac{6}{\pi}$   C) $\frac{1}{\pi}$   D) $\frac{\pi}{4}$   E) $\frac{3}{\pi}$

F) $\frac{1}{2}$   G) $\frac{3}{2\pi}$   H) $2\pi$   I) $\frac{3\pi}{4}$   J) $\frac{4\pi}{3}$

$f(x) = 3 \ln(\arctan x)$ so $f'(x) = 3 \cdot \frac{1}{1 + x^2}$, so $f'(1) = 3/2 \cdot \pi = \frac{6}{\pi}$

12. On the interval $[1, 3]$, the absolute minimum of the function $f(x) = \frac{x}{a} + \frac{a^2}{2x^2}$ occurs

at $x = 2$. What is the absolute maximum value of $f(x)$ on $[1, 3]$?
\[ f'(x) = \frac{1}{a} - \frac{a^2}{x^2}. \]
Since the absolute minimum occurs at \( x = 2 \) and since \( f'(x) \) exists at every point in \((1,3)\), it must be that \( f'(2) = \frac{1}{a} - \frac{a^2}{8} = 0 \), so \( a = 2 \) and therefore
\[ f(x) = \frac{x}{2} + \frac{4}{2x^2} = \frac{x}{2} + \frac{2}{x^2}. \]
Since \( f \) has no other critical numbers in \((1,3)\), the absolute maximum for \( f \) must occur at one of the two endpoints. Since \( f(1) = \frac{1}{2} + \frac{2}{1} = \frac{5}{2} = 2.5 \)
and \( f(3) = \frac{3}{2} + \frac{2}{9} = \frac{27+4}{18} = \frac{31}{18} \approx 1.72 \), the absolute maximum value must be \( \frac{5}{2} \).

13. For \( f(x) = 3x(x - 4)^{\frac{3}{2}} \), the derivative \( f'(x) = (x - 4)^{-\frac{1}{2}}(5x - 12) \). What is the largest interval listed on which \( f(x) \) is concave down? (For your convenience: the right endpoints in the intervals listed below are increasing as you advance through the list.)

A) \(( -\infty, -\frac{5}{12}) \)  B) \(( -\infty, 0) \)  C) \(( -\infty, \frac{6}{5}) \)  D) \(( -\infty, 4^{1/3}) \)
E) \(( -\infty, \frac{7}{4}) \)  F) \(( -\infty, \frac{12}{5}) \)  G) \(( -\infty, 3) \)  H) \(( -\infty, 4) \)
I) \(( -\infty, \frac{24}{5}) \)  J) \(( -\infty, \frac{36}{5}) \)

\[ f''(x) = (x - 4)^{-\frac{1}{2}}(5x - 12) - \frac{1}{3}(x - 4)^{-\frac{3}{2}} \]
\[ = (x - 4)^{-\frac{1}{2}}(5x - 20 - 5 \frac{5}{3}x + 4) = (x - 4)^{-\frac{1}{2}}(10 - \frac{10}{3}x - 16) \]
\[ = \frac{1}{(x - 4)^{1/2}} \left(10 - \frac{10}{3}x - 16 \right). \]
f''(x) does not exist for \( x = 4 \), and \( f'(x) = 0 \) for \( x = \frac{24}{5} \).

We have

<table>
<thead>
<tr>
<th>( x &lt; 4 )</th>
<th>( \frac{1}{(x - 4)^{1/2}} )</th>
<th>( (\frac{10}{3}x - 16) )</th>
<th>( f''(x) )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 &lt; x &lt; \frac{24}{5} )</td>
<td>( + )</td>
<td>( - )</td>
<td>( - )</td>
<td>concave down</td>
</tr>
<tr>
<td>( \frac{24}{5} &lt; x )</td>
<td>( + )</td>
<td>( + )</td>
<td>( + )</td>
<td>concave up</td>
</tr>
</tbody>
</table>

Therefore \( f \) is concave down on \(( -\infty, \frac{24}{5}) \) and \( f \) turns concave up to the right of \( \frac{24}{5} \).
14. If \( \lim_{x \to \infty} (2x + 3)^{\frac{1}{\ln x}} = 10 \), what is \( a \)?

A) 1  
B) \ln 2  
C) \frac{1}{\ln 2}  
D) \ln 10  
E) \ln 3  
F) \frac{1}{\ln 10}  
G) \frac{1}{\ln 3}  
H) \frac{1}{\ln 6}  
I) e  
J) e \ln 10

If \( y = (2x + 3)^{\frac{1}{\ln x}} \), then \( \ln y = \ln (2x + 3)^{\frac{1}{\ln x}} = \frac{1}{a \ln x} \ln (2x + 3) \).

\[
\lim_{x \to \infty} \frac{\ln (2x + 3)}{a \ln x}
\]

is of the form \( \frac{\infty}{\infty} \) so LH’spitita’s Rule gives

\[
\lim_{x \to \infty} \frac{\ln (2x + 3)}{a \ln x} = \lim_{x \to \infty} \frac{2x + 3}{a(2x + 3)} = \lim_{x \to \infty} \frac{2x}{a(2 + \frac{3}{x})} = \frac{2}{a}.
\]

Therefore \( \ln y \to \frac{1}{a} \) as \( x \to \infty \), and therefore \( y \to e^{1/a} = 10 \).

Solving for \( a \) gives \( \frac{1}{a} = \ln 10 \), so \( a = \frac{1}{\ln 10} \).

**Questions 15-19 are “true/false” questions**

15. \( \lim_{x \to \infty} (1 + \frac{1}{x})^{6x} = \infty \)

A) True  
B) False

**true**

\[
\lim_{x \to \infty} (1 + \frac{1}{x})^{6x} = \lim_{x \to \infty} ((1 + \frac{1}{x})^x)^6 = e^6
\]

16. Mary drives the 280 miles from St. Louis to Kansas City in 5 hours. At some time during the trip she was traveling 56 miles/hr.

A) True  
B) False

Her average velocity during the trip was \( \frac{280}{5} = 56 \) mph. In this setting, the Mean Value Theorem states that there was some time \( c \) during the trip where her instantaneous velocity = average velocity for the trip = 56 mph.
17. If $c$ is a critical point of $f$ and $f''(c) > 0$, then $f(x)$ has an absolute minimum at $x = c$.

A) True  
B) False

$f$ would have a local minimum at $x = c$, but there's no reason a local minimum has to also be an absolute minimum.

18. There exists a differentiable function $f$ such that $f(5) = 200$, $f(1) = 0$ and $f'(x) > 60$ for all $x$.

A) True  
B) False

Since $\frac{f(5) - f(1)}{5 - 1} = \frac{200}{4} = 50$, the Mean Value Theorem says there must be a number $c$ between 1 and 5 where $f'(c) = 50$.

19. $\frac{d}{dx} \ln(8) = \frac{1}{8}$

A) True  
B) False

$\ln 8$ is a constant, so $\frac{d}{dx} \ln 8 = 0$. 
Part II: (25 points) In each problem, clearly show your solution in the space provided. “Show your solution” does not simply mean “show your scratch work” — you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. Generally, a correct answer without supporting work may not receive full credit.

20. a) Find all the critical numbers for the function \( f(x) = e^x(x^2 - 3) \).

\[
f'(x) = e^x(x^2 - 3) + e^x(2x) = e^x(x^2 + 2x - 3)
= e^x(x + 3)(x - 1).
\]

Since \( f'(x) \) exists for all \( x \), the only critical numbers are those \( x \)'s for which
\( f'(x) = e^x(x + 3)(x - 1) = 0 \), that is, \( x = -3 \) and \( x = 1 \).

b) What are the absolute maximum and minimum values for \( f(x) = e^x(x^2 - 3) \) on the interval \([0, 2]\)? (Be sure to give the exact max and min values — although you can also include a decimal approximation for these values if you like.)

\( f \) is continuous on the closed interval \([0, 2]\), so the Extreme Value Theorem guarantees the existence of both absolute maximum and absolute minimum values. These must occur at either an endpoint or a critical number in \((0, 2)\) — and by part a), the only such critical number is \( x = 1 \).

Testing at these points, we get
\[
\begin{align*}
f(0) &= e^0(0 - 3) = -3 \\
f(1) &= e^1(1 - 3) = -2e \approx -5.4 \\
f(2) &= e^2(4 - 3) = e^2 \approx 7.4
\end{align*}
\]

so the maximum value is \( e^2 \) and the minimum value is \(-2e\).
21. a) Find \( \frac{dy}{dx} \) if \( y = \log_2 \left( \frac{2x}{x^2+1} \right) \) (No simplification is necessary after you get to a correct formula \( \frac{dy}{dx} = \ldots \))

We can simplify first. \( y = \log_2 \left( \frac{2x}{x^2+1} \right) = \log_2 2 + \log_2 x - \log_2 (x^2+1) \), so
\[
\frac{dy}{dx} = \frac{1}{\ln 2} \frac{2}{x} - \frac{2x}{(\ln 2)(x^2+1)}
\]

b) Find \( \frac{dy}{dx} \) if \( y = x \arctan x \) (No simplification is necessary after you get to a correct formula \( \frac{dy}{dx} = \ldots \))

Use logarithmic differentiation: \( \ln y = \ln (x \arctan x) = (\arctan x)(\ln x) \), so
\[
\frac{1}{y} \frac{dy}{dx} = (\arctan x) \left( \frac{1}{x} \right) + (\ln x) \left( \frac{1}{1+x^2} \right) = \frac{\arctan x}{x} + \frac{\ln x}{1+x^2} , \text{ so}
\]
\[
\frac{dy}{dx} = y \left( \frac{\arctan x}{x} + \frac{\ln x}{1+x^2} \right) = x \arctan x \left( \frac{\arctan x}{x} + \frac{\ln x}{1+x^2} \right)
\]

c) Find \( \lim_{x \to 0^+} \left( \frac{1}{x} - \csc x \right) \)

(Note: this exact problem was done in the lecture.)

\( \lim_{x \to 0^+} \left( \frac{1}{x} - \csc x \right) \) is of the form “\( \infty - \infty \)”. We can rewrite this as the limit of a fraction:
\[
\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} . \text{ This limit is of the form “0/0”. We use}
\]
LH’spitita’s rule (twice):
\[
\lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0^+} \frac{\cos x - 1}{x} = \lim_{x \to 0^+} \frac{\cos x - 1}{x \cos x + \sin x} = \lim_{x \to 0^+} \frac{\cos x - 1}{x \cos x + \sin x}
\]
\[
= \lim_{x \to 0^+} \frac{-\sin x}{\cos x - x \sin x + \cos x} = 0/2 = 0
\]