1. Find the volume of the solid generated by rotating the curve \( y = x^2 \) from \( x = 0 \) to \( x = 1 \) about the x-axis.

\[
V = \pi \int_0^1 (x^2)^2 \, dx
= \pi \int_0^1 x^4 \, dx
= \frac{\pi}{5}
\]

\( \text{Solid of Revolution} \)
2. The finite region enclosed by the curves \( y = x \) and \( y = x^2 \) is rotated about the x-axis. Find the volume of the resulting solid.

A) \( \pi/5 \)
B) \( 5/\pi \)
C) \( \pi/6 \)
D) \( 6/\pi \)

\( \boxed{\text{E) } 2\pi/15} \)
F) \( 8\pi/15 \)
G) \( 4\pi/3 \)
H) \( 4/3\pi \)
I) \( 15\pi/2 \)
J) \( 15\pi/8 \)

cross section at \( x \):

\[
A(x) = \pi x^2 - \pi (x^2)^2 = \pi (x^2 - x^4)
\]

\[
V = \int_a^b A(x) \, dx = \int_0^1 \pi (x^2 - x^4) \, dx
\]

\[
= \pi \left[ \int_0^1 x^2 \, dx - \int_0^1 x^4 \, dx \right]
\]

\[
= \pi \left[ \frac{1}{3} - \frac{1}{5} \right] = \boxed{\frac{2\pi}{15}}
\]
3. Find the resulting volume if the region of problem 2 is rotated about the y-axis instead.

\[ A(y) = \pi (\sqrt{y})^2 = \pi y \quad V = \int_0^1 A(y) \, dy = \int_0^1 \pi (y - y^2) \, dy = \pi \left[ \frac{1}{2} - \frac{1}{3} \right] = \frac{\pi}{6} \]
4. Suppose the curve \( y = x^2 \) from \( x = 0 \) to \( x = 1 \) is rotated about the line \( x = 1 \). Find the resulting volume.

\[
A(y) = \pi \left( r_y \right)^2 = \pi \left(1 - \sqrt{y} \right)^2
\]

\[
V = \int_0^1 A(y) \, dy = \pi \int_0^1 \left(1 - \sqrt{y} \right)^2 \, dy
\]

\[
= \pi \int_0^1 \left(1 - 2\sqrt{y} + y \right) \, dy = \left[ \frac{\pi}{6} \right]
\]
5. Suppose the curve \( y = x^2 \) from \( x = 0 \) to \( x = 1 \) is rotated about the line \( y = 1 \). Find the resulting volume.

A) \( \pi/5 \)
B) \( 5/\pi \)
C) \( \pi/6 \)
D) \( 6/\pi \)
E) \( 2\pi/15 \)
F) \( 8\pi/15 \)
G) \( 4\pi/3 \)
H) \( 4/3\pi \)
I) \( 15\pi/2 \)
J) \( 15\pi/8 \)

\[
A(x) = \pi (y) \varepsilon = \pi (1 - x^2)^2
\]

\[
V = \int_a^b A(x) \, dx = \pi \int_0^1 (1-x^2)^2 \, dx = \frac{8\pi}{15}
\]
6. Find the arc length of the parametric curve \[ \begin{align*} x &= 1 + t^2 \\ y &= 4 + \frac{2}{3} t^3 \end{align*}, \quad 0 \leq t \leq 1. \] Choose the closest of the following.

A) 0.10
B) 0.30
C) 0.80
D) 1.1
E) 1.15
F) 1.20
G) 1.25
H) 1.30
I) 1.35
J) 1.40

\[ x' = 2t \implies x'^2 = 4t^2 \]
\[ y' = 2t^2 \implies y'^2 = 4t^4 \]
\[ \sqrt{x'^2 + y'^2} = \sqrt{4t^2 + 4t^4} = 2t\sqrt{1 + t^2} \]
\[ l = \int_0^1 \sqrt{1 + t^2} \, dt = 2 \int_0^1 t \sqrt{1 + t^2} \, dt \]
\[ = \int_1^{32} \sqrt{u} \, du = \frac{2}{3} \left[ \frac{32^{3/2}}{3} - 1 \right] \]
\[ = \overline{1.21895} \]
7. We know that if \( f(x) \) is continuous on \([a, b]\), then there exists a number \( c \in [a, b] \) so that 
\[
 f(c) = \text{the average value of } f(x) \text{ on } [a, b].
\]
Find such a number \( c \) in the case \( f(x) = x^2 \) and 
\([a, b] = [0, 2] \).

\[
\begin{align*}
\text{AV}(f) &= \frac{\int_a^b f(x) \, dx}{b - a} = \frac{\int_0^2 x^2 \, dx}{2 - 0} = \frac{\frac{8}{3}}{2} = \frac{4}{3} \\
\text{So } f(c) &= \text{AV}(f) \\
15. \quad c^2 &= \frac{4}{3} \implies c = \sqrt{\frac{2}{3}} = 1.1547
\end{align*}
\]

A) 0.10 
B) 0.30 
C) 0.80 
D) 1.1 
E) 1.15 
F) 1.20 
G) 1.25 
H) 1.30 
I) 1.35 
J) 1.40
Note: In problems 8 - 11 and in the hand-graded problems, we use the SI system: distance in meters, mass in kilograms, time in seconds, force in newtons, work in joules. These units are completely consistent. You do not need to worry about which units to use.

8. If a certain spring is stretched from neutral position by an elongation (in meters) \( x \), the force it exert (in newtons) is \( F = 100x \). How much work (in joules) is done in stretching the spring from neutral to an elongation of 1 m?

\[
W = \int_a^b F(x) \, dx = \int_0^1 100x \, dx
\]

\[
= 100 \int_0^1 x \, dx = \left[ 50x^2 \right]_0^1 = 50
\]
9. A 100 meter chain which has a mass of 1 Kg/meter hangs from the roof to the ground alongside a 100 meter tall building. How much work (in joules) is done if the entire chain is hauled up onto the roof? Pick the closest answer below.

A) 5000  
B) 10,000  
C) 20,000  
D) 30,000  
E) 35,000  
F) 40,000  
G) 45,000  
H) 50,000  
I) 55,000  
J) 60,000

\[ \Delta x = \frac{x_i - x_{i-1}}{\Delta x} \]

Let \( p_i \) be the \( i^{th} \) piece

Mass: \( m_i = \Delta x = \Delta x \)

\[ F_i = m_i g = g \Delta x \]

Dist to raise \( p_i \): \( x_i \)

\[ W_i = g \int x_i \Delta x \]

\[ W = g \int_0^{100} x \, dx = \left[ g \cdot \frac{x^2}{2} \right]_0^{100} = \frac{(9.8)(50000)}{2} = 490000 \text{ J} \]
10. A tank is in the form of a right circular cone with the point at the bottom. It has a radius of 1 m and a height of 2 m, and is filled to the top with water (density $\rho = 1000 \text{ Kg/m}^3$). How much work is done in pumping the tank out completely?

A) 5000
B) 10,000
C) 20,000
D) 30,000
E) 35,000
F) 40,000
G) 45,000
H) 50,000
I) 55,000
J) 60,000

By similar $\Delta x$,

$$\frac{V_i}{2-\chi_i^{*}} = \frac{1}{2} \Rightarrow V_i = \frac{1}{2} (2-\chi_i^{*})$$

$$V_i = \pi \chi_i^{*} r_i^{2} \Delta x = \frac{\pi}{4} (2-\chi_i^{*})^2 \Delta x$$

$$M_i = \rho V_i = 1000 V_i = 2450 \pi (2-\chi_i^{*})^2 \Delta x$$

$$F_i = M_i g = (9.8) M_i = 2450 \pi (2-\chi_i^{*})^2 \Delta x$$

$$W_i = F_i \chi_i = 2450 \pi (2-\chi_i^{*})^2 \Delta x$$

$$W = 2450 \pi \sum (2-\chi_i^{*})^2 \Delta x$$

$$W = 2450 \pi \int_{0}^{2} \chi (2-\chi)^2 \, d\chi$$

$$W = 10262.6 \pi$$
11. A large rectangular aquarium is 4 m long, and has ends which are squares 2 meters on a side. It is completely filled with water (density $\rho = 1000 \text{ Kg/m}^3$). What is the total hydrostatic force (in newtons) on one end of the tank?

A) 5000  
B) 10,000  
C) 20,000  
D) 30,000  
E) 35,000  
F) 40,000  
G) 45,000  
H) 50,000  
I) 55,000  
J) 60,000

\[ A_i = 2 \Delta x \]
\[ \rho \Delta g = (1000) \times 9.8 \times 2 = 19600 \times \Delta x \]
\[ F_i = \rho \cdot A_i = (19600 \times \Delta x) \times (2 \Delta x) = 19600 \times \Delta x \]
\[ F \approx 19600 \sum \Delta x \]
\[ F = 19600 \int_0^2 x \, dx = 39200 \text{ N} \]
12. Let $R$ be the semicircular region in the top half of the plane bounded above by the circle
$x^2 + y^2 = 1$ and below by the x-axis. Find the y coordinate $\bar{y}$ of $R$.

A) $\frac{\pi}{5}$  
B) $\frac{5}{\pi}$  
C) $\frac{\pi}{6}$  
D) $\frac{6}{\pi}$  
E) $\frac{2\pi}{15}$  
F) $\frac{8\pi}{15}$  
G) $\frac{4\pi}{3}$  
H) $\frac{4}{3\pi}$  
I) $\frac{15\pi}{2}$  
J) $\frac{15\pi}{8}$

\[
\bar{y} = \frac{\int_{-a}^{a} \frac{1}{2} \left[ f(x) \right]^2 \, dx}{A} = \frac{\frac{1}{2} \int_{-1}^{1} (\sqrt{1-x^2})^2 \, dx}{\frac{\pi}{2}}
\]

\[
= \frac{\int_{-1}^{1} (1-x^2) \, dx}{\pi} = \frac{2 \int_{0}^{1} (1-x^2) \, dx}{\pi}
\]

\[
= \frac{2 \left( \frac{2}{3} \right)}{\pi} = \frac{4}{3\pi}
\]
13. Which of the following functions $f(x)$ are the density function of some random variable $X$?

I) $f(x) = \begin{cases} 
1/2, & x \in [0,2] \\
0, & x \in [0,2] 
\end{cases}$

II) $f(x) = \begin{cases} 
0, & x < 0 \\
e^{-x}, & x \geq 0 
\end{cases}$

III) $f(x) = \frac{1}{\pi(1+x^2)}$

A) I

B) II

C) III

D) I, II

E) I, III

F) II, III

G) I, II, III

H) Insufficient information to answer

\[ \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \, dx = 2 \int_{0}^{\infty} \frac{1}{\pi(1+x^2)} \, dx \]

\[= \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+x^2} \, dx = \frac{2}{\pi} \lim_{A \to \infty} \int_{0}^{A} \frac{1}{1+x^2} \, dx \]

\[= \frac{2}{\pi} \lim_{A \to \infty} \tan^{-1}x \bigg|_{0}^{A} = \frac{2}{\pi} \lim_{A \to \infty} \left( \tan^{-1}A - \tan^{-1}0 \right) \]

\[= \left( \frac{2}{\pi} \right) \left( \frac{\pi}{2} \right) = 1 \]
14. The random variable $X$ has probability density function $f(x) = \begin{cases} \frac{x}{2}, & x \in [0,2] \\ 0, & x \notin [0,2] \end{cases}$.

Find $\mu = \text{the mean of } X$.

A) $\frac{1}{6}$

B) $\frac{1}{3}$

C) $\frac{1}{4}$

\[
\mu = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{2} x \left( \frac{1}{2} x \right) \, dx = \frac{1}{2} \int_{0}^{2} x^2 \, dx = \left( \frac{1}{2} \right) \left( \frac{8}{3} \right) = \frac{4}{3}
\]

D) $\frac{1}{2}$

E) $\frac{2}{3}$

F) $\frac{3}{4}$

G) 1

H) $\frac{5}{4}$

I) $\frac{4}{3}$

J) $\frac{3}{2}$
15. The average waiting time on the tech support line of a computer company is 10 minutes. Assume that the waiting time in minutes for a random call follows an exponential distribution. What is the probability that your next call to the line will be answered within 1 minute. Pick the closest of the following.

A) 0.10
B) 0.30
C) 0.80
D) 1.1
E) 1.15
F) 1.20
G) 1.25
H) 1.30
I) 1.35
J) 1.40
3. An aquarium in the form of a rectangular solid is 2 meters long, and has ends which are rectangles 1 meter wide and \( h \) meters tall. To pump all of the water (\( \rho = 1000 \text{ Kg/m}^3 \)) out of the tank requires 4900 J of work. Find the height of the tank, \( h \).

\[
V_i = (1)(2)(\Delta x) = 2\Delta x
\]

\[
m_i = \rho V_i = (1000)(2\Delta x) = 2000\Delta x
\]

\[
F_i = m_i g = (2000\Delta x)(9.8) = 19600\Delta x
\]

\[
W_i \approx F_i x_i \Delta x = 19600 x_i \Delta x
\]

\[
W \approx \sum W_i = 19600 \sum x_i \Delta x
\]

\[
W = 19600 \int_0^h x \, dx = (19600)(\frac{h^2}{2})
\]

So \( W = 9800h^2 \)

If \( W = 4900 \)

\[
9800h^2 = 4900
\]

\[
h = \frac{1}{\sqrt{2}}
\]
Part II Hand Graded Problems.

25 points. Please sign these sheets legibly.

1. Pappus' theorem states that if a region $S$ of the plane, whose centroid is the point $C$, is rotated about a line $l$ which does not intersect $S$, the volume of the resulting solid is given by:

$$
\text{Volume} = (\text{area of } S) \times (\text{distance traveled by } C \text{ during the rotation}).
$$

Note that the distance traveled by $C$ is equal to $2\pi R$, where $R$ is the perpendicular distance from the point $C$ to the line $l$.

Use Pappus' theorem to find the volume of the solid generated when the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$ is rotated about the line $y = -x + 4$.

\[A_{\text{net}} = 2^2 = 4\]

Centroid = $(1, 1)$

\[\text{DIST } R \text{ to } l = \sqrt{(2-1)^2 + (2-1)^2} = \sqrt{2}\]

So centroid travels $2\pi R = 2\sqrt{2}\pi$

\[V = (A_{\text{net}}) \times (\text{DIST}) = (4)(2\sqrt{2}\pi) = \frac{8\sqrt{2}\pi}{\pi}\]
2. Set up, but do not attempt to evaluate, integrals which give the arc lengths of

a) the graph of \( y = \sin x \), \( 0 \leq x \leq 2\pi \).

b) the parametric curve \( \begin{cases} x = \sin t \\ y = t^2 \end{cases} \), \( 0 \leq t \leq 2\pi \).

\[
\text{a)} \quad y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left( \frac{dy}{dx} \right)^2 = \cos^2 x
\]

\[
\ell = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
\ell = \int_0^{2\pi} \sqrt{1 + \cos^2 x} \, dx
\]

\[
\text{b)} \quad x = \sin t \Rightarrow x' = \cos t \Rightarrow x'^2 = \cos^2 t
\]

\[
y = t^2 \Rightarrow y' = 2t \Rightarrow y'^2 = 4t^2
\]

\[
\ell = \int_0^b \sqrt{x'^2 + y'^2} \, dt
\]

\[
\ell = \int_0^{2\pi} \sqrt{\cos^2 t + 4t^2} \, dt
\]