This exam contains ten multiple-choice problems worth two points each, five true-false problems worth one point each, and four problems to be hand-graded worth 25 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, choose the letter corresponding to the only correct answer.

1. Consider the following first-order differential equation. \((x + y)dx + (x - 2)dy = 0\)

   This differential equation is which of the following? (Note: "Linear" could mean linear in \(y\) or linear in \(x\) or both.)

   (A) separable, linear, and exact
   (B) separable and linear, but not exact
   (C) separable and exact, but not linear
   (D) linear and exact, but not separable
   (E) separable, but neither linear nor exact
   (F) linear, but neither separable nor exact
   (G) exact, but neither separable nor linear
   (H) neither separable nor linear nor exact

2. Consider the linear initial value problem \(y' + \frac{x}{x-5}y = \frac{x^2}{x-1}, y(2) = 0\). The Existence and Uniqueness theorem for linear initial value problems guarantees that this initial value problem has a unique solution. Use this same theorem to determine the largest interval on which this solution is guaranteed to exist.

   (A) \((-\infty, -5)\)
   (B) \((-\infty, 0)\)
   (C) \((-\infty, 1)\)
   (D) \((-\infty, \infty)\)
   (E) \((-5, 0)\)
   (F) \((-5, 1)\)
   (G) \((-5, \infty)\)
   (H) \((0, 1)\)
   (I) \((0, \infty)\)
   (J) \((1, \infty)\)
3. Consider the initial value problem \( \frac{dy}{dx} = y^{\frac{1}{2}}, \ y(0) = 1 \). Use Euler's method with \( h = 0.5 \) to approximate the value of \( y(1) \) to three decimal places.

(A) 1.145  
(B) 1.203  
(C) 1.273  
(D) 1.413  
(E) 1.770  
(F) 2.072  
(G) 2.111  
(H) 2.152  
(I) 2.363  
(J) 2.394

4. The population \( y \) of rabbits on a small island grows according to the following logistic equation, where \( t \) is measured in years.

\[
\frac{dy}{dt} = (4 - 0.01y)y
\]

If the initial population is 100 rabbits, what will the population be after one year?

(A) 25 rabbits  
(B) 99 rabbits  
(C) 101 rabbits  
(D) 140 rabbits  
(E) 200 rabbits  
(F) 288 rabbits  
(G) 326 rabbits  
(H) 379 rabbits  
(I) 400 rabbits  
(J) 1600 rabbits
5. Refer back to the previous problem. What is the limiting rabbit population? (In other words, what is the saturation level or environmental carrying capacity for this population?)
(Hint: You can get the answer from the solution to the initial value problem, or, if you are unsure about your solution, you can get the answer from the logistic equation itself.)

(A) 25 rabbits
(B) 99 rabbits
(C) 101 rabbits
(D) 140 rabbits
(E) 200 rabbits
(F) 288 rabbits
(G) 326 rabbits
(H) 379 rabbits
(I) 400 rabbits
(J) 1600 rabbits

6. An object weighing 8 lbs. is dropped from a helicopter. Suppose the air resistance is equivalent to half the instantaneous velocity. Assuming that the object has not yet reached the ground, what is its velocity after 1 second? (Hint: Take the positive direction to be downward.)

(A) 3.459 feet per second
(B) 3.999 feet per second
(C) 7.533 feet per second
(D) 7.853 feet per second
(E) 13.835 feet per second
(F) 15.707 feet per second
(G) 30.081 feet per second
(H) 31.021 feet per second
7. In a certain forest, the rate constant for the birth rate of the squirrel population is 0.2 per month, (in other words, 0.2 \( \frac{\text{squirrels per month}}{\text{squirrel}} \) are born), and the death rate is 50 \( \frac{\text{squirrels}}{\text{month}} \). Let \( t \) represent time in months, and let \( y(t) \) represent the squirrel population at any time \( t \). Which of the following differential equations correctly models the changes in this population?

(A) \( \frac{dy}{dt} = y + 0.2 - 50 \)

(B) \( \frac{dy}{dt} = 0.2 - 50y \)

(C) \( \frac{dy}{dt} = 0.2 - \frac{y}{50} \)

(D) \( \frac{dy}{dt} = 0.2y - 50 \)

(E) \( \frac{dy}{dt} = 0.2y - 50y \)

(F) \( \frac{dy}{dt} = 0.2y - \frac{y}{50} \)

(G) \( \frac{dy}{dt} = \frac{y}{0.2} - 50 \)

(H) \( \frac{dy}{dt} = \frac{y}{50} - 50y \)

(I) \( \frac{dy}{dt} = \frac{y}{0.2} - \frac{y}{50} \)

8. Consider the second-order differential equation \( y'' - ty' + 3y = t^2 \), and suppose that \( y_1 \) and \( y_2 \) are solutions of this differential equation. Which of the following are also solutions?

(A) \( y_1 + y_2 \)

(B) \( 5y_1 \)

(C) both A and B

(D) neither A nor B
9. Consider the following sets of functions.

\[ S_1 = \{e^t, e^{2t}\} \quad S_2 = \{t^2, t^{-2}\} \quad S_3 = \{\sin t, 0\} \]

Which of these is/are linearly independent?

(A) \( S_1, S_2, \) and \( S_3 \)
(B) \( S_1 \) and \( S_2 \) only
(C) \( S_1 \) and \( S_3 \) only
(D) \( S_2 \) and \( S_3 \) only
(E) \( S_1 \) only
(F) \( S_2 \) only
(G) \( S_3 \) only
(H) none of these

10. Find the general solution of the following differential equation.

\[ y'' + 4y' + 5y = 0 \]

(A) \( y = c_1 e^t + c_2 e^{3t} \)
(B) \( y = c_1 e^{-t} + c_2 e^{-4t} \)
(C) \( y = c_1 e^t + c_2 e^{-5t} \)
(D) \( y = c_1 e^{-t} + c_2 e^{5t} \)
(E) \( y = c_1 e^{t}\cos 2t + c_2 e^{t}\sin 2t \)
(F) \( y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t \)
(G) \( y = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t \)
(H) \( y = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t \)
Part II. True-False (1 point each)

Mark your answer card "A" if the statement is true and "B" if the statement is false.

11. The function \( y = e^{\sin x} \) is a solution of the following initial value problem.

\[
\frac{dy}{dx} = \sqrt{y^2 - \sin^2 x \cdot e^{2\sin x}}, \quad y(0) = 1
\]

12. The differential equation \( \frac{dy}{dx} = -y^2 + x^2 + 1 \) has the following direction field.
13. The following differential equation is exact.

\[ y^2 e^{xy} dx + (e^{xy} + xye^{xy})dy = 0 \]

14. The Existence and Uniqueness Theorem for first-order initial value problems guarantees that the following initial value problem has a unique solution.

\[ y' = y^1, \quad y(0) = 0 \]

15. The following differential equation describes logistic growth.

\[ y' = 5y - 6y^2 \]
Part III. Hand-Graded.

In each problem in part III, follow directions carefully and show all your work. Be sure to write your name and section number at the top of this page and the next page.

15. (6 points)

The following differential equation is exact. (You do not need to verify this.) Find its general solution. You do not need to solve for $y$ in your answer.

$$[2x + y\cos(xy)]dx + [y^2 + x\cos(xy)]dy = 0$$
17. (6 points)

Find the general solution of the following differential equation. You do not need to solve for $y$ in your answer.

$$y \sqrt{x} \frac{dy}{dx} = e^{-x} \sin \sqrt{x}$$
15. (9 points)

A tank with capacity 1000 gal initially contains 600 gal of pure water. A brine solution with concentration 10 lbs per gal flows into the tank at a rate of 3 gal per min, and the well-stirred mixture flows out of the tank at a rate of 2 gal per min.

(a) Let \( t \) represent time in minutes, and let \( y \) represent the amount of salt in lbs. Write the initial value problem which describes the change in \( y \) over time.

(b) Solve the initial value problem to determine the amount of salt \( y(t) \) in the tank at any time \( t \) (until the tank overflows).

(c) Determine the concentration of salt in the brine solution at the instant that the tank fills to capacity. Include the correct units in your answer.
19. (4 points)

The half-life of a radioactive element is 5 years. How long will it take for the substance to decay to one-tenth its original amount? Round your answer to the nearest tenth, and include the correct units.