This exam contains twelve multiple-choice problems worth two points each, six true-false problems worth one point each, and three free-response problems worth 20 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, choose the letter corresponding to the only correct answer.

1. The general solution of the differential equation \( x^2 y'' = 2y \) is \( y = A x^{-1} + B x^2 \). (You do not need to verify this.) Find the solution of the following initial value problem.

\( x^2 y'' = 2y \quad y(1) = 3, \; y'(1) = 2 \)

(A) \( y = 3x^{-1} \)
(B) \( y = 3x^2 \)
(C) \( y = 2x^{-1} + 3x^2 \)
(D) \( y = 3x^{-1} + 2x^2 \)
(E) \( y = \frac{1}{2} x^{-1} + \frac{5}{2} x^2 \)
(F) \( y = \frac{5}{2} x^{-1} + \frac{1}{2} x^2 \)
(G) \( y = \frac{1}{3} x^{-1} + \frac{8}{3} x^2 \)
(H) \( y = \frac{5}{3} x^{-1} + \frac{1}{3} x^2 \)
(I) \( y = \frac{4}{3} x^{-1} + \frac{8}{3} x^2 \)
(J) \( y = \frac{4}{3} x^{-1} + \frac{8}{3} x^2 \)

2. The direction field pictured corresponds to which of the following differential equations?

(A) \( \frac{dy}{dx} = x + y \)
(B) \( \frac{dy}{dx} = xy \)
(C) \( \frac{dy}{dx} = \frac{x}{y} \)
(D) \( \frac{dy}{dx} = \frac{y}{x} \)

For example, at \( (3, -1) \),

\[ m = \frac{dy}{dx} = \frac{y}{x} = \frac{-1}{3}, \]
3. Solve the following initial value problem by separating variables and integrating.

\[ \frac{dy}{dx} = x^3 e^{\frac{1}{2}x^2}, \quad y(0) = 0 \]

What is the value of C?

\[ \int dy = \int x^3 e^{\frac{1}{2}x^2} \, dx \]

\[ u = x^2 \quad dv = x e^{\frac{1}{2}x^2} \, dx \]

\[ du = 2x \, dx \quad v = e^{\frac{1}{2}x^2} \]

\[ y = x^2 e^{\frac{1}{2}x^2} - \int 2x e^{\frac{1}{2}x^2} \, dx \]

\[ y = x^2 e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2} + C \]

\[ y(0) = 0 \]

\[ 0 = 0 - 2 + C \quad C = 2 \]

\[ y = x^2 e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2} + 2 \]

4. Find an integrating factor which could be used to solve the following linear differential equation.

\[ (x^2 + 1) \frac{dy}{dx} - 2xy = 4 \]

\[ (A) \frac{1}{x^2+1} \]

\[ (B) x^2 + 1 \]

\[ (C) (x^2 + 1)^2 \]

\[ (D) e^{-x^2} \]

\[ (E) e^{x^2} \]

\[ (F) e^{x^2+1} \]

\[ (G) \ln(x^2) \]

\[ (H) \ln(x^2 + 1) \]

\[ (I) \frac{1}{\ln(x^2+1)} \]
5. The following differential equation is exact. (You do not need to verify this.) Find its general solution.

\[
\frac{e^x \sin(e^y) \, dx + e^x e^y \cos(e^y) \, dy}{M} = 0
\]

(A) \( y = e^x \cos(e^y) \)
(B) \( y = e^x \sin(e^y) \)
(C) \( y = e^x \sin(e^y) \cos(e^y) \)
(D) \( F = e^x \cos(e^y) + C \)
(E) \( F = e^x \sin(e^y) + C \)
(F) \( F = e^x \sin(e^y) \cos(e^y) + C \)
(G) \( e^x \cos(e^y) = C \)
(H) \( e^x \sin(e^y) = C \)
(I) \( e^x \sin(e^y) \cos(e^y) = C \)

\[
F = \int e^x \sin(e^y) \, dx = \sin(e^y) \int e^x \, dx = \sin(e^y) \cdot e^x + g(y)
\]

\[
\frac{\partial F}{\partial y} = \cos(e^y) \cdot e^x \cdot e^x + g'(y)
\]

\[
g'(y) = 0 \quad g(y) = 0
\]

\[
F = e^x \sin(e^y)
\]

solution: \( e^x \sin(e^y) = C \)

6. The following differential equation can be put into the proper form and solved as which of the following types?

\[
\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}
\]

(I) separable \quad (II) linear in \( x \) \quad (III) linear in \( y \) \quad (IV) exact

(A) none of these
(B) I only
(C) II only
(D) III only
(E) IV only
(F) I and II only
(G) I and III only
(H) I and IV only
(I) II and IV only
(J) III and IV only

2xy \, dy = (x^2 - y^2) \, dx

\( y^2 - x^2 \, dx + 2xy \, dy = 0 \)

\[
\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2y \quad \text{exact}
\]

not separable because of the factor \( x^2 - y^2 \)

not linear in \( x \) because of \( x^2 \)

not linear in \( y \) because of \( y^2 \)
7. The following differential equation can be put into the proper form and solved as which of the following types?

\[ 8e^x dy = y \cos x \, dx \]

(I) separable   (II) linear in \( x \)   (III) linear in \( y \)   (IV) exact

(A) none of these
(B) I only
(C) II only
(D) III only
(E) IV only
(F) I and II only
(G) I and III only
(H) I and IV only
(I) II and IV only
(J) III and IV only

\[ \frac{8}{y} \, dy = e^{-x} \cos x \, dx \text{ separable} \]

\[ 8e^x \frac{dy}{dx} - \cos x \cdot y = 0 \text{ linear in } y \]

Not linear in \( x \) because of \( e^x \) and \( \cos x \)

\[ -y \cos x \, dx + 8e^x \, dy = 0 \]

\[ \frac{\partial M}{\partial y} = -\cos x \quad \frac{\partial N}{\partial x} = 8e^x \text{ not exact} \]

8. Consider the following linear initial value problem.

\[ xy' + x^2 y = \sin x, \; y(1) = 2 \]

The Existence and Uniqueness Theorem for linear initial value problems guarantees that this initial value problem has a unique solution. Use this same theorem to determine the largest interval on which this solution is guaranteed to exist.

(A) \((-\infty, -\pi)\)
(B) \((-\infty, 0)\)
(C) \((-\infty, \pi)\)
(D) \((-\infty, \infty)\)
(E) \((-\pi, 0)\)
(F) \((-\pi, \pi)\)
(G) \((-\pi, \infty)\)
(H) \((0, \pi)\)
(I) \((0, \infty)\)
(J) \((\pi, \infty)\)

Standard form:

\[ y' + xy = \frac{\sin x}{x} \]

\[ p(x) = x \text{ and } q(x) = \frac{\sin x}{x} \text{ continuous except at } x = 0 \]

\[ 0 \quad \boxed{\star} \quad 1 \quad (0, \infty) \]
9. Consider the initial value problem \( \frac{dy}{dx} = x + y - 1 \), \( y(1) = 0 \). Use Euler's method with \( h = 0.1 \) to approximate the value of \( y(1.3) \) to three decimal places.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( m_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>1.1</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>1.2</td>
<td>0.01</td>
<td>0.21</td>
</tr>
<tr>
<td>3</td>
<td>1.3</td>
<td>0.031</td>
<td></td>
</tr>
</tbody>
</table>

10. The population of turkeys in a certain wooded area obeys the following logistic equation, where \( y \) represents the number of turkeys in the area after \( t \) months.

\[
\frac{dy}{dt} = (0.8 - 0.01y)y
\]

If there are initially 20 turkeys, what will be the population (to the nearest turkey) after three months?

\[
y = \frac{ry_0}{ay_0 + (r-ay_0)e^{-rt}}
\]

\[
y = \frac{(0.8)(20)}{(0.01)(20) + (0.8-0.2)e^{-0.8t}}
\]

\[y_0 = 20\]

\[
y(3) \approx 62.89
\]
11. Classify the following second-order differential equation.

\[ ty'' - 8y' = 12y \quad ty'' - 8y' - 12y = 0 \]

(A) linear, homogeneous, has constant coefficients
(B) nonlinear, homogeneous, has constant coefficients
(C) linear, nonhomogeneous, has constant coefficients
(D) nonlinear, nonhomogeneous, has constant coefficients
(E) linear, homogeneous, does not have constant coefficients
(F) nonlinear, homogeneous, does not have constant coefficients
(G) linear, nonhomogeneous, does not have constant coefficients
(H) nonlinear, nonhomogeneous, does not have constant coefficients

12. Consider the following sets of functions.

(I) \( \{t^3, t^4\} \)  
(II) \( \{t^3, t^{-3}\} \)  
(III) \( \{t^3, -t^3\} \)  
(IV) \( \{t^3, 0\} \)

Which of these is/are linearly dependent?

(A) none of these
(B) I only
(C) II only
(D) III only
(E) IV only
(F) I and II only
(G) I and III only
(H) I and IV only
(I) II and IV only
(J) III and IV only

\[ (\text{III}) - t^3 = (-1)t^3 \quad \text{dependent} \]
\[ (\text{IV}) \quad 0 = (0)t^3 \quad \text{dependent} \]

or check the Wronskians
Part II. True-False (1 point each)

Mark your answer card “A” if the statement is true and “B” if the statement is false.

13. The function \( y = -3\sin 3t + 2\cos 3t \) is a solution of the DE \( y'' - 2y' = 39\sin 3t \).
    
    \[
    y' = -9\cos 3t - 6\sin 3t \\
    y'' = 27\sin 3t - 18\cos 3t \\
    (27\sin 3t - 18\cos 3t) - 2(-9\cos 3t - 6\sin 3t) = 39\sin 3t
    \]
    
    \[ \boxed{A} \]

14. The following differential equation is exact.
    
    \[
    e^{x-y} dx + e^{y-x} dy = 0 \\
    \frac{\partial M}{\partial y} = e^{x-y} (-1) \\
    \frac{\partial N}{\partial x} = e^{y-x} (-1) \\
    \text{not equal}
    \]
    
    \[ \boxed{B} \]

15. The Existence and Uniqueness Theorem for first-order initial value problems guarantees that the following initial value problem has a unique solution on some interval.
    
    \[
    \frac{dy}{dx} = y\ln x, \quad y(1) = 0
    \]
    
    \[ F = y\ln x \quad \frac{\partial F}{\partial y} = \ln x \quad \text{both continuous around } (1,0) \]
    
    \[ \boxed{A} \]

16. The carrying capacity for the population represented by the following logistic differential equation is 2000.
    
    \[
    \frac{dy}{dx} = (0.8 - 0.01y)y, \quad y(0) = 20
    \]
    
    \[ r = 0.8 \quad a = 0.01 \quad y_0 = 20 \quad \text{carrying capacity: } \frac{a}{r} = 80 \]
    
    \[ \boxed{B} \]

17. Consider the second order differential equation \( y'' + \frac{1}{y^2} = 0 \). If \( y_1 \) and \( y_2 \) are both solutions of this differential equation, then so is \( y_1 + y_2 \).
    
    \[ \boxed{B} \]
    
    The DE is nonlinear, so the Principle of Superposition does not apply.

18. Consider the second order differential equation \( y'' + \frac{1}{y^2} = 0 \). If \( y_1 \) is a solution of this differential equation, then so is \( 2y_1 \).
    
    \[ \boxed{A} \]
    
    The DE is linear and homogeneous, so the Principle of Superposition applies.
Part III. Free-Response

19. (5 points)

Find the general solution of the following differential equation. You do not need to solve for \( y \) or to simplify your answer in any other way. Show your work.

\[
x \sinh 6y \frac{dy}{dx} = \ln x \quad x > 0
\]

Separable

\[
\sinh 6y \, dy = \frac{\ln x}{x} \, dx
\]

\[
u = 6y
\]

\[
du = 6 \, dy
\]

\[
\omega = \ln x
\]

\[
d\omega = \frac{1}{x} \, dx
\]

\[
\frac{1}{6} \int \sinh u \, du = \int \omega \, d\omega
\]

\[
\frac{1}{6} \cosh u = \frac{1}{2} \omega^2 + C
\]

\[
\frac{1}{6} \cosh 6y = \frac{1}{2} (\ln x)^2 + C
\]
20. (10 points)

A tank contains 1000 gal of water in which 200 lbs of salt are initially dissolved. Salt water with a concentration of 3 lbs per gal flows into the tank at the rate of 1 gal per min, and the well-stirred mixture flows out of the tank at the rate of 2 gal per min.

(a) Let $t$ represent time in minutes, and let $y$ represent the amount of salt in lbs. Write the initial value problem which models the above situation. Be sure to give a differential equation AND an initial condition.

$$\frac{dy}{dt} = (3)(1) - \left(\frac{y}{1000-t}\right)(2) \quad y_0 = 200$$

(b) Solve the initial value problem to determine the amount of salt $y(t)$ in the tank at time $t$. Show all the steps needed to arrive at your solution.

$$\frac{dy}{dt} + \frac{2}{1000-t}y = 3 \quad \mu(t) = e^{\int \frac{2}{1000-t} \, dt} = e^{-2\ln(1000-t)} = (1000-t)^{-2}$$

$$(1000-t)^{-2} \frac{dy}{dt} + 2(1000-t)^{-3}y = 3(1000-t)^{-2}$$

$$\int (1000-t)^{-2}y \, dt = \int 3(1000-t)^{-2} \, dt = 3(1000-t)^{-1} + C$$

$$y = 3(1000-t) + C(1000-t)^2$$

$$200 = 3000 + C \cdot 10,000 \quad C = \frac{-2800}{10,000} = -0.028$$

$$y = 3(1000-t) - 0.028(1000-t)^2$$

(c) Determine the concentration of salt in the brine solution at the instant when the tank contains only 100 gal. Include the correct units in your answer.

$$t = 900$$

$$\frac{y(900)}{V(900)} = \frac{3(100) - 0.028(100)^2}{100} = \frac{272}{100} = 2.72 \frac{\text{lbs}}{\text{gal}}$$
21. (5 points)

Consider the following differential equation.

\[ t^2 y'' + ty' - 9y = 0 \quad t > 0 \]

One solution of this differential equation is \( y_1 = t^3 \). (You do not need to verify this.) Use the method of reduction of order to find a second solution \( y_2 \) such that \( \{y_1, y_2\} \) is linearly independent. Show all the steps needed to arrive at your solution. (You do not need to verify that \( y_2 \) is a solution or check the independence. Just find \( y_2 \).)

**Standard form:** \( y'' + \frac{1}{t} y' - \frac{9}{t^2} y = 0 \)

\[
\omega = \frac{e^{\int -\frac{1}{t} \, dt}}{(t^3)^2} = \frac{e^{-\ln t}}{t^6} = \frac{t^{-1}}{t^6} = t^{-7}
\]

\[ v = \int t^{-7} \, dt = -\frac{1}{6} t^{-6} \]

\[ y_2 = \left( -\frac{1}{6} t^{-6} \right) (t^3) = -\frac{1}{6} t^{-3} \]

We may take \( y_2 = -\frac{1}{6} t^{-3} \), or just take \( y_2 = t^{-3} \).