This exam contains twelve multiple-choice problems worth two points each, six true-false problems worth one point each, and three free-response problems worth 20 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, choose the letter corresponding to the only correct answer.

1. Which of the following is a direction field for the differential equation $\frac{dy}{dx} = x^2$?

   (A) ![Diagram A]

   (B) ![Diagram B]

   (C) ![Diagram C]

   (D) ![Diagram D]

There are a number of ways to tell that the first direction field is the correct one. One way is to observe that it is the only one in which all slopes are positive.
2. The general solution of the differential equation \( x^2y'' - 4xy' + 6y = 0 \) is \( y = Ax^2 + Bx^3 \). (You do not need to verify this.) Use this to find the solution \( y \) of the following initial value problem. Then find \( y(2) \).

\[
x^2y'' - 4xy' + 6y = 0 \quad y(1) = -3, \; y'(1) = 0
\]

(A) \(-15\)

(B) \(-12\)

(C) \(-9\)

(D) \(-6\)

(E) \(-3\)

(F) \(3\)

(G) \(6\)

(H) \(9\)

(I) \(12\)

(J) \(15\)

\[\begin{align*}
y &= Ax^2 + Bx^3 \\
y' &= 2Ax + 3Bx^2 \quad \begin{cases} -3 = A + B \\ 0 = 2A + 3B \end{cases} \quad \begin{cases} A = -9 \\ B = 6 \end{cases}
\end{align*}\]

\[y = -9x^2 + 6x^3, \quad y(2) = -36 + 48 = 12\]

3. The following differential equation can be put into the proper form and solved as which of the following types?

\[
[\sin(e^y)] \, dx + [xe^y \cos(e^y) - 3y] \, dy = 0
\]

(I) separable \qquad (II) linear in \( x \) \qquad (III) linear in \( y \) \qquad (IV) exact

(A) none of these

(B) I only

(C) II only

(D) III only

(E) IV only

(F) I and II only

(G) I and III only

(H) I and IV only

(I) II and IV only

(J) III and IV only

\(\begin{align*}
&\text{(I) not separable because of the expression} \\
&x e^y \cos(e^y) - 3y
\end{align*}\)

\(\begin{align*}
&\text{(II) linear in } x \\
&(\pi) \sin(e^y) \frac{dx}{dy} + e^y \cos(e^y) \, x = 3y
\end{align*}\)

\(\begin{align*}
&\text{linear in } x
\end{align*}\)

\(\begin{align*}
&\text{(III) not linear in } y \text{ because of } \sin(e^y) \text{ and other complicated expressions}
\end{align*}\)

\(\begin{align*}
&\text{(IV) exact}
\end{align*}\)
4. Consider the first-order differential equation \( \frac{dy}{dx} = x^{\frac{1}{3}} y^{\frac{1}{3}} \). What is the best conclusion that can be drawn about its solutions from the first Existence and Uniqueness Theorem, (the one which applies to any first-order differential equation in the form \( \frac{dy}{dx} = f(x, y) \))? (The "best" conclusion must be accurate and must be as precise as possible.)

(A) The theorem guarantees that there exists a unique solution through any point \((x, y)\).
(B) The theorem guarantees that there exists a solution through any point \((x, y)\), and it guarantees uniqueness as long as \(x \neq 0\).
(C) The theorem guarantees that there exists a solution through any point \((x, y)\), and it guarantees uniqueness as long as \(y \neq 0\).
(D) The theorem guarantees that there exists a solution through any point \((x, y)\), and it guarantees uniqueness as long as \(x \neq 0\) and \(y \neq 0\).
(E) The theorem guarantees that there exists a unique solution through any point \((x, y)\) as long as \(x \neq 0\).
(F) The theorem guarantees that there exists a unique solution through any point \((x, y)\) as long as \(y \neq 0\).
(G) The theorem guarantees that there exists a unique solution through any point \((x, y)\) as long as \(x \neq 0\) and \(y \neq 0\).

\[ f(x, y) = x^{\frac{1}{3}} y^{\frac{1}{3}}, \quad \text{always continuous} \]

\[ \frac{\partial f}{\partial y} = x^{\frac{1}{3}} \cdot \frac{1}{3} y^{-\frac{2}{3}}, \quad \text{continuous except when } y = 0 \]

5. Consider the following linear initial value problem. \( y' + (\ln x)y = \frac{x-4}{x-10}, \quad y(2) = 7 \)

The Existence and Uniqueness Theorem (the one which applies specifically to linear differential equations) guarantees that this initial value problem has a unique solution. Use this same theorem to determine the largest interval on which this solution is guaranteed to exist.

(A) \((-\infty, 0)\)
(B) \((-\infty, 4)\)
(C) \((-\infty, 10)\)
(D) \((-\infty, \infty)\)
(E) \((0, 4)\)
(F) \((0, 10)\)
(G) \((0, \infty)\)
(H) \((4, 10)\)
(I) \((4, \infty)\)
(J) \((10, \infty)\)
6. Consider the initial value problem \( \frac{dy}{dx} = 2x - y, \ y(0) = 2 \). Use Euler's method with \( h = 0.25 \) to approximate the value of \( y(1) \) to three decimal places.

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(A) 1.081  
(B) 1.109  
(C) 1.164  
(D) 1.188  
(E) 1.207  
(F) 1.250  
(G) 1.266  
(H) 1.313  
(I) 1.454  
(J) 1.472  

7. A rock weighing 4 lbs is projected directly upward with an initial velocity of 600 ft per sec. The rock encounters air resistance equal to one-fourth its instantaneous velocity. Which of the following initial value problems correctly models the motion of the rock?

(A) \( \frac{dv}{dt} - \frac{1}{16}v = -32 \)   
(B) \( \frac{dv}{dt} - \frac{1}{16}v = 32 \)   
(C) \( \frac{dv}{dt} + \frac{1}{16}v = -32 \)   
(D) \( \frac{dv}{dt} + \frac{1}{16}v = 32 \)   
(E) \( \frac{dv}{dt} - 2v = -32 \)   
(F) \( \frac{dv}{dt} - 2v = 32 \)   
(G) \( \frac{dv}{dt} + 2v = -32 \)   
(H) \( \frac{dv}{dt} + 2v = 32 \)
8. A tank contains 1000 gal of a brine solution in which 50 lbs of salt are initially dissolved. A brine solution with concentration 1 lb per gal flows into the tank at a rate of 5 gal per min, and the well-stirred mixture flows out of the tank at a rate of 5 gal per min. How much salt (to the nearest lb) is in the tank after one hour (60 min)?

\[
\frac{dy}{dt} = (1)(5) - \left(\frac{y}{1000}\right)(5) \quad V = 1000
\]
\[
y_0 = 50
\]
\[
\frac{dy}{dt} + 0.005y = 5
\]
\[
\mu(t) = e^{0.005t} = e^{0.005t}
\]
\[
e^{0.005t} \frac{dy}{dt} + 0.005e^{0.005t}y = 5e^{0.005t}
\]
\[
e^{0.005t}y = 1000e^{0.005t} + C
\]
\[
y = 1000 + Ce^{-0.005t}
\]
\[
50 = 1000 + C \quad C = -950
\]
\[
y = 1000 - 950e^{-0.005t}
\]
\[
y(60) = 1000 - 950e^{-0.3}
\]
\[
\approx 296
\]
9. What condition(s) must a differential equation satisfy in order for the Principle of Superposition to apply to its solutions?

(I) It must have order two.
(II) It must be linear.
(III) It must be homogeneous.
(IV) It must have constant coefficients.

(A) I and II only
(B) I and III only
(C) I and IV only
(D) II and III only
(E) II and IV only
(F) III and IV only
(G) I, II, and III only
(H) I, II, and IV only
(I) I, III, and IV only
(J) II, III, and IV only

10. Find the Wronskian of the set \( \{t \cos 3t, tsin 3t\} \).

(A) \( W = 3 \)
(B) \( W = t^2 \)
(C) \( \overline{W = 3t^2} \)
(D) \( W = 9t^2 \)
(E) \( W = t^2(\cos^2 3t - \sin^2 3t) \)
(F) \( W = 3t^2(\cos^2 3t - \sin^2 3t) \)
(G) \( W = 3t^2 + 2t \cos 3t \sin 3t \)
(H) \( W = 9t^2 + 2t \cos 3t \sin 3t \)
(I) \( W = t^2(\cos^2 3t - \sin^2 3t) + 2t \cos 3t \sin 3t \)
(J) \( W = 3t^2(\cos^2 3t - \sin^2 3t) + 2t \cos 3t \sin 3t \)

\[
W = \begin{vmatrix}
    t \cos 3t & tsin 3t \\
    -3t \sin 3t + \cos 3t & 3t \cos 3t + \sin 3t
\end{vmatrix}
\]

\[
W = \left(3t^2 \cos^2 3t + t \cos 3t \sin 3t \right) - \left(-3t^2 \sin^2 3t + t \cos 3t \sin 3t \right)
\]

\[
= 3t^2 \cos^2 3t + 3t^2 \sin^2 3t
\]

\[
= 3t^2 \left(\cos^2 3t + \sin^2 3t \right)
\]

\[
= 3t^2
\]
11. A fundamental set of solutions for a certain second-order, linear, homogeneous differential equation is \{\cos 2t, \sin 2t\}. How many of the following functions are solutions of this differential equation?

\[
\begin{align*}
\text{(I)} & \quad y = 0 & \text{(II)} & \quad y = 3\cos 2t & \text{(III)} & \quad y = -8\sin 2t \\
\text{(IV)} & \quad y = \cos 2t + \sin 2t & \text{(V)} & \quad y = \cos 2t - \sin 2t & \text{(VI)} & \quad y = \cos 2t \cdot \sin 2t \\
\text{(VII)} & \quad y = \frac{\cos 2t}{\sin 2t} & \text{(VIII)} & \quad y = \cos^3 2t \cdot \sin^2 2t & \text{(IX)} & \quad y = 4\cos 2t + 9\sin 2t
\end{align*}
\]

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5
(G) 6
(H) 7
(I) 8
(J) 9

Any linear combination of \cos 2t and \sin 2t is a solution; anything else is not.

12. The function \(y_1 = t^2\) is a solution of the following linear homogeneous differential equation.

\[y'' - \frac{3}{t} y' + \frac{4}{t^3} y = 0 \quad t > 0\]

(You do not need to verify this.)

Use reduction of order to find a second solution \(y_2\) of the differential equation such that \(\{y_1, y_2\}\) is a linearly independent set.

\[
\begin{align*}
\omega &= e^{-\int -\frac{3}{t} \, dt} = e^{\frac{3\ln t}{t^4}} = \frac{3}{t^4} = t^{-1} \\
\omega &= e^{-\int -\frac{3}{t} \, dt} = e^{\frac{3\ln t}{t^4}} = \frac{3}{t^4} = t^{-1} \\
V &= \int t^{-1} \, dt = \ln t \\
y_2 &= t^2 \cdot \ln t \\
y_2 &= t^2 \cdot \ln t
\end{align*}
\]

(A) \(y_2 = -\frac{1}{2}\)
(B) \(y_2 = -\frac{1}{6} t^{-6}\)
(C) \(y_2 = -\frac{1}{3} t^{-4}\)
(D) \(y_2 = -\frac{1}{2} t^{-2}\)
(E) \(y_2 = -\frac{1}{3} t^2\)
(F) \(y_2 = \frac{1}{4} t^4\)
(G) \(y_2 = \frac{1}{4} t^6\)
(H) \(y_2 = \ln t\)
(I) \(y_2 = t^{-2}\ln t\)
(J) \(y_2 = t^2 \ln t\)
Part II. True-False (1 point each)

Mark your answer card “A” if the statement is true and “B” if the statement is false.

13. The function \( y = te^{2t} \) is a solution of the differential equation \( y'' - 4y' + 4y = 0 \).

\[
\begin{align*}
  y &= te^{2t} \\
  y' &= 2te^{2t} + e^{2t} \\
  y'' &= 4te^{2t} + 2e^{2t} + 2e^{2t} \\
  \left[4te^{2t} + 4e^{2t}\right] - 4\left[2te^{2t} + e^{2t}\right] + 4\left[te^{2t}\right] = 0 
\end{align*}
\]

\[\checkmark\]

14. The function \( y = x - 1 \) is a solution of the differential equation represented by the direction field shown below. (You may accept this first statement as true; the next statement is the one which you must judge.) This solution \( y = x - 1 \) is a stable equilibrium solution.

\[\text{(Diagram)}\]

An equilibrium solution is a solution which is a constant function.

15. If the general solution of a differential equation is given by \( \frac{y}{x} = e^{3x} + C \), then the general solution can also be written correctly as \( y = xe^{3x} + Cx \).

It should be \( y = xe^{3x} + Cx \).

\[\checkmark\]
16. If the general solution of a differential equation is given by \( \ln|y| = x^2 + C \), then the general solution can also be written correctly as \( y = \pm e^C e^{x^2} \).

\[
\begin{align*}
\ln|y| & = x^2 + C \\
\left|y\right| & = e^{x^2+C} = e^{x^2} e^C \\
y & = \pm e^C e^{x^2}
\end{align*}
\]

17. If the general solution of a differential equation is given by \( y = Ae^{-2\ln|x|} \), then the general solution can also be written correctly as \( y = -2A|x| \).

It should be \( y = A \cdot x^{-2} \).

18. The set \( \{e^t, e^{2t}\} \) is a fundamental set of solutions of the linear, homogeneous differential equation \( y'' - 3y' + 2y = 0 \). (You may accept this first statement as true; the next statement is the one which you must judge.) The set \( \{e^t + e^{2t}, e^t - e^{2t}\} \) is also a fundamental set of solutions for this differential equation.

A) correct number of functions? yes - there are two
B) each function a solution? yes - each is a linear combination of known solutions
C) linearly independent? yes - neither is a constant multiple of the other (or check the Wronskian)
Part III. Free-Response

19. (6 points)

The following differential equation is both linear and separable. Find its general solution using the technique for a linear differential equation. Show all the steps needed to arrive at your solution, and be sure to solve for $y$ in your final answer.

$$(x^2 + 4) \frac{dy}{dx} + 3xy = x$$

$$\frac{dy}{dx} + \frac{3x}{x^2+4} y = \frac{x}{x^2+4}$$

$$\mu(x) = e^{\int \frac{3x}{x^2+4} \, dx} = e^{\frac{3}{2} \ln(x^2+4)} = (x^2+4)^{3/2}$$

$$u = x^2+4 \quad du = 2x \, dx$$

$$u = x^2+4 \quad du = 2x \, dx$$

$$(x^2+4)^{3/2} \frac{dy}{dx} + 3x \frac{(x^2+4)^{1/2}}{y} = x \frac{(x^2+4)^{1/2}}{x}$$

$$(x^2+4)^{3/2} y = \int x (x^2+4)^{1/2} \, dx$$

$$= \frac{1}{2} \cdot \frac{2}{3} (x^2+4)^{3/2} + C$$

$$y = \frac{1}{3} + C (x^2+4)^{-3/2}$$
20. (7 points)

The following differential equation is exact. (You do not need to verify this.) Find its general solution. Show all the steps needed to arrive at your solution. You do not need to solve for $y$ or to simplify your solution in any other way.

$$[-y\cos(y-x)+x+y] \, dx + [\sin(y-x)+y\cos(y-x)+x+y] \, dy = 0$$

$$F = \int [-y\cos(y-x)+x+y] \, dx$$

$$= y\sin(y-x)+\frac{1}{2}x^2+xy+g(y)$$

$u = y-x$

$du = -dx$

$$\frac{\partial F}{\partial y} = y\cos(y-x)+\sin(y-x)+x+g'(y)$$

$g'(y) = y$

$$g(y) = \frac{1}{2}y^2$$

$$F = y\sin(y-x)+\frac{1}{2}x^2+xy+\frac{1}{2}y^2$$

$y\sin(y-x)+\frac{1}{2}x^2+xy+\frac{1}{2}y^2 = C$
21. (7 points)

Determine the time of death (to the nearest minute) if a corpse is at 90.0°F when discovered at 8:00 a.m. and 86.0°F one hour later. (Assume that room temperature is 70.0°F, and note that normal body temperature is 98.6°F.)

\[ T - T_s = (T_0 - T_s) e^{kt} \]

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<td>?</td>
<td>98.6</td>
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\[ T - 70 = 20e^{kt} \]

\[ 16 = 20e^{k \cdot 1} \]

\[ k = \ln .8 \]

\[ T - 70 = 20e^{(\ln .8)t} \]

\[ 28.6 = 20e^{(\ln .8)t} \]

\[ 1.43 = e^{(\ln .8)t} \]

\[ \ln 1.43 = (\ln .8)t \]

\[ t = \frac{\ln 1.43}{\ln .8} \approx -1.60289 \text{ hr.} \]

\[ -1 \text{ hr. 36 min.} \]

\[ 6:24 \text{ a.m.} \]