Differential Equations (Math 217)  
Exam 2  
October 25, 2005

This exam contains twelve multiple-choice problems worth two points each, six true-false problems worth one point each, and two free-response problems worth 20 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, choose the letter corresponding to the only correct answer.

1. The roots of the differential equation \( y^{(4)} + 8y'' + 16y = 0 \) are \( r = \pm 2i, \pm 2i \). (You do not need to verify this.) What is the general solution of this differential equation?

(A) \( y = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 2t + c_4 \sin 2t \)

(B) \( y = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t \)

(C) \( y = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t \cos 2t + c_4 e^t \sin 2t \)

(D) \( y = c_1 t \cos 2t + c_2 t \sin 2t + c_3 e^t \cos 2t + c_4 e^t \sin 2t \)

(E) \( y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + c_3 e^t \cos 2t + c_4 e^t \sin 2t \)

(F) \( y = c_1 \cos (2\ln t) + c_2 \sin (2\ln t) + c_3 \cos (2\ln t) + c_4 \sin (2\ln t) \)

(G) \( y = c_1 \cos (2\ln t) + c_2 \sin (2\ln t) + c_3 t \cos (2\ln t) + c_4 t \sin (2\ln t) \)

(H) \( y = c_1 \cos (2\ln t) + c_2 \sin (2\ln t) + c_3 (\ln t) \cos (2\ln t) + c_4 (\ln t) \sin (2\ln t) \)

(I) \( y = c_1 t \cos (2\ln t) + c_2 t \sin (2\ln t) + c_3 (\ln t) \cos (2\ln t) + c_4 (\ln t) \sin (2\ln t) \)

(J) \( y = c_1 (\ln t) \cos (2\ln t) + c_2 (\ln t) \sin (2\ln t) + c_3 t (\ln t) \cos (2\ln t) + c_4 t (\ln t) \sin (2\ln t) \)

2. Consider the differential equation \( y^{(4)} - 3y'' - 2y' + 12y = 0 \). Find all the roots of the characteristic equation. What is their sum?

(A) -5

(B) -4

(C) -3

(D) -2

(E) -1

(F) 1

(G) 2

(H) 3

(I) 4

(J) 5

\[ r^4 - 3r^3 - 2r^2 + 2r + 12 = 0 \]

\[
\begin{array}{c|cccc}
2 & 1 & -3 & -2 & 2 & 12 \\
 & 2 & -2 & -8 & -12 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
3 & 1 & -1 & -4 & -6 & 0 \\
 & 3 & 6 & 6 & 0 \\
\end{array}
\]

\[ r^2 + 2r + 2 = 0 \]

\[ r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i \]

\[ 2 + 3 + (-1 + i) + (-1 - i) = 3 \]
3. Suppose \( y_1 \) is a solution of a nonhomogeneous linear differential equation, and \( y_2 \) is a solution of the corresponding homogeneous equation. Which of the following statements is/are true?

- (I) \( -y_1 \) is a solution of the nonhomogeneous equation.  \( F \)
- (II) \( -y_2 \) is a solution of the homogeneous equation.  \( T \)
- (III) \( y_1 - y_2 \) is a solution of the nonhomogeneous equation.  \( T \)
- (IV) \( y_2 - y_1 \) is a solution of the homogeneous equation.  \( F \)

(A) I only  
(B) II only  
(C) III only  
(D) IV only  
(E) I and II only  
(F) I and III only  
(G) I and IV only  
(H) II and III only  
(I) II and IV only  
(J) III and IV only

- (I) The principle of superposition does not apply.  
- (II) The principle of superposition applies.  
- (III) Every solution of the nonhomogeneous DE is the sum of \( y_1 \) and some solution of the homogeneous DE.  
- (IV) Since \( -y_1 \) is a solution of neither, \( y_2 - y_1 \) is a solution of neither.

4. Suppose you were to use the method of undetermined coefficients to find a particular solution of the following nonhomogeneous linear differential equation.

\[ y'' - 3y' + 2y = e^t \sin 6t + 8t \]

How many terms would be in the correct expression for the form of \( y_p \)?

- (A) 1  
- (B) 2  
- (C) 3  
- (D) 4  
- (E) 5  
- (F) 6  
- (G) 7  
- (H) 8  
- (I) 9  
- (J) 10
5. Find a particular solution of the following nonhomogeneous differential equation.

\[ y'' - 2y' + y = 3e^t \]

(A) \( y_p = 0 \)  
(B) \( y_p = e^t \)  
(C) \( y_p = -2e^t \)  
(D) \( y_p = 3e^t \)  
(E) \( y_p = te^t \)  
(F) \( y_p = -\frac{1}{3}te^t \)  
(G) \( y_p = 2te^t \)  
(H) \( y_p = t^2e^t \)  
(I) \( y_p = -3t^2e^t \)  
(J) \( y_p = \frac{3}{2}t^2e^t \)

\[ y_p = Ate^t \]

\[ y_p' = At^2e^t + 2Ate^t \]

\[ y_p'' = At^2e^t + 2Ae^t + 2Ate^t + 2Ae^t \]

\[ 2Ae^t = 3e^t \quad A = \frac{3}{2} \]

6. The motion of a spring system is described by the differential equation \( \frac{d^2x}{dt^2} + 16x = 0 \), where \( x \) is measured in feet and \( t \) is measured in seconds. The object is set into motion from equilibrium with an initial upward velocity of 8 ft/sec. After how many seconds will the object first return to equilibrium?

\( x = \pm 4t \)

\( x = c_1 \cos 4t + c_2 \sin 4t \quad x_0 = 0, \quad v_0 = -8 \)

(A) \( \frac{\pi}{16} \)  
(B) \( \frac{\pi}{8} \)  
(C) \( \frac{\pi}{4} \)  
(D) \( \frac{\pi}{2} \)  
(E) \( \pi \)  
(F) \( 2\pi \)  
(G) \( 4\pi \)  
(H) \( 8\pi \)  
(I) \( 16\pi \)

\( v = 4c_2 \cos 4t \)

\(-8 = 4c_2 \quad c_2 = -2 \)

\( x = -2 \sin 4t \)

\( 0 = -2 \sin 4t \)

\( 4t = \pi \quad t = \frac{\pi}{4} \)
7. A 16-lb object stretches a spring 1.6 ft. If the motion of the spring system is damped, where the resistive force is \( F_R = 2 \frac{dx}{dt} \), find the quasiperiod of the motion (in seconds).

(A) \( \pi \)

\[ 16 = m \cdot 32 \cdot \frac{1}{2} \quad 16 = k (l \cdot 6) \quad k = 10 \]

(B) \( \frac{\pi}{4} \)

\[ \frac{1}{2} \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 10x = 0 \quad \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 20x = 0 \]

(C) \( \frac{\pi}{3} \)

\[ \lambda^2 + 4\lambda + 20 = 0 \quad \lambda = -2 \pm \frac{2 \sqrt{16 - 80}}{2} = -2 \pm 4i \]

(D) \( \frac{\pi}{2} \)

\[ x = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t \]

(E) \( \frac{3\pi}{4} \)

\[ \text{quasiperiod: } \frac{2\pi}{4} \]

(F) \( \pi \)

\[ \text{quasiperiod: } 2\pi/4 \]

(G) \( 3\pi \)

\[ \text{The quasiperiod cannot be determined since initial conditions are not given.} \]

8. The motion of a spring system is described by the equation \( \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 8x = 0 \), where \( x \) is measured in feet and \( t \) is measured in seconds. The object is set into motion from equilibrium with an initial downward velocity of 8 ft/sec. Find the maximum displacement of the object (in feet).

(A) \( \frac{1}{2} \)

\[ \lambda^2 + 6\lambda + 8 = 0 \quad \lambda = -2, -4 \]

\[ x = c_1 e^{-2t} + c_2 e^{-4t} \quad v = -2c_1 e^{-2t} - 4c_2 e^{-4t} \]

(B) \( 1 \)

\[ 0 = c_1 + c_2 \quad 8 = -2c_1 - 4c_2 \quad c_1 = 4 \quad c_2 = -4 \]

(C) \( 2 \)

\[ x = 4e^{-2t} - 4e^{-4t} \quad v = -8e^{-2t} + 16e^{-4t} \]

(D) \( 4 \)

\[ 0 = -8e^{-2t} + 16e^{-4t} \quad 8e^{-2t} = 16e^{-4t} \]

(E) \( \frac{1}{2}e^2 \)

\[ e^{2t} = 2 \quad t = \frac{1}{2} \ln 2 \]

(F) \( e^2 \)

\[ e^{2t} = 2 \quad t = \frac{1}{2} \ln 2 \]

(G) \( 2e^2 \)

\[ x(\frac{1}{2} \ln 2) = 4e^{-\ln 2} - 4e^{-2\ln 2} \]

\[ = 4(\frac{1}{2}) - 4(\frac{1}{4}) \]

\[ = 1 \]
9. Consider the following differential equation. Classify the point \( x_0 = 1 \).

\[(x - 1)y'' + 2y' + \frac{1}{(x-1)^2}y = 0\]

\[(A)\] \( x_0 = 1 \) is an ordinary point of the differential equation.

\[(B)\] \( x_0 = 1 \) is a regular singular point of the differential equation.

\[(C)\] \( x_0 = 1 \) is an irregular singular point of the differential equation.

\[(D)\] none of the above

\[
\text{standard form: } y'' + \frac{2}{x-1}y' + \frac{1}{(x-1)^2}y = 0
\]

\( x_0 = 1 \) is a singular point.

\( (x-1)p(x) = 2 \)

\( (x-1)^2 q(x) = \frac{1}{x-1} \)

\( x_0 = 1 \) is an irregular singular point.

10. Consider the differential equation \((x^2 - 9)y'' + (x - 3)y' - 6y = 0\). Determine a lower bound for the radius of convergence of power series solutions about the ordinary point \( x_0 = 1 \).

\[(A)\] 0

\[(B)\] 1

\[(C)\] 2

\[(D)\] 3

\[(E)\] 4

\[(F)\] 5

\[(G)\] 6

\[(H)\] 7

\[(I)\] 8

\[(J)\] \( \infty \)
11. Which of the following differential equations would require a power series solution (as opposed to a solution obtained through simpler means)?

(I) $5y'' - 2y' - 7y = 0$ \textit{constant coefficients}
(II) $x^2y'' - 2xy' - 4y = 0$ \textit{Cauchy-Euler}
(III) $xy'' + 2y' + xy = 0$

(A) I only \\
(B) II only \\
(C) III only \\
(D) I and II only \\
(E) I and III only \\
(F) II and III only \\
(G) I, II, and III \\
(H) none of these \\

12. Find the indicial roots of the following differential equation. (Note that $x_0 = 0$ is a regular singular point.)

$$xy'' + (x - 1)y' + 3y = 0$$

$$y'' + \frac{x-1}{x} y' + \frac{3}{x} y = 0$$

$$x \rho(x) = x - 1 \quad x^2 q(x) = 3x$$

$$\rho_0 = -1 \quad q_0 = 0$$

$$\lambda(\lambda - 1) - \lambda = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda = 0, 2$$
Part II. True-False (1 point each)

Mark your answer card "A" if the statement is true and "B" if the statement is false.

13. The set of functions $S = \{1 - t^2, 1 + t^2, 1\}$ is linearly independent.
   \[
   W = \begin{vmatrix}
   1 - t^2 & 1 + t^2 & 1 \\
   -2t & 2t & 0 \\
   -2 & 2 & 0
   \end{vmatrix} = (-4t) - (-4t) = 0
   
   F
   
14. A set containing four functions can never be a fundamental set of solutions for a third-order linear homogeneous differential equation.
   
   The numbers of functions in a FSOS must match the order of the DE.
   
   T

15. The spring motion represented by the differential equation $\frac{d^2x}{dt^2} + x = 0$ has a greater frequency than the spring motion represented by the differential equation $\frac{d^2x}{dt^2} + x = 0$.
   
   When the mass increases, the frequency decreases.
   
   F

16. The spring motion represented by the differential equation $\frac{d^2x}{dt^2} + 9x = 0$ has a greater frequency than the spring motion represented by the differential equation $\frac{d^2x}{dt^2} + x = 0$.
   
   When the spring constant increases, the frequency increases.
   
   T

17. The spring motion represented by the differential equation $\frac{d^2x}{dt^2} + 4x = 2\cos 4t$ exhibits the phenomenon of resonance.
   
   $r^2 + 4 = 0 \implies r = \pm 2i
   
   x(t) = c_1 \cos 2t + c_2 \sin 2t
   
   \text{natural frequency: 2 \ forced frequency: 4}$
   
   F

18. The spring motion represented by the differential equation $\frac{1}{4} \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 8x = 0$ is underdamped.
   
   $c^2 - 4mk = 16 - 8 = 8 > 0$ overdamped
   
   F
Part III. Free-Response

19. (10 points)

Find the general solution of the following nonhomogeneous Cauchy-Euler differential equation. Use variation of parameters. Show all the steps needed to arrive at your solution, and simplify your final answer completely. In particular, if any terms can be absorbed at the end, go ahead and let them be absorbed.

\[ x^2y'' - 5xy' + 9y = x^6 \quad x > 0 \]

\[
x^2y'' - 5xy' + 9y = 0
\]

\[
\lambda (\lambda - 1) - 5\lambda + 9 = 0 \quad \lambda^2 - 6\lambda + 9 = 0 \quad \lambda = 3, 3
\]

\[
y_1 = x^3 \quad y_2 = x^3 \ln x
\]

\[
W = \begin{vmatrix}
    x^3 & x^3 \ln x \\
    3x^2 & x^2 + 3x^2 \ln x
\end{vmatrix} = x^5
\]

standard form: \[ y'' - \frac{5}{x}y' + \frac{9}{x^2}y = x^4 \]

\[
u_1' = \frac{(-x^3 \ln x)(x^4)}{x^5} = -x^2 \ln x
\]

\[
u_2' = \frac{(x^3)(x^4)}{x^5} = x^2 \quad u_2 = \frac{1}{3} x^3
\]

\[
u = \ln x \quad dv = -x^2 dx
\]

\[
udu = x^{-1} dx \quad v = -\frac{1}{3} x^3
\]

\[
y_p = u_1y_1 + u_2y_2 = \left[ -\frac{1}{3} x^3 \ln x + \frac{1}{9} x^3 \right] [x^3] + \left[ \frac{1}{3} x^3 \right] [x^3 \ln x]
\]

\[
y = c_1 x^3 + c_2 x^3 \ln x + \frac{1}{9} x^4
\]
20. (10 points)

Consider the following differential equation: \( x^2 y'' - x^2 y' - 2y = 0 \) for \( x > 0 \).

The point \( x_0 = 0 \) is a regular singular point of this equation. Furthermore, there is a Frobenius series solution of the form \( y = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2} \). (You do not have to verify these facts.)

Find such a solution. (Your answer should be a specific solution with no arbitrary constants.)

Work out the coefficients through \( a_3 \). Show all the steps needed to arrive at your solution.

\[
\begin{align*}
&y = \sum_{n=0}^{\infty} a_n x^{n+2} \quad y' = \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n \\
&x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n - x^2 \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\
&\sum_{n=0}^{\infty} (n+2)(n+1) a_n x^{n+2} - \sum_{n=0}^{\infty} (n+2)a_n x^{n+3} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\
&\sum_{n=0}^{\infty} (n+2)(n+1) a_n x^{n+2} - \sum_{n=1}^{\infty} (n+1) a_{n-1} x^{n+2} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\
&2a_0 x^2 - 2a_0 x^2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_n x^{n+2} - \sum_{n=1}^{\infty} (n+1) a_{n-1} x^{n+2} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\
&\sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_n - (n+1) a_{n-1} - 2a_n \right] x^{n+2} = 0
\end{align*}
\]

For \( n \geq 1 \), \( (n+2)(n+1) a_n - (n+1) a_{n-1} - 2a_n = 0 \)

For \( n \geq 1 \), \( (n^2 + 3n + 2 - 2) a_n = (n+1) a_{n-1} \)

For \( n \geq 1 \), \( a_n = \frac{n+1}{n(n+3)} a_{n-1} \)

\[
\begin{align*}
n = 1 : & \quad a_1 = \frac{1}{2} a_0 \\
n = 2 : & \quad a_2 = \frac{3}{10} a_1 = \frac{3}{20} a_0 \\
n = 3 : & \quad a_3 = \frac{2}{9} a_2 = \frac{1}{30} a_0 \\
\end{align*}
\]

\[
y = x^2 \sum_{n=0}^{\infty} a_n x^n = x^2 \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right) \\
= x^2 \left( a_0 + \frac{1}{2} a_0 x + \frac{3}{20} a_0 x^2 + \frac{1}{30} a_0 x^3 + \cdots \right)
\]

Let \( a_0 = 1 \).

\[
y = x^2 \left( 1 + \frac{1}{2} x + \frac{3}{20} x^2 + \frac{1}{30} x^3 + \cdots \right)
\]