This exam contains eleven multiple-choice problems worth two points each, eight true-false problems worth one point each, and three free-response problems worth 20 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, choose the letter corresponding to the only correct answer.

1. Legendre's equation is as follows.

\[(1 - x^2)y'' - 2xy' + k(k + 1)y = 0\]

Its general solution in power series about the ordinary point \(x_0 = 0\) is as follows.

\[y = a_0 \left[ 1 - \frac{k(k+1)}{2!} x^2 + \frac{k(k+1)(k-2)(k+3)}{4!} x^4 + \frac{k(k+1)(k-2)(k+3)(k-4)(k+5)}{6!} x^6 + \ldots \right] + a_1 \left[ x - \frac{(k-1)(k+2)}{3!} x^3 + \frac{(k-1)(k+2)(k-3)(k+4)}{5!} x^5 + \frac{(k-1)(k+2)(k-3)(k+4)(k-5)(k+6)}{7!} x^7 + \ldots \right]\]

(You do not need to verify this.)

Use this information to find the Legendre polynomial \(P_4(x)\) of degree four.

(A) \(y = 1 - 10x^2\)
(B) \(y = -\frac{1}{9} (1 - 10x^2)\)
(C) \(y = 1 - 10x^2 + \frac{35}{3} x^4\)
(D) \(y = \frac{3}{8} (1 - 10x^2 + \frac{35}{3} x^4)\)
(E) \(y = \frac{3}{5} (1 - 10x^2 + \frac{35}{3} x^4)\)
(F) \(y = 1 - 10x^2 + \ldots\)
(G) \(y = -\frac{1}{9} (1 - 10x^2) + \ldots\)
(H) \(y = 1 - 10x^2 + \frac{35}{3} x^4 + \ldots\)
(I) \(y = \frac{3}{8} (1 - 10x^2 + \frac{35}{3} x^4) + \ldots\)
(J) \(y = \frac{3}{5} (1 - 10x^2 + \frac{35}{3} x^4) + \ldots\)

(Note that the five choices without the dots indicate that the series terminates, and the five choices with the dots indicate that there are further nonzero terms.)
2. Consider the following differential equation. \[ x^2 y'' - xy' + (x^2 + 1)y = 0 \]

During the process of finding a series solution about the regular singular point \( x_0 = 0 \), the following information is derived. (You do not need to verify this information.)

\[ a_n = 0 \text{ for all odd } n. \]

For \( n \geq 2 \), \[ a_n = \frac{(-1)^{n-1}}{n^2} a_{n-2} \]

Set \( n = 2m \), and find a general formula for \( a_n = a_{2m} \). (Take \( a_0 = 1 \).)

(A) \[ a_{2m} = \frac{(-1)^m}{m!} \]

(B) \[ a_{2m} = \frac{(-1)^m}{(2m)!} \]

(C) \[ a_{2m} = \frac{(-1)^m}{2^m(m!)^2} \]

(D) \[ a_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2} \]

(E) \[ a_{2m} = \frac{(-1)^m}{(m!)^2} \]

(F) \[ a_{2m} = \frac{(-1)^m}{((2m)!)^2} \]

(G) \[ a_{2m} = \frac{(-1)^m}{2^m(m!)^2} \]

(H) \[ a_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2} \]

3. Find the inverse of the matrix \( \mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 5 \\ -1 & 0 & 1 \end{pmatrix} \). What is the element in the third row and third column (lower right corner) of \( \mathbf{A}^{-1} \)?

(A) \(-4\)

(B) \(-3\)

(C) \(-2\)

(D) \(-1\)

(E) \(0\)

(F) \(1\)

(G) \(2\)

(H) \(3\)

(I) \(4\)

(J) \( \mathbf{A} \) does not have an inverse.
4. The following system of first-order differential equations can be transformed into a single second-order differential equation. What is this new differential equation?

\[
\begin{align*}
\begin{cases}
x' &= 4x + 2y \\
y' &= -x + y
\end{cases}
\end{align*}
\]

(A) \(x'' - 5x' - 2x = 0\)
(B) \(x'' - 5x' + 6x = 0\)
(C) \(x'' - 3x' - 2x = 0\)
(D) \(x'' - 3x' + 6x = 0\)
(E) \(x'' + 3x' - 6x = 0\)
(F) \(x'' + 3x' + 2x = 0\)
(G) \(x'' + 5x' - 6x = 0\)
(H) \(x'' + 5x' + 2x = 0\)

5. Only one of the following vector functions is a solution of the system \(X' = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} X\).

Which one is it? (Hint: There are at least three different ways to figure out the answer. With a little reflection, you may be able to choose a way which is not too time-consuming.)

(A) \(X = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t\)
(B) \(X = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}\)
(C) \(X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t}\)
(D) \(X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t\)
(E) \(X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}\)
(F) \(X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}\)
(G) \(X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t\)
(H) \(X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{2t}\)
(I) \(X = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{3t}\)
6. The matrix \( A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \) has an eigenvalue \( \lambda = 3 \) of multiplicity two; however, this eigenvalue does not have two linearly independent eigenvectors corresponding to it. One eigenvector corresponding to \( \lambda = 3 \) is \( v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), so that one solution of the system \( X' = AX \) is \( X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \). (You do not need to verify any of this information.) Find a second solution \( X_2 \) of the system \( X' = AX \) so that \( X_1 \) and \( X_2 \) are linearly independent.

(A) \( X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} \)

(B) \( X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{3t} \)

(C) \( X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^{3t} \)

(D) \( X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} t e^{3t} \)

(E) \( X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^{3t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \)

(F) \( X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} t e^{3t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \)

(G) \( X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} \)

(H) \( X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{3t} \)

7. Which of the following nonhomogeneous systems can be solved using the method of undetermined coefficients?

(I) \( X' = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ \ln t \end{pmatrix} \)

(II) \( X' = \begin{pmatrix} t & 1 \\ -1 & -t \end{pmatrix} X + \begin{pmatrix} 2e^{-2t} \\ 3e^{-3t} \end{pmatrix} \)

(III) \( X' = \begin{pmatrix} 4 & t^2 \\ t^2 & 2 \end{pmatrix} X + \begin{pmatrix} \sin t \\ \tan t \end{pmatrix} \)

(A) I only 

(B) II only 

(C) III only 

(D) I and II only 

(E) I and III only 

(F) II and III only 

(G) I, II, and III 

(H) none of these
8. Use the method of variation of parameters to find a particular solution $X_p$ of the following nonhomogeneous system.

$$X' = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 0 \\ t^{-1} \end{pmatrix}, \quad t > 0$$

Hint: $X_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2t + 1 \\ t \end{pmatrix}$  \hspace{1cm} (You do not need to verify this.)

(A) $\begin{pmatrix} 2t^{-1} \\ t^{-1} \end{pmatrix}$

(B) $\begin{pmatrix} -2t^{-1} \\ -t^{-1} \end{pmatrix}$

(C) $\begin{pmatrix} -t \ln t + t \\ \ln t \end{pmatrix}$

(D) $\begin{pmatrix} t \ln t - t \\ -\ln t \end{pmatrix}$

(E) $\begin{pmatrix} 4t - 4t \ln t \\ 2t + \ln t - 2t \ln t \end{pmatrix}$

(F) $\begin{pmatrix} -4t + 4t \ln t \\ -2t - \ln t + 2t \ln t \end{pmatrix}$

(G) $\begin{pmatrix} -2t^2 - 5t \ln t - 2t \ln t \\ 2t + 5t \ln t \end{pmatrix}$

(H) $\begin{pmatrix} 2t^2 + 5t \ln t + 2t \ln t \\ -2t - 5t \ln t \end{pmatrix}$

(I) $\begin{pmatrix} 5t + 2t \ln t + 2t^2 \\ 2t + \ln t + t^2 \end{pmatrix}$

(J) $\begin{pmatrix} -5t - 2t \ln t - 2t^2 \\ -2t - \ln t - t^2 \end{pmatrix}$
9. The following is the general solution of a linear homogeneous system of differential equations. Which of the pictures below shows the correct phase portrait for this system?

\[ \mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \]
10. The following is the general solution of a linear homogeneous system of differential equations. Which of the pictures below shows the correct phase portrait for this system?

\[
\mathbf{X} = c_1 e^{-t} \begin{pmatrix} 2 \cos 2t - \sin 2t \\ -2 \cos 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \sin 2t + \cos 2t \\ -2 \sin 2t \end{pmatrix}
\]
11. Consider the changing population of a certain species in three neighboring territories.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$: time</td>
<td>$a_1$: birth rate for $x$</td>
</tr>
<tr>
<td>$x$: population in first territory</td>
<td>$b_1$: birth rate for $y$</td>
</tr>
<tr>
<td>$y$: population in second territory</td>
<td>$c_1$: birth rate for $z$</td>
</tr>
<tr>
<td>$z$: population in third territory</td>
<td>$a_2$: rate at which individuals move from the first territory to the second</td>
</tr>
<tr>
<td></td>
<td>$a_3$: rate at which individuals move from the first territory to the third</td>
</tr>
<tr>
<td></td>
<td>$b_2$: rate at which individuals move from the second territory to the first</td>
</tr>
<tr>
<td></td>
<td>$b_3$: rate at which individuals move from the second territory to the third</td>
</tr>
<tr>
<td></td>
<td>$c_2$: rate at which individuals move from the third territory to the first</td>
</tr>
<tr>
<td></td>
<td>$c_3$: rate at which individuals move from the third territory to the second</td>
</tr>
</tbody>
</table>

Find the differential equation which models the rate of change of the population in the first territory over time.

(A) $\frac{dx}{dt} = a_1x - a_2x - a_3x + b_2y + c_2z$

(B) $\frac{dx}{dt} = a_1x + a_2x + a_3x - b_2y - c_2z$

(C) $\frac{dx}{dt} = a_1x - a_2x - b_3x + b_2y + c_2z$

(D) $\frac{dx}{dt} = a_1x + a_2x + a_3x - b_2x - c_2z$

(E) $\frac{dx}{dt} = a_1x - a_2x - a_3x + b_2y + c_2z + b_1b_2y + c_1c_2z$

(F) $\frac{dx}{dt} = a_1x + a_2x + a_3x - b_2y - c_2z - b_1b_2y - c_1c_2z$

(G) $\frac{dx}{dt} = a_1x - a_2x - a_3x + b_2x + c_2x + b_1b_2x + c_1c_2x$

(H) $\frac{dx}{dt} = a_1x + a_2x + a_3x - b_2x - c_2x - b_1b_2x - c_1c_2x$
Part II. True-False (1 point each)

Mark your answer card “A” if the statement is true and “B” if the statement is false.

12. For any real number \( k \), the interval of convergence of the series solutions of Legendre's equation is \((-\infty, \infty)\). (If it helps, you may refer to problem 1.)

13. Consider the system of algebraic equations given in matrix form by \( Ax = b \). If \( A \) is nonsingular, then the system has a unique solution.

14. The following set of vectors is linearly independent.

\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\}
\]

15. If \( \lambda = 3 \) is an eigenvalue of the matrix \( \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \), then

\[
\det \begin{pmatrix} a_{11} - 3 & a_{12} & a_{13} \\ a_{21} & a_{22} - 3 & a_{23} \\ a_{31} & a_{32} & a_{33} - 3 \end{pmatrix} = 0.
\]
16. $v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ -14 & -2 & -7 & 11 & -9 & -8 \\ -9 & -3 & -3 & 7 & -6 & -4 \\ -19 & -5 & -9 & 17 & -12 & -9 \\ -29 & -7 & -13 & 23 & -16 & -15 \\ 19 & 5 & 9 & -15 & 12 & 11 \end{pmatrix}$.

17. If an eigenvalue $\lambda$ has multiplicity $m$, it will always have $m$ linearly independent eigenvectors associated with it.

18. The following set of vector functions is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{4t}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{4t}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} \right\}$$

19. The matrix $\begin{pmatrix} 3 & -2e^{-5t} \\ 1 & e^{-5t} \end{pmatrix}$ is a fundamental matrix for the system $X' = \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} X$. (This first statement is true; you do not need to verify it. The second statement is the one which you must judge.) The general solution of the system $X' = \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} X$ is therefore given by $X = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix}$. 
Part III. Free-Response

20. (8 points)

Consider the following homogeneous linear system. Find the general solution (involving real-valued functions only) of this system. Show all the steps needed to arrive at the solution. You do not need to simplify your answer.

\[
\begin{pmatrix}
-1 & -2 \\
4 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
x'
\end{pmatrix}
\]
21. (11 points)

Consider the nonhomogeneous system \( \mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6e^t \\ 0 \end{pmatrix} \).

(a) Find the general solution \( \mathbf{x}_h \) of the corresponding homogeneous system. (Hint: The eigenvalues of the coefficient matrix are "nice.")

(b) Use the method of undetermined coefficients to find \( \mathbf{x}_p \), a particular solution of the nonhomogeneous system. Show all the steps needed to arrive at the solution.
22. (1 point)

Let $A$ and $B$ be $n \times n$ matrices. Although it is true that $(a + b)^2 = a^2 + 2ab + b^2$ for any numbers $a$ and $b$, it is not generally true that $(A + B)^2 = A^2 + 2AB + B^2$.

(You may take $A^2$ to mean $A \cdot A$, $(A + B)^2$ to mean $(A + B)(A + B)$, etc.)

Find the flaw in the following "proof" of the statement $(A + B)^2 = A^2 + 2AB + B^2$. In other words, briefly say exactly what is wrong with this "proof."

Hint: Do not panic. Just look over the "proof" step by step and see if you find something which does not make sense. (Also, don't forget that this is worth only one point!)

$$(A + B)^2 = (A + B)(A + B)$$
$$= A(A + B) + B(A + B)$$
$$= A \cdot A + A \cdot B + A \cdot B + B \cdot B$$
$$= A^2 + 2AB + B^2$$