EXAM 3 AND ANSWERS, MATH 233
WEDNESDAY, NOVEMBER 15, 2000

This examination has 30 multiple choice questions. Problems are worth one point apiece, for a total of 30 points for the whole examination.

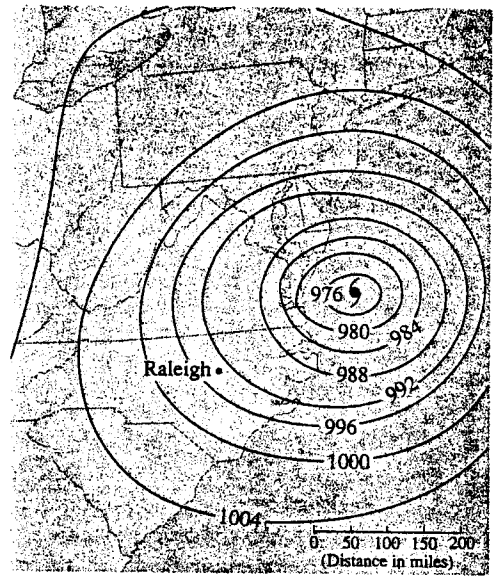
You may use a scientific calculator, such as the TI-83, but not any calculator that has a CAS. You may also use both sides of a 3×5 notecard which you have prepared beforehand with notes and formulas.

An answer key appears on the last page. Solutions are given for some problems.

1. Find $\frac{dz}{dt}$ at $t = 1$, where $z = x^3 + y^2$, $x = t^3$ and $y = 1 + t^2$.
   (A) 17
   (B) 18
   (C) 19
   (D) 20
   (E) 21
   (F) 22
   (G) 23
   (H) 24
   (I) 25
   (J) 26

2. A contour map of barometric pressure (in millibars) is shown for 7:00 a.m. on Sept. 12, 1960, when Hurricane Donna was raging. Find the best estimate of the value (in millibars/mile) of the directional derivative of the pressure function at Raleigh, North Carolina, in the direction of the eye of the hurricane.

(A) 0
(B) $-.02$
(C) .04
(D) $-.06$
(E) .3
(F) $-.1$
(G) .12
(H) $-.14$
(I) .16
(J) $-.18$



3. If $f(x, y) = x^3 - 4x^2y + y^2$, find the directional derivative of f at the point $P(0, -1)$ in the direction of the vector $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

(A) -2
(B) $9/5$
(C) $-8/5$
(D) $7/5$
(E) $-6/5$
(F) 1
(G) $-4/5$
(H) $3/5$
(I) $-2/5$
(J) $1/5$

4. Find the critical points of $f(x,y) = x^2 + y^2 + x^2y + 4$.

(A) $(0,0)$, $(-\sqrt{2},-1)$, $(\sqrt{2},-1)$
(B) $(0,0)$, $(\sqrt{2},1)$, $(1,\sqrt{2})$
(C) $(0,0)$, $(2,1)$, $(-2,1)$
(D) $(0,0)$, $(-1,1)$, $(-1,2)$
(E) $(0,1)$, $(\sqrt{2},1)$, $(-\sqrt{2},1)$
(F) $(1,1)$, $(\sqrt{2},0)$, $(0,-\sqrt{)}$
(G) $(-1,-1)$, $(\sqrt{2},\sqrt{3})$, $(-\sqrt{2},\sqrt{3})$
(H) $(-1,-1)$, $(\sqrt{2},4)$, $(3,\sqrt{2})$
(I) $(0,1)$, $(\sqrt{3},4)$, $(-\sqrt{3},4)$
(J) $(0,1)$, $(4,4)$, $(-4,-4)$

5. Use the second derivative test to classify the critical points i) $(-1,1)$ and ii) $(0,0)$ of the function $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. You may use that $f_{xx} = 6y - 6$, $f_{xy} = 6x$ and $f_{yy} = 6y - 6$.

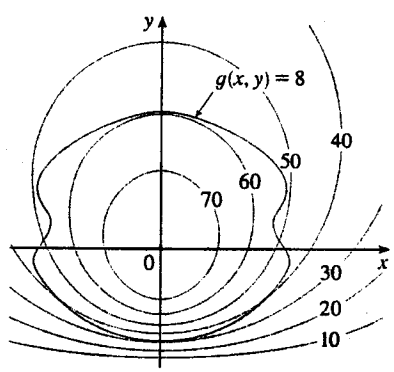
(A) i) local max., ii) local max.
(B) i) local max., ii) local min.
(C) i) local max., ii) saddle
(D) i) local max., ii)undetermined
(E) i) local min., ii) local max.
(F) i) local min., ii) local min.
(G) i) local min., ii) saddle
(H) i) local min., ii) undetermined
(I) i) saddle, ii) local min.
(J) i) saddle, ii) local max.

6. The point on the ellipsoid $x^2 + \frac{1}{4}y^2 + 9z^2 = 1$ closest to the point $(1, 2, 1)$ can be found by finding the minimum point of the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$. Find f and g.

(A) $f = x + 2y + z, \quad g = x^2 + \frac{1}{4}y^2 + 9z^2$
(B) $f = x^2 + \frac{1}{4}y^2 + 9z^2, \quad g = 1$
(C) $f = (x-1)^2 + (y-2)^2 + (z-1)^2, \quad g = z$
(D) $f = x^2 + y^2 + z^2, \quad g = (x-1)^2 + (y-2)^2 + (z-1)^2$
(E) $f = z, \quad g = x^2 + \frac{1}{4}y^2 + 9z^2 - 1$
(F) $f = x^2 + \frac{1}{4}y^2 + 9z^2, \quad g = (x-1)^2 + (y-2)^2 + (z-1)^2$
(G) $f = z, \quad g = (x-1)^2 + (y-2)^2 + (z-1)^2$
(H) $f = x^2 + y^2 + z^2, \quad g = x^2 + \frac{1}{4}y^2 + 9z^2$
(I) $f = (x-1)^2 + (y-2)^2 + (z-1)^2, \quad g = x^2 + \frac{1}{4}y^2 + 9z^2 - 1$
(J) $f = y, \quad g = x^2 + \frac{1}{4}y^2 + 9z^2$

7. Pictured below are a contour map of a function $f(x, y)$ and a curve defined by the equation $g(x, y) = 8$. Find the best estimate of the **maximum** value of f subject to the constraint $g(x, y) = 8$.

(A) 80
(B) 70
(C) 60
(D) 50
(E) 40
(F) 30
(G) 20
(H) 10
(I) 8
(J) 0



8. The following MATLAB script calculates the Riemann sum by the midpoint rule with $m = 70$ and $n = 50$ for a double integral. Evaluate this double integral to find the best approximation to the value of RS which will be returned by MATLAB.

```
h = (2 - 0)/70;  k = (2 - 1)/50;
[x,y] = meshgrid(0+h/2:h:2-h/2, 1+k/2:k:2-k/2);
RS = sum(sum(x.^2.*y))*h*k
```

(A) 11
(B) 10
(C) 9
(D) 8
(E) 7
(F) 6
(G) 5
(H) 4
(I) 3
(J) 2

9. Find $\iint_R x^2 y \, dA$, where $R = [0, 2] \times [1, 2]$.

(A) 2
(B) 3
(C) 4
(D) 5
(E) 6
(F) 7
(G) 8
(H) 9
(I) 10
(J) 11

10. Find $\iint_D 2x \cos y \, dA$, where D is bounded by $y = 0$, $y = x^2$ and $x = 2$.

(A) $1 + \cos 3$
(B) $1 - \cos 4$
(C) $1 + \cos 5$
(D) $2 - \cos 6$
(E) $2 \cos 1$
(F) $3 \cos 2$
(G) $4 \cos 3$
(H) $5 \cos 4$
(I) 4
(J) 1

11. In polar coordinates, the integral $\iint_R xy \, dA$, where R is the region in the first quadrant that lies between $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$, is $\int_0^a \int_2^b f(r, \theta) \, dr d\theta$. Find a, b and f.

(A) $a = \frac{\pi}{2}, \quad b = 5, \quad f = r^2\theta$
(B) $a = \pi, \quad b = 25, \quad f = r\theta$
(C) $a = \frac{\pi}{4}, \quad b = 5, \quad f = \frac{1}{2}r^2 \sin 2\theta$
(D) $a = 2\pi, \quad b = 5, \quad f = r^2 \cos\theta \sin\theta$
(E) $a = \frac{\pi}{2}, \quad b = 25, \quad f = \frac{1}{2}r^3 \sin 2\theta$
(F) $a = \frac{\pi}{2}, \quad b = 5, \quad f = r \cos\theta \sin\theta$
(G) $a = \pi, \quad b = 25, \quad f = r^2 \cos\theta \sin\theta$
(H) $a = \frac{\pi}{2}, \quad b = 5, \quad f = r^3 \cos\theta \sin\theta$
(I) $a = 2\pi, \quad b = \cos\theta, \quad f = r^2 \sin\theta$
(J) $a = \frac{\pi}{4}, \quad b = \sin\theta, \quad f = r^3 \cos\theta$

12. A lamina occupies the disk $x^2 + y^2 \leq 9$. Find its mass, if the density at any point is proportional to its distance from the origin, where the proportionality constant is $\frac{2}{3}$.

(A) 8π
(B) 10π
(C) 12π
(D) 14π
(E) 16π
(F) 18π
(G) 20π
(H) 22π
(I) 24π
(J) 26π

Solution: $m = \int_0^{2\pi} \int_0^3 \frac{2}{3} r \, r dr d\theta = 12\pi$.

13. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} C & \text{if } 0 \leq x \leq 2,\ 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the constant C.

(A) 1
(B) 1/2
(C) 1/3
(D) 1/4
(E) 1/5
(F) 1/6
(G) 1/7
(H) 1/8
(I) 1/9
(J) 1/10

Solution: Must have $1 = \int_0^2 \int_0^3 C \, dy dx = 6C$, so $C = 1/6$.

14. Find the area of the part of the plane $2x - 2y + z = 5$ that lies over the rectangle $0 \le x \le 2$, $0 \le y \le 3$.

(A) 5
(B) 6
(C) 10
(D) 12
(E) $6\sqrt{5}$
(F) $5\sqrt{6}$
(G) $5\sqrt{7}$
(H) $12\sqrt{2}$
(I) 18
(J) $6\sqrt{10}$

Solution: Area $= \int_0^2 \int_0^3 \sqrt{(-2)^2 + (2)^2 + 1}\, dydx = 18$.

15. If the function $z = z(x, y)$ satisfies the equation $xy + yz - xz = 0$, find $\frac{\partial z}{\partial x}$ when $x = 2$ and $y = 1$.

(A) 3
(B) $-3/2$
(C) $4/3$
(D) $-5/4$
(E) 2
(F) $-1/2$
(G) $2/3$
(H) $-3/4$
(I) $4/5$
(J) -1

16. Suppose $f(x, y)$ is a differentiable function whose gradient at the point $P(5, 1)$ is $\nabla f(5, 1) = \langle 1, -2\rangle$. Find $\frac{d}{dt}|_{t=0} f(5 + \frac{3}{5}t, 1 + \frac{4}{5}t)$.

(A) $-3/5$
(B) $7/10$
(C) $-4/5$
(D) $9/10$
(E) -1
(F) $11/10$
(G) $-6/5$
(H) $13/10$
(I) $-7/5$
(J) $3/2$

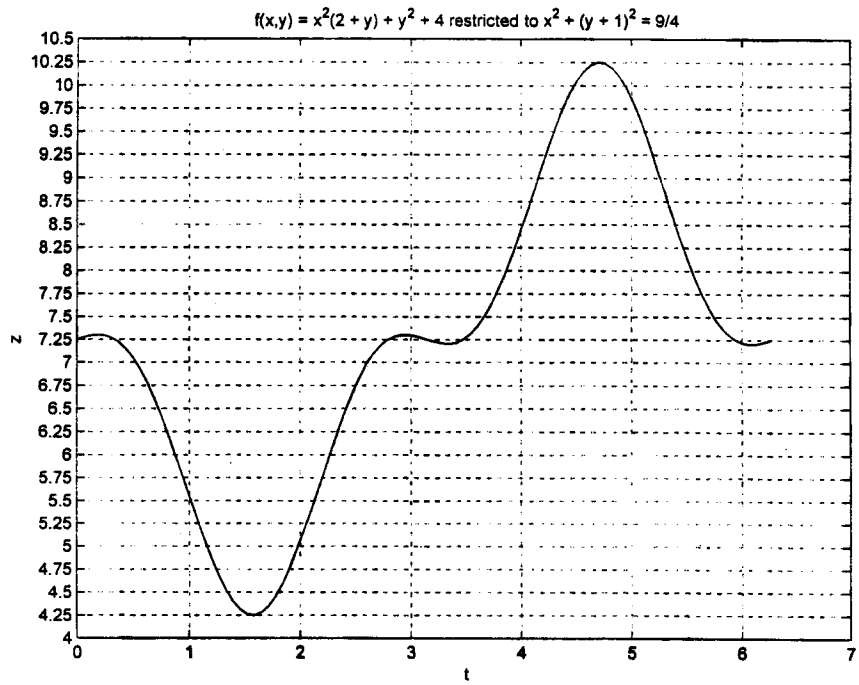
Solution: $\frac{d}{dt}|_{t=0} f(5 + \frac{3}{5}t, 1 + \frac{4}{5}t)$ is the directional derivative of f at the point $(5, 1)$ in the direction of $\langle \frac{3}{5}, \frac{4}{5}\rangle$, which is $\nabla f(5, 1) \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \langle 1, -2\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = -1$.

17. Find the maximum rate of change of $f(x, y, z) = x + \frac{y}{z}$ at the point $P(4, 3, -1)$.

(A) $\sqrt{11}$
(B) $2\sqrt{10}$
(C) 3
(D) $2\sqrt{2}$
(E) $\sqrt{7}$
(F) $\sqrt{6}$
(G) $\sqrt{5}$
(H) 2
(I) $\sqrt{3}$
(J) $\sqrt{2}$

18. The function $f(x,y) = x^2(2+y)+y^2+4$ on the disk $D = \{x^2+(y+1)^2 \leq 9/4\}$ has critical points at $(0,0)$, $(2,-2)$ and $(-2,-2)$. The following graph is of $z = f(x,y)$ restricted to the boundary of D, parameterized by $x = \frac{3}{2}\cos t$, $y = -1 + \frac{3}{2}\sin t$, $0 \leq t \leq 2\pi$. Find the absolute minimum value of $f(x,y)$ on D.

(A) 0
(B) 1.5
(C) 2.15
(D) 3
(E) 4
(F) 4.25
(G) 7.2
(H) 7.3
(I) 8
(J) 10.25



19. Refer to problem 18. Find the absolute maximum value of $f(x,y)$ on D.

(A) 0
(B) 1.5
(C) 2.15
(D) 3
(E) 4
(F) 4.25
(G) 7.2
(H) 7.3
(I) 8
(J) 10.25

20. Find the maximum value of $f(x, y) = xy$ subject to the constraint $x^2 + 4y^2 = 2$.

(A) .3
(B) .34
(C) .4
(D) .5
(E) .6
(F) .7
(G) .74
(H) .8
(I) 0
(J) 1

Solution: $\nabla f = \langle y, x\rangle$ and $\nabla(x^2 + 4y^2) = \langle 2x, 8y\rangle$. By the method of Lagrange multipliers, the maximum point occurs among the solutions to the three equations $y = \lambda 2x$, $x = \lambda 8y$ and $x^2 + 4y^2 = 2$. The four solution points are $(\pm 1, \pm\frac{1}{2})$, and $f(\pm 1, \pm\frac{1}{2}) = \pm\frac{1}{2}$, so the maximum value is $1/2$.

21. Find the value of $f(x, y) = xy$ at its critical point in the interior of the region $x^2 + 4y^2 \leq 2$.

(A) .3
(B) .34
(C) .4
(D) .5
(E) .6
(F) .7
(G) .74
(H) .8
(I) 0
(J) 1

22. Let $V = \iint_R (52 - x^2 - y^2)\, dA$, where $R = [0, 2] \times [0, 4]$. Use the lines $x = 1$ and $y = 2$ to divide R into rectangles (i.e., the case $m = 2 = n$). Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Find the correct ordering of the numbers V, L and U.

(A) $V > L = U$
(B) $L = U > V$
(C) $V > U > L$
(D) $V > L > U$
(E) $U > V > L$
(F) $U > L > V$
(G) $L > U > V$
(H) $L > V > U$
(I) $L > U = V$
(J) $U = V = L$

Solution: See exercise 7 on page 848 of the text.

23. Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ and above the rectangle $[0, 1] \times [0, 2]$.

(A) 14
(B) 13
(C) 12
(D) 11
(E) 10
(F) 9
(G) 8
(H) 7
(I) 6
(J) 5

24. Find a and b if $\int_1^2 \int_0^{\ln x} f(x,y)\,dydx = \int_0^{\ln 2} \int_a^b f(x,y)\,dxdy$.

(A) $a = 1$, $b = 2$
(B) $a = 0$, $b = 1/\ln 2$
(C) $a = 1$, $b = e^y$
(D) $a = e^y$, $b = 2$
(E) $a = e^x$, $b = \ln 2$
(F) $a = 0$, $b = 2x$
(G) $a = 1$, $b = x^2$
(H) $a = e^y$, $b = 0$
(I) $a = \ln x$, $b = 2$
(J) $a = \ln x$, $b = e^x$

Solution: The first iterated integral equals the double integral of f over the region $D = \{(x,y) : 1 \le x \le 2, 0 \le y \le \ln x\} = \{(x,y) : 0 \le y \le \ln 2, e^y \le x \le 2\}$. Therefore, $a = e^y$ and $b = 2$.

25. The volume above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 8$ is $\int_0^{2\pi} \int_0^b (\sqrt{8-r^2} - r) r\,drd\theta$. Find b.

(A) 1
(B) $\sqrt{2}$
(C) $\sqrt{3}$
(D) 2
(E) $\sqrt{5}$
(F) $\sqrt{6}$
(G) $\sqrt{7}$
(H) $2\sqrt{2}$
(I) 3
(J) $\sqrt{10}$

Solution: The cone and sphere intersect in the curve whose vertical projection onto the xy-plane is $x^2+y^2+x^2+y^2 = 8$, which is the circle of radius 2 centered at the origin. The intergral must be taken over the disk bounded by this circle. Therefore, $b = 2$.

26. A lamina with density function $\rho(x, y) = x$ occupies the region D which is the region in the first quadrant bounded by $y = x^2$ and $y = 1$. For any constant c,

$$\iint_D (x-c)x\,dA = \frac{2}{15} - \frac{c}{4} \quad \text{and} \quad \iint_D (y-c)x\,dA = \frac{1}{6} - \frac{c}{4}$$

Find the center of mass $(\bar{x}, \bar{y})$ of this lamina.

(A) $(\frac{2}{15}, \frac{1}{6})$
(B) $(\frac{4}{15}, \frac{1}{3})$
(C) $(\frac{8}{15}, \frac{2}{3})$
(D) $(\frac{2}{5}, \frac{1}{2})$
(E) $(\frac{2}{3}, 1)$
(F) $(\frac{4}{5}, \frac{4}{3})$
(G) $(\frac{1}{4}, \frac{1}{4})$
(H) $(\frac{7}{15}, \frac{3}{4})$
(I) $(\frac{1}{3}, \frac{4}{5})$
(J) $(\frac{1}{2}, \frac{1}{2})$

Solution: The given integrals are the moments about the line $x = c$ and about the line $y = c$, respectively. Therefore, $\bar{x}$ is the solution of $\frac{2}{15} - \frac{c}{4} = 0$ and $\bar{y}$ is the solution of $\frac{1}{6} - \frac{c}{4} = 0$, so that $\bar{x} = \frac{8}{15}$ and $\bar{y} = \frac{2}{3}$.

27. Find the integral whose value is the area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 2x$.

(A) $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4 + r^2}\,dr d\theta$
(B) $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{4r^2 + 1}\,r\,dr d\theta$
(C) $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} 2r\,dr d\theta$
(D) $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}}\,dr d\theta$
(E) $\int_{-\pi/2}^{\pi/2} \int_0^{1} \sqrt{4 - r^2}\,dr d\theta$
(F) $\int_0^{2\pi} \int_0^{1} \sqrt{4r^2 + 1}\,r\,dr d\theta$
(G) $\int_0^{2\pi} \int_0^{2\cos\theta} \sqrt{4r^2 + 1}\,dr d\theta$
(H) $\int_0^{2\pi} \int_0^{1} 2\sqrt{2\cos\theta}\,r\,dr d\theta$
(I) $\int_0^{\pi} \int_0^{2} 2r\,dr d\theta$
(J) $\int_0^{\pi} \int_0^{2} \sqrt{4r^2 + 1}\,r\,dr d\theta$

Solution: Area $= \iint_D \sqrt{1 + 4x^2 + 4y^2}\,dA$, where D is the disk enclosed by the circle $x^2 + y^2 = 2x$, whose equation in polar coordinates is $r = 2\cos\theta$. In polar coordinates, the double integral becomes $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \sqrt{1 + 4r^2} r\,dr d\theta$.

28. The joint density function for a pair of random variables X and Y is

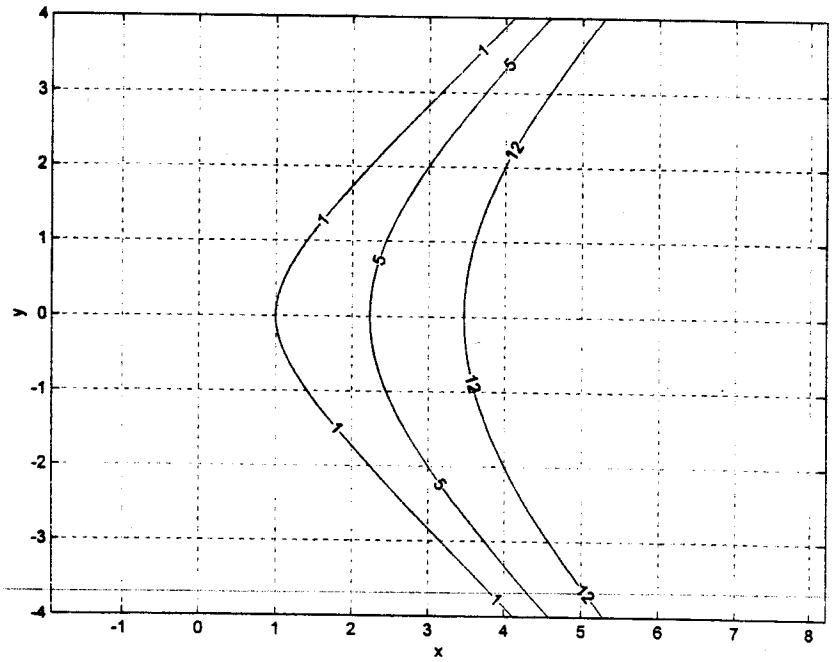
$$f(x,y) = \begin{cases} xy & \text{if } 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability $P(X \le \frac{1}{2}, Y \le 1)$.

(A) $\frac{1}{18}$
(B) $\frac{1}{17}$
(C) $\frac{1}{16}$
(D) $\frac{1}{15}$
(E) $\frac{1}{14}$
(F) $\frac{1}{12}$
(G) $\frac{1}{11}$
(H) $\frac{1}{10}$
(I) 1
(J) 2

29. The following contour map shows 3 level curves of the function $f(x,y)$. From the information given, find the most likely gradient vector for f at the point $P(3,2)$.

(A) $\langle 2, 0\rangle$
(B) $\langle 0, 3\rangle$
(C) $\langle 2, 3\rangle$
(D) $\langle -2, 3\rangle$
(E) $\langle -2, -3\rangle$
(F) $\langle 0, 0\rangle$
(G) $\langle 6, 4\rangle$
(H) $\langle 6, -4\rangle$
(I) $\langle -6, 4\rangle$
(J) $\langle -6, -4\rangle$



30. Find the area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies over $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

(A) $\pi/3$
(B) $\pi/2$
(C) $\pi/\sqrt{2}$
(D) π
(E) $\sqrt{2}\,\pi$
(F) 2π
(G) $2\sqrt{2}\,\pi$
(H) $\sqrt{3}\,\pi$
(I) $4\pi/3$
(J) 4π

Solution: $z_x = x/z$ and $z_y = y/z$ so that $z_x^2 + z_y^2 + 1 = \frac{x^2+y^2+z^2}{z^2} = \frac{2(x^2+y^2)}{x^2+y^2} = 2$. Therefore, the area is $\int_0^{2\pi} \int_0^1 \sqrt{2}\, r dr d\theta = \sqrt{2}\,\pi$.