1. Find the gradient of the function $f(x,y,z) = xe^{yz}$ at the point (1, 0, 1).

A) < 0, 1, 1>

B) < 1, 1, 0>

C) < -1, 1, 1>

D) < e, 1, 0>

E) < e, -e, 1>

F) < 1, 1, e>

G) < 2, -e, 0>

H) < -e, -1, 0>

I) < 2, e, 0>

J) None of the above

$\nabla f(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

$\nabla f(1,0,1) = \langle 1, 1, 0 \rangle$

2. Find the directional derivative of the function $f(x,y,z,) = \dfrac{x}{y+z}$ in the direction of $\vec{v} = \langle 3,-4,0 \rangle$ at the point (4, 1, 1).

A) 1/10

B) 1/5

C) 1/2

D) 3/4

E) 1

F) 11/10

G) 6/5

H) 3/2

I) 7/4

J) None of the above

$\nabla f(x,y,z) = \left\langle \frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2} \right\rangle$

$\nabla f(4,1,1) = \left\langle \frac{1}{2}, -1, -1 \right\rangle$

UNIT VEC. IN DIR. OF $\vec{v}$ IS

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,-4,0 \rangle}{5} = \left\langle \frac{3}{5}, -\frac{4}{5}, 0 \right\rangle$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \left\langle \frac{1}{2}, -1, -1 \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}, 0 \right\rangle = \boxed{\frac{11}{10}}$

3. Find the equation of the tangent plane to the surface $yz = \ln(x+z)$ at the point (0, 0, 1).

A) $x+z=1$

B) $x+y+z=1$

C) $x+y+z=-1$

D) $x-y+z=1$

E) $y-z=2$

F) $x-y+z=2$

G) $2x+y-z=1$

H) $2x-y+z=1$

I) $x+2y+z=0$

J) None of the above

THIS IS A LEVEL SURFACE OF

$f(x,y,z) = yz - \ln(x+z)$

$f_x = -\frac{1}{x+z}, \quad f_y = y - \frac{1}{x+z}, \quad f_z = z$

$f_x(0,0,1) = -1$

$f_y(0,0,1) = 1$

$f_z(0,0,1) = -1$

TANGENT PLANE IS

$f_x(x-a) + f_y(y-b) + f_z(z-c) = 0$

$-1(x-0) + 1(y-0) - 1(z-1) = 0$

$\vdots$

$\boxed{x - y + z = 1}$

4. Find all points at which the function $f(x,y) = x^3y + 12x^2 - 8y$ has a relative maximum.

A) (1, 0)

B) (1, 0) and (-1, 2)

C) (-1,0)

D) (-1, 0) and (-1, 2)

E) (0, 0)

F) (0, 0) and (1, 1)

G) (1, 1) and (-1, -1)

H) all points of the form (1, y)

I) f has no relative maxima

J) None of the above

$f_x = 3x^2y + 24x$

$f_y = x^3 - 8$

$f_{xx} = 6xy + 24$

$f_{yy} = 0$

$f_{xy} = 3x^2$

AT C.P.

(1) $0 = f_x = 3x^2y + 24x$

(2) $0 = f_y = x^3 - 8$

BY (2), $x = 2$.

SO (1) SAYS $xy + 8 = 0 \Rightarrow 2y + 8 = 0 \Rightarrow y = -4$

HENCE ONLY C.P. IS $(2, -4)$

$f_{xx}(2,-4) = -24$

$f_{yy}(2,-4) = 0$

$f_{xy}(2,-4) = 12$

SO $D = (-24)(0) - (12)^2 = -144 < 0$

SO $(2,-4)$ IS A SADDLE PT.

$\rightarrow$ NO REL MAX

5. Find the maximum value attained by the function $f(x,y) = xy$ on the circle $x^2 + y^2 = 2$.

A) 0

B) 1/2

C) $1/\sqrt{2}$

D) 2/3

E) $3/\sqrt{2}$

F) 1

G) $\sqrt{2}$

H) 3/2

I) $2\sqrt{2}$

J) None of the above

LAGRANGE PROBLEM,

$f(x,y) = xy \Rightarrow \nabla f = \langle y, x \rangle$

$g(x,y) = x^2+y^2 \Rightarrow \nabla g = \langle 2x, 2y \rangle$

SO SYSTEM.

$\nabla f = \lambda \nabla g$

$g(x,y) = 2$

IS

① $y = 2\lambda x$

② $x = 2\lambda y$

③ $x^2 + y^2 = 2$

NOTE $x \neq 0$, $y \neq 0$ LEST BY ①,② $x = y = 0$

SO ③ SAYS $0 = 2$ ✗

HENCE $x \neq 0$, $y \neq 0$.

② $x = 2\lambda y \overset{①}{=} 2\lambda(2\lambda x) = 4\lambda^2 x$

$\therefore x = 4\lambda^2 x \overset{x\neq 0}{\Longrightarrow} 1 = 4\lambda^2 \Rightarrow \lambda = \pm 1/2$

IF $\lambda = \frac{1}{2}$

① $y = x$

② $x = y$

③ $x^2 + y^2 = 2$

$\Rightarrow$ ③ $2x^2 = 2 \Rightarrow y = x = \pm 1$

SOL'NS $(-1, -1, 1/2)$, $(1, 1, 1/2)$

IF $\lambda = -\frac{1}{2}$

① $y = -x$

② $x = -y$

③ $x^2 + y^2 = 2$

$\Rightarrow$ ③ $2x^2 = 2 \Rightarrow x = \pm 1$, $y = -x$

SOL'NS $(1, -1, -1/2)$, $(-1, 1, -1/2)$

POSSIBLE MAX PTS: $(1,1)$, $(-1,-1)$, $(-1,1)$, $(1,-1)$

CHECK ALL:

(x,y)	$f(x,y) = xy$
$(1,1)$	1
$(-1,-1)$	1
$(-1,1)$	-1
$(1,-1)$	-1

MAX VAL OF f IS 1

6. Evaluate $\iint_R \cos(x+y)dA$ where $R = \{(x,y) | 0 \le x \le \pi, 0 \le y \le \pi\}$.

$R = [0,\pi] \times [0,\pi]$

$I = \int_0^\pi \int_0^\pi \cos(x+y)\, dy\, dx$

$\int_0^\pi \cos(x+y)\, dy = \sin(x+\pi) - \sin x$

$I = \int_0^\pi [\sin(x+\pi) - \sin x]\, dx$

$= \int_0^\pi \sin(x+\pi)\, dx - \int_0^\pi \sin x\, dx$

$= -2 - 2 = \boxed{-4}$

A) -4

B) -3

C) -2

D) -3/2

E) -1

F) 0

G) 1

H) 3/2

I) 4

J) None of the above

7. *To the nearest integer*, what is the value of $\iint_R e^{x+y}\,dA$, where $R = [0,2] \times [0,1]$.

A) -2

B) 0

C) 2

D) 3

E) 4

F) 7

G) 10

H) 11

I) 13

J) 15

$$\iint_R e^{x+y}\,dA = \int_0^2 \int_0^1 e^{x+y}\,dy\,dx$$

$$= \int_0^2 \int_0^1 e^x e^y\,dy\,dx$$

$$= \left(\int_0^1 e^y\,dy\right)\left(\int_0^2 e^x\,dx\right)$$

$$= (e-1)(e^2-1) = 10.978 \approx \boxed{11}$$

8. Find the volume under the plane $x + y + z = 2$ and above the square $[0,1] \times [0,1]$ in the x - y plane.

A) -2

B) -3/2

C) -1

D) 0

E) 1/2

F) 1

G) $\sqrt{2}$

H) $\sqrt{3}$

I) 7/2

J) None of the above

$x+y+z=2$ IS THE GRAPH OF

$$z = f(x,y) = 2 - x - y$$

So $$V = \iint_R (2 - x - y)\, dA$$

$$= \int_0^1 \int_0^1 (2 - x - y)\, dy\, dx$$

$$\int_0^1 (2 - x - y)\, dy = \frac{3}{2} - x$$

So

$$I = \int_0^1 \left(\frac{3}{2} - x\right) dx = \boxed{1}$$

9. Find $\iint_R xy\,dA$, where R is the unit disc $\{(x,y) | x^2 + y^2 \le 1\}$.

A) -2

B) -1

C) 0

D) 1/2

E) $1/\sqrt{2}$

F) $\sqrt{2}$

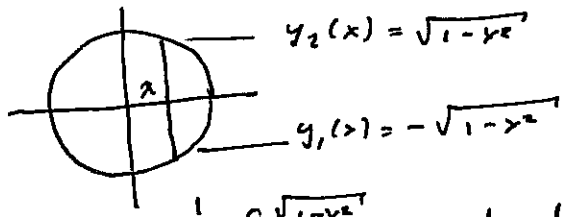
G) 1

H) 2

I) π

J) None of the above

R IS RECT TYPE I:



$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dy = x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\,dy$$

$= x \cdot 0$ SINCE y IS ODD.

$$\text{so } I = \int_{-1}^{1} 0\,dx = \boxed{0}$$

OR

R IS POLAR RECTANGLE: $[0,1] \times [0,2\pi]$ (r θ)

$$I = \iint_R xy\,dA = \int_0^{2\pi} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3 \sin\theta\cos\theta\,dr\,d\theta)$$

$$= \left(\int_0^{2\pi} \sin\theta\cos\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$$\int_0^{2\pi} \sin\theta\cos\theta\,d\theta = \int_0^0 u\,du = 0$$

$u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$

$$\text{so } I = \boxed{0}$$

10. Find the volume under the surface $z = x^2 + y^3$ above the finite region in the x-y plane bounded by the curves $y = x$ and $y = x^2$.

A) 0

B) 2/18

C) 3/180

D) 3/18

E) 2/9

(F) 13/180

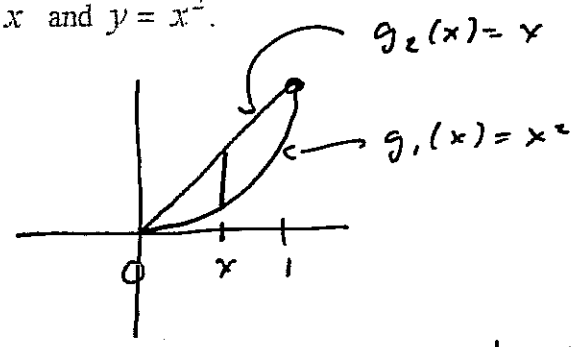
G) 1/6

H) 17/18

I) 1

J) None of the above

R:



$$V = \iint_R (x^2+y^3)\,dA = \int_0^1 \int_{x^2}^{x} (x^2+y^3)\,dy\,dx$$

$$\int_{x^2}^{x} (x^2+y^3)\,dy = \int_{x^2}^{x} x^2\,dy + \int_{x^2}^{x} y^3\,dy$$

$$= x^2(x-x^2) + \frac{1}{4}y^4\Big|_{x^2}^{x} = \cdots = x^3 - \frac{3}{4}x^4 - \frac{1}{4}x^8$$

$$\text{So } V = \int_0^1 \left(x^3 - \frac{3}{4}x^4 - \frac{1}{4}x^8\right)dx = \boxed{\frac{13}{180}}$$

11. Find the volume under the plane z = y and above the region of the x-y plane bounded by the curve $x = y^2 + 1$ and the lines y = -x, y = 0 and y = 1.

A) 0

B) 1/12

C) 1/4

D) 1/3

E) 1/2

F) 5/6

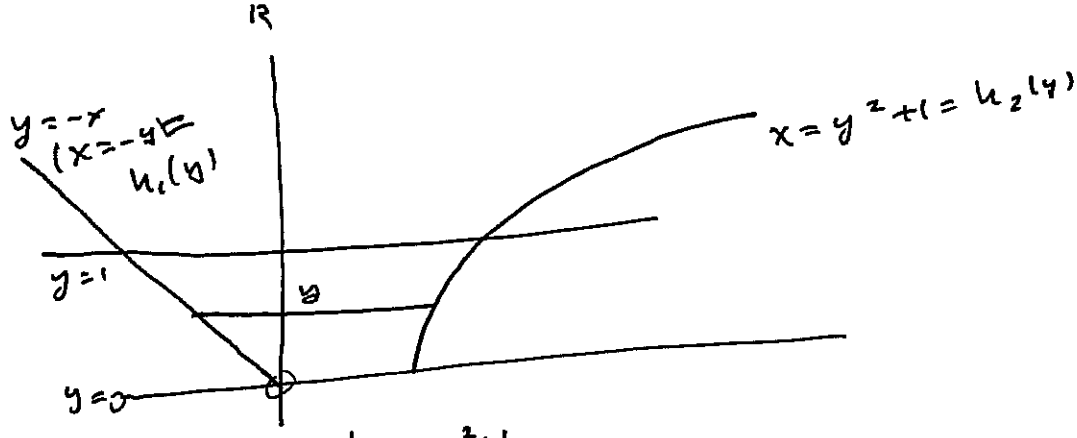
G) 1

H) 13/12

I) 7/6

J) None of the above

$V = \iint_R y \, dA$ where R is type II:



$$V = \int_0^1 \int_{-y}^{y^2+1} y \, dx \, dy$$

$$\int_{-y}^{y^2+1} y \, dx = y\left[(y^2+1)-(-y)\right] = y^3 + y^2 + y$$

$$V = \int_0^1 (y^3 + y^2 + y) \, dy = \boxed{\frac{13}{12}}$$

12. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the circular sector in the first quadrant bounded by the x-axis, the y-axis and the circle $x^2 + y^2 = 4$.

A) $\pi/3$

B) $\pi/2$

C) π

D) $4\pi/3$

E) 2π

F) $5\pi/2$

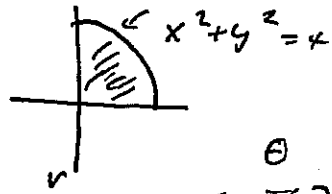
G) 3π

H) $7\pi/2$

I) 4π

J) None of the above

$V = \iint_R \sqrt{x^2+y^2}$, R :



R IS POLAR RECT $[0, 2] \times [0, \pi/2]$

$V = \int_0^2 \int_0^{\pi/2} \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \cdot r \, d\theta \, dr$

$= \int_0^2 \int_0^{\pi/2} r^2 \, d\theta \, dr = \int_0^2 \frac{\pi}{2} r^2 \, dr$

$= \frac{\pi}{2} \int_0^2 r^2 \, dr = \boxed{\frac{4\pi}{3}}$

13. Evaluate $\iint_R x^2 + y^2 dA$, where R is the region of the plane above the x-axis and inside the spiral $r = \theta$, where θ is in the interval $[0, \pi]$.

MUST DO IN POLAR COORDS

A) 0

B) $\pi/20$

C) $\pi/15$

D) $\pi/2$

E) $\pi^2/15$

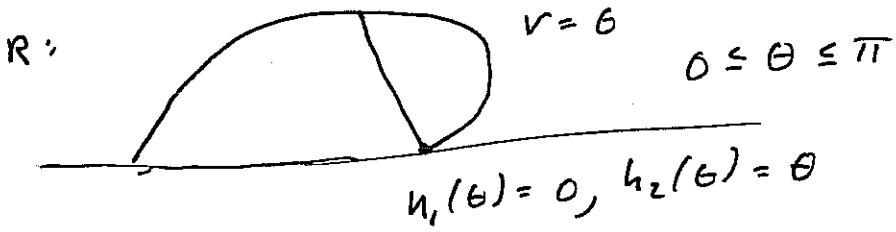
F) $\pi^3/20$

G) $\pi^4/15$

H) $\pi^4/20$

I) $\pi^5/20$

J) None of the above



$f(r\cos\theta, r\sin\theta) = r$

So

$$I = \int_0^{\pi} \int_0^{\theta} r^2 \cdot r \, dr \, d\theta = \int_0^{\pi} \left[\int_0^{\theta} r^3 \, dr \right] d\theta$$

$$= \int_0^{\pi} \frac{\theta^4}{4} d\theta = \boxed{\frac{\pi^5}{20}}$$

14. Find the surface area of that part of the paraboloid $z = x^2 + y^2$ which is inside the cylinder $x^2 + y^2 = 1$. *Pick the closest value below.*

A) 3.50

B) 3.56

C) 3.73

D) 3.90

E) 4.22

F) 4.75

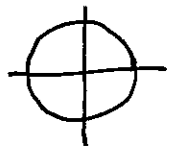
G) 4.98

H) 5.13

I) 5.33

J) 5.71

USE $A(S) = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA$

WHERE R:  $= [0,1] \times [0,2\pi]$ (r, θ)

$z_x = 2x, \quad z_y = 2y$

SO $A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA$

$= \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2} \cdot r\, dr\, d\theta = 2\pi \int_0^1 \sqrt{1+4r^2}\,(r\,dr)$

$= 2\pi \cdot \frac{1}{8} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5$

SUBS
$u = 1 + 4r^2$
$du = 8r\,dr$
$r\,dr = \frac{1}{8} du$

$= \frac{\pi}{6}\left[5^{3/2} - 1\right] \approx 5.330$

15. The graph of the function $y = x^3$ in the x-y plane between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the surface area of the resulting surface. *Pick the closest value below.*

A) 3.50

B) 3.56

C) 3.73

D) 3.90

E) 4.22

F) 4.75

G) 4.98

H) 5.13

I) 5.33

J) 5.71

USE $A(S) = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow 1+f'(x)^2 = 1+9x^4$

$A = 2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx$

$= 2\pi \cdot \frac{1}{36} \int_1^{10} \sqrt{u}\,du$

$u = 1+9x^4$

$du = 36x^3\,dx$

$x^3\,dx = \frac{1}{36}du$

$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2}\Big|_1^{10}$

$= \frac{\pi}{27}\left[10^{3/2}-1\right] \approx 3.563$

Hand-graded problem Name_____ ID_____
Please write legibly above, and show your work below.

We know that the continuous function $f(x,y) = x^2 + y^2 + x^2y + 4$ must attain a maximum and a minimum value on the closed set $S = [0,1] \times [0,1]$. Find these values, and the points of S at which they occur.

FIRST FIND CRIT. PTS.

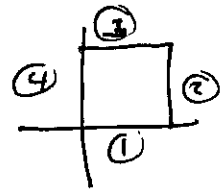
$0 = f_x = 2x + 2xy \Rightarrow$ (1) $x + xy = 0$

$0 = f_y = 2y + x^2$ (2) $2y + x^2 = 0$

IF $x = 0$ (2) $\Rightarrow y = 0 \rightarrow$ C.P. AT $(0,0)$

SO W.L.O.G. $x \neq 0$ THEN (1) $1 + y = 0 \Rightarrow y = -1 \Rightarrow$ C.P. NOT IN S.

NOW CONSIDER ∂S:



USE CALC I METHODS:

ON (1) $g(x) = f(x,0) = x^2 + 4$, $x \in [0,1] \rightarrow$ MIN = 4 AT $(0,0)$, MAX = 5 AT $(1,0)$

ON (2) $g(y) = f(1,y) = y^2 + y + 5$, $y \in [0,1] \rightarrow$ MIN = 5 AT $(1,0)$, MAX = 7 AT $(1,1)$

ON (3) $g(x) = f(x,1) = 2x^2 + 5$, $x \in [0,1] \rightarrow$ MIN = 5 AT $(0,1)$, MAX = 7 AT $(1,1)$

ON (4) $g(y) = f(0,y) = y^2 + 4$, $y \in [0,1] \rightarrow$ MIN = 4 AT $(0,0)$, MAX = 5 AT $(0,1)$

CHECK ALL POSSIBILITIES:

(x,y)	$f(x,y)$	
$(0,0)$	4	← MIN = 4 AT $(0,0)$
$(1,0)$	5	
$(0,1)$	5	
$(1,1)$	7	← MAX = 7 AT $(1,1)$