Solutions to Second Midterm

1. The integers $\mathbb{Z}$ form a countable set. The one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$ is:

   $f: \mathbb{Z} \to \mathbb{N}$

   $f(j) = 2j$ if $j > 0$

   $f(j) = -2j + 1$ if $j \leq 0$.

2. Assume to the contrary that $S$ is countable. Then we have an enumeration:

   $S^1, S^2, S^3, \ldots$

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This is an exhaustive list of all sequences of 0s and 1s. Now define
\[ t_1 = \begin{cases} 0 & \text{if } s_1 = 1 \\ 1 & \text{if } s_1 = 0 \end{cases} \]
\[ t_2 = \begin{cases} 0 & \text{if } s_2 = 1 \\ 1 & \text{if } s_2 = 0 \end{cases} \]
\[ t_3 = \begin{cases} 0 & \text{if } s_3 = 1 \\ 1 & \text{if } s_3 = 0 \end{cases} \]
and so forth. Then \( T = \langle t_1, t_2, t_3, \ldots \rangle \) is a sequence of 0s and 1s. That is not in the enumeration, so \( S \) is uncountable.

3. Let \( p \) be any polynomial. Then \( p \) has the same degree as \( p \). So the relation is reflexive.

If \( p, q \) are polynomials and \( p \) has the same degree as \( q \), then \( q \) has the same degree as \( p \). So the relation is symmetric.

If \( p, q, r \) are polynomials and (i) \( p \) has the same degree as \( q \) and (ii) \( q \) has the same degree as \( r \), then clearly \( p \) has the same degree as \( r \). So the relation is transitive, hence it is an
equivalence relation. The equivalence classes are, for each nonnegative integer $k$, the collection

$$\mathcal{P}_k = \{ \text{all polynomials of degree } k \}.$$

4. If $(a, b) \in S$ then $a + b = a + b$ so $(a, b) \sim (a, b)$ and the relation is reflexive.

If $(a, b) \in S$, $(c, d) \in S$ and $(a, b) \sim (c, d)$ then $a + d = b + c$ so $c + b = d + a$. Thus $(c, d) \sim (a, b)$. Hence the relation is symmetric.

If $(a, b)$, $(c, d)$, $(e, f)$ are in $S$ and $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then

$$a + d = b + c$$

$$c + f = d + e$$

Adding gives

$$a + d + c + f = b + c + d + e$$

Cancelling $c$ and $d$ from both sides yields

$$a + f = b + e$$

So $(a, b) \sim (e, f)$ and the relation is transitive.
b) Suppose that $(a_1, a_2) \sim (z_1^*, a_2^*), \text{ and } (b_1, b_2) \sim (b_1^*, b_2^*)$.

Then we know that

1. $a_1 + a_2 = z_1 + a_2^* \Rightarrow a_1 + a_2 = z_1 + a_2^*$
2. $b_1 + b_2 = b_1 + b_1^*$

We need to see that

3. $(a_1, a_2) + (b_1, b_2) \sim (z_1^*, a_2^*) + (b_1^*, b_2^*)$
   \[= (z_1 + a_2^* + b_1 + b_2^*)
   \[= (z_1 + b_1 + a_2 + b_2^*)
   \[= (z_1 + b_1 + a_2 + b_2^*) = (z_1 + b_1 + a_2 + b_1^*)

Adding (1) and (2) together yields (3).

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So addition is indeed well defined.

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d) We define multiplication by

$$(z_1, a_2) \cdot (b_1, b_2) = (z_1 b_2 + z_2 b_1, a_2 b_1 + z_2 b_2).$$
5. a) A set $S \subseteq \mathbb{R}$ has an upper bound if there is a real number $M$ such that $s \leq M$ for all $s \in S$.

b) A set $S \subseteq \mathbb{R}$ has a least upper bound $m$ if $m$ is an upper bound for $S$ and there is no other upper bound $n$ for $S$ such that $m < n$.

c) The set $\{x \in \mathbb{R} : x^2 < 2\}$ has 3 as an upper bound, but there is no real least upper bound.

d) The real number $\mathbb{R}$ is a ordered field that contains $\mathbb{Q}$ and such that every set with an upper bound has a least upper bound.

6. The complex numbers are $\mathbb{R} \times \mathbb{R}$ equipped with the arithmetic operations:

\[(a,b) + (c,d) = (a+c, b+d)\]
\[(a,b) - (c,d) = (a-c, b-d)\]
The pair $(0, 1)$ plays the role of $i$.

Observe that

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1,$$

so $i^2 = -1$.

7. a) A field is a set $F$ equipped with operations of addition $(+)$ and multiplication $(\cdot)$. Both these operations are commutative and associative.

There are clear field axioms.

b) eleven.

c) There are two: addition and multiplication.

d) If $x \in F$ then there is an element $-x \in F$ such that $x + (-x) = 0$.

e) If $y \in F$, $y \neq 0$, then there is an element $y^{-1} \in F$ such that $y \cdot y^{-1} = 1$. 
8. A relation $R$ on sets $S$ and $T$ is a function if
   a) For each $s \in S$ there is a $t \in T$ such that $(s, t) \in R$.
   b) If $(s, t_1)$ and $(s, t_2)$ are in $R$ then $t_1 = t_2$.

9. a) This is not one-to-one because $f(-1) = f(1)$. It is not onto because $f(x) \neq -5$.
   b) This is one-to-one because each real number has a unique cube root.
   c) This is onto because every value between $-1$ and 1 inclusive is assumed by sine. It is not one-to-one because $\sin x = 5 \sin 2x$.
   d) This is not onto because $T^{-1}$ does not assume the value 0. It is increasing.
   \[ \frac{d}{dx} \left( T^{-1} \right) = \frac{1}{12x^2} > 0 \]
10. a) \( \mathbb{R} \) is uncountable.
\[
\phi : \mathbb{R} \to \mathbb{R} \times \mathbb{Z}
\]
\[
x \mapsto (x, 0)
\]
is one-to-one. So \( \mathbb{R} \times \mathbb{Z} \) is uncountable.

b) \( \mathbb{Z} \) is countable, \( \mathbb{N} \) is countable, so \( \mathbb{N} \times \mathbb{N} \) is countable. The union of two countable sets is countable. Thus \( \mathbb{Z} \cup (\mathbb{N} \times \mathbb{N}) \) is countable.

c) \( (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{C} \times \mathbb{Z}) = (\mathbb{R} \times \mathbb{Z}) \).
This is uncountable by part (a).

d) \( \mathbb{C} \) is uncountable,
\[
\psi : \mathbb{C} \to \mathbb{C} \times \mathbb{C} \times \mathbb{N}
\]
\[
z \mapsto (z, 0, 0)
\]
is injective, so \( \mathbb{C} \times \mathbb{C} \times \mathbb{N} \) is uncountable.
EXTRA CREDIT: We may identify this set with the set of all sequences of indices.
This is uncountable by Cantor diagonalization.