Second Midterm

**General Instructions:** Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Provide a complete solution to each problem. If you only write the answer then you will not get full credit. If you need extra room for your work then use the backs of the pages.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

1. Sketch the graphs of these functions of two variables:

   (6 points) (a) \( f(x, y) = x^2 + y^2 \)
   - Level sets are \( x^2 + y^2 = c \) with center the origin and increasing radius as \( c \) increases.
   - Slice with \( x-z \) plane is \( z = x^2 \), a parabola.
   - Same with \( y-z \) plane.

   (6 points) (b) \( g(x, y) = \sqrt{x^2 + y^2} \)
   - Level sets are \( \sqrt{x^2 + y^2} = c \) or \( x^2 + y^2 = c^2 \) with center the origin and radius increasing as \( c \) increases.
   - Slice with \( x-z \) plane is \( z = \sqrt{x^2} = |x| \), a line.
   - Same with \( y-z \) plane.
2. We are given functions $f(x, y)$, $g(s, t)$, and $h(x, t)$ such that $F(s, t) = f(g(s, t), h(s, t))$ makes sense. We also know that

- $g(1, 2) = 3,$
- $h(1, 2) = 4,$
- $\frac{\partial g}{\partial s}(1, 2) = 5,$
- $\frac{\partial g}{\partial t}(1, 2) = 6,$
- $\frac{\partial h}{\partial s}(1, 2) = 7,$
- $\frac{\partial h}{\partial t}(1, 2) = 8,$
- $\frac{\partial f}{\partial x}(3, 4) = 9,$
- $\frac{\partial f}{\partial y}(3, 4) = 10.$

Calculate $\frac{\partial F}{\partial s}(1, 2)$ and $\frac{\partial F}{\partial t}(1, 2)$. [Hint: Your answer in each instance should be a number.]

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial h}{\partial s} = 9 \cdot 5 + 10 \cdot 7 = 115$$

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial h}{\partial t} = 9 \cdot 6 + 10 \cdot 8 = 134.$$
3. Use Taylor’s formula to find a quadratic polynomial that approximates the function \( g(x, y) = \cos(x + y) \) near the origin.

\[
\begin{align*}
\cos(x, y) &= g(x, y) \approx g(0, 0) + x \cdot g_x(0, 0) + y \cdot g_y(0, 0) \\
&\quad + \frac{1}{2} \left( x^2 g_{xx}(0, 0) + 2xy g_{xy}(0, 0) + y^2 g_{yy}(0, 0) \right) \\
&= 1 + x \cdot (-\sin 0) + y \cdot (-\sin 0) \\
&\quad + \frac{1}{2} \left( x^2 \cdot (-\cos 0) + 2xy \cdot (-\cos 0) + y^2 \cdot (-\cos 0) \right) \\
&= 1 - \frac{1}{2} x^2 - xy - \frac{1}{2} y^2
\end{align*}
\]

4. Let \( F(x, y, z) = x^2 - y^2 z \). What is the direction of greatest increase for this function at the point \( (1, 2, 3) \) of its domain?

\[
\nabla F = \langle 2x, -2yz, -y^2 \rangle
\]

\[
\nabla F (1, 2, 3) = \langle 2, -12, -9 \rangle
\]

is the direction of greatest increase.
(10 points) 5. Find the tangent plane to the graph of the function \( f(x, y) = x^2 + y^3 \) at the point \((1, 1, 2)\).

Consider \( h(x, y, z) = z - x^2 - y^3 \). Then
\[
\nabla h = \langle -2x, -3y^2, 1 \rangle.
\]
At the point \((1, 1, 2)\) then
\[
\text{The normal vector is} \quad \nabla h(1, 1, 2) = \langle -2, -3, 1 \rangle.
\]
The plane then has equation
\[
\langle -2, -3, 1 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 1, 2 \rangle) = 0
\]
or
\[
-2x - 3y + z = -3.
\]

(12 points) 6. Find and identify all local maxima and local minima for the function \( f(x, y) = (x-2)(y+3)y \).

\[
\frac{\partial f}{\partial x} = y^2 + 3y = 0 \quad \Rightarrow \quad y(y+3) = 0
\]
\[
\frac{\partial f}{\partial y} = (x-2)(2y+3) = 0
\]
If \( y = 0 \) then \( (x-2)3 = 0 \) \( \Rightarrow \) \( x = 2 \)
If \( y = -3 \) then \( (x-2)(-3) = 0 \) \( \Rightarrow \) \( x = 2 \)

So we have \((2, 0), (2, -3)\) as critical points.

The Hessian is
\[
\left(\begin{array}{cc}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{array}\right) = \left(\begin{array}{cc}
0 & 2y+3 \\
2y+3 & 2x-4
\end{array}\right).
\]
The determinant is \(- (2y+3)^2 < 0\).

Thus both points are saddle points.
(10 points) 7. Use the method of Lagrange multipliers to find the point on the plane \( x + 2y + 3z = 6 \) that is nearest to the origin.

We wish to minimize the function \( f(x, y, z) = x^2 + y^2 + z^2 \)
subject to the constraint \( g(x, y, z) = x + 2y + 3z = 6 \),

We set \( \nabla f = \lambda \nabla g \) or
\[
\begin{pmatrix}
2x \\
2y \\
2z
\end{pmatrix} = \lambda \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]

Thus
\[
2x = \lambda \\
2y = 2\lambda \\
2z = 3\lambda
\]

and \( x + 2y + 3z = 6 \) \( \star \)

So \( x = \frac{\lambda}{2}, y = \lambda, z = \frac{3\lambda}{2} \). Plugging these into \( \star \) yields
\[
x + 2x + 9\lambda/2 = 6 \quad \Rightarrow \quad \lambda = 6/7.
\]
The nearest stationary point is
\[
\left( \frac{3}{7}, \frac{6}{7}, \frac{9}{7} \right).
\]

(10 points) 8. Reverse the order of integration and then evaluate the integral

\[
\int_0^1 \int_y^1 e^{x^2} \, dx \, dy.
\]

\[
= \int_0^1 \int_0^x e^{x^2} \, dx \, dy \\
= \int_0^1 x e^{x^2} \, dx \\
= \frac{1}{2} e^{x^2} \bigg|_0^1 = \frac{1}{2} e - \frac{1}{2}.
\]
(10 points) 9. Set up but do not evaluate the integral to calculate the solid bounded by the two paraboloids \( y = x^2 + y^2 + 1 \) and \( y = -2x^2 - 2y^2 + 10 \).

The surfaces intersect when \( x^2 + y^2 + 1 = -2x^2 - 2y^2 + 10 \)
\[ \Rightarrow 3x^2 + 3y^2 = 9 \quad \Rightarrow x^2 + y^2 = 3. \]

The second surface lies above the first. So the integral is

\[ \int \int \left[ -2x^2 - 2y^2 + 10 \right] - \left[ x^2 + y^2 + 1 \right] \, dx \, dy \]
\[ \text{over } x^2 + y^2 \leq 3 \]
\[ \Rightarrow \int_0^3 \int_{\sqrt{3-x^2}}^{\sqrt{3-x^2}} \left[ -2x^2 - 2y^2 + 10 \right] - \left[ x^2 + y^2 + 1 \right] \, dy \, dx. \]

(6 points) 10. Give all possible polar coordinates for the point \((-1, \sqrt{3})\) specified in rectangular coordinates.

The point has distance \( \sqrt{1 + 3} = 2 \) from the origin. Its ray subtends an angle of \( 2\pi/3 \). Thus the possible polar coordinates are

\( (2, \frac{2\pi}{3} + 2k\pi) \).

Extra Credit: Is the function

\[ f(x, y) = \begin{cases} x^2 + y^2 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

continuous at the origin? Why or why not? [Answer on the back of the page.]
Answer to Extra Credit Problem:

Note that if \(|x| < 6, |y| < 6\) then

\[
\left( xy^2 \right) < 8 y^2 < 8 \left( x^2 + y^2 \right).
\]

Hence \(\left| \frac{xy^2}{x^2 + y^2} \right| < 8\).

It follows that

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = 0 = f(0, 0).
\]

So, the function is continuous at \((0, 0)\).