

Part I, True or False, 3 points each

Answer A for True, B for False.

1. If \mathbf{u} and \mathbf{v} are nonzero vectors then the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to \mathbf{u} .

yes

2. If the vector function $\mathbf{r}(t)$ has constant length, $|\mathbf{r}(t)| = \text{constant}$, then $\mathbf{r}(t)$ is perpendicular to $\frac{d}{dt}\mathbf{r}(t)$.

yes

3. The only curves in three dimensional space with constant curvature (that is, the curvature function $\kappa(\mathbf{u})$ takes the same value at every point of the curve) are circles and straight lines.

no : a helix also has constant curvature.

4. If two planes are not parallel then they intersect in a line and the direction of that line is parallel to the cross product of the normal vectors for the two planes.

yes

5. If $L(t)$ is the length of the curve $\mathbf{r}(s)$ for $0 \leq s \leq t$ then $L'(t) = |\mathbf{r}'(t)|$.

yes

*rate of change of distance
traveled = speed*

$$L'(t) = \frac{d}{dt} \int_0^t |\mathbf{r}'(s)| ds \stackrel{\text{FTC}}{=} |\mathbf{r}'(t)|$$

Part II, Multiple Choice, 5 points each

1. Find the center and radius of the sphere with equation $x^2 - 2x + y^2 + z^2 + 6z = 10$

- a. Center $(2, 3, -1)$, radius $\sqrt{22}$,
- b. Center $(-2, 0, -1)$, radius $\sqrt{14}$,
- c. Center $(1, 0, -3)$, radius $\sqrt{20}$,
- d. Center $(-2, -3, -1)$, radius $\sqrt{18}$,
- e. Center $(2, 0, -1)$, radius $\sqrt{19}$.

$$x^2 - 2x + y^2 + z^2 + 6z = 10$$

$$x^2 - 2x + 1 + y^2 + z^2 + 6z + 9 = 10 + 1 + 9$$

$$(x-1)^2 + y^2 + (z+3)^2 = 20$$

2. The vector $\overrightarrow{P_1 P_2}$ goes from initial point $P_1 = (2, 0, 8)$ to terminal point $P_2 = (-1, -2, -3)$. Find the vector of length 1 which points in the same direction as $\overrightarrow{P_1 P_2}$.

- a. $\frac{1}{\sqrt{68}}(4, 4, 6)$,
- b. $\frac{1}{\sqrt{30}}(1, -2, 5)$,
- c. $\frac{1}{\sqrt{68}}(4, 4, 6)$,
- d. $\frac{1}{\sqrt{134}}(-3, -2, -11)$,
- e. $\frac{1}{\sqrt{30/4}}\left(\frac{1}{2}, -1, \frac{5}{2}\right)$.

$$\begin{aligned} P_2 - P_1 &= (-1-2, -2-0, -3-8) \\ &= (-3, -2, -11) \end{aligned}$$

$$\begin{aligned} \text{Ans} &= \frac{(-3, -2, -11)}{\sqrt{3^2 + 2^2 + 11^2}} \end{aligned}$$

3. Suppose $\mathbf{u} = (2, -2, 1)$, $\mathbf{v} = (1, -6, 3)$. Compute the length $|3\mathbf{u} - 2\mathbf{v}|$.

- a. 7,
- b. 8,
- c. $\sqrt{23}$,
- d. $\sqrt{61}$,
- e. $\sqrt{31}$.

$$3\mathbf{u} - 2\mathbf{v} = (3 \cdot 2 - 2 \cdot 1, 3(-2) - 2(-6))$$

$$= (4, 6, -3)$$

$$\text{Length} = \sqrt{16 + 36 + 9}$$

4. Suppose $\mathbf{u} = (2, -2, 1)$, $\mathbf{v} = (1, -6, 3)$, $\mathbf{w} = (2, -2, 1)$. Compute $(\mathbf{u} + \mathbf{v}) \times \mathbf{w}$.

- a. $(0, 5, 10)$,
- b. $(4, 18, -23)$,
- c. $(4, 1, 0)$,
- d. $(-6, 2, -18)$,
- e. $(-12, 0, 14)$.

$$\mathbf{u} + \mathbf{v} = (3, -8, 4)$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -8 & 4 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= (-8 - (-8)) \vec{i}$$

$$- (3 - 8) \vec{j}$$

$$+ (-6 - (-16)) \vec{k}$$

$$= 0 \vec{i} - (-5) \vec{j} + 10 \vec{k}$$

5. Find the volume of the (box) parallelepiped determined by the three edges

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{w} = \mathbf{i} + \mathbf{k}.$$

- a. 3,
- b. 4,
- c. 1,
- d. 5,
- e. 8.

$$\begin{aligned} & \left| \det \begin{pmatrix} 2 & -3 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \right| \\ &= \left| -2(-6+0) - (-1)(-6) - 0 \right| \\ &= |-2-6+1+6| = |-1| = 1 \end{aligned}$$

6. The line L through the point $(3, 1, 7)$ and parallel to the vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$ passes through the point $(1, b, c)$. Find b and c .

- a. $b = 1, c = -1,$
- b. $b = 3, c = 5,$
- c. $b = -3, c = -15,$
- d. $b = 3, c = -9,$
- e. $b = -1, c = 9.$

$$L = (3, 1, 7) + t(1, 1, -1)$$

$$x = 1 \rightarrow t = -2$$

$$\text{so } b = 1 + (-2)1 = -1$$

$$c = 7 + (-2)(-1) = 9$$

7. Find the cosine of the angle at which the planes $3x - 4y + 2z = 7$ and $2x + 3y + 4z = 8$ intersect.

a. $\frac{8}{\sqrt{17}}$,

b. $\frac{2}{29}$,

c. $\frac{8}{29}$,

d. $\frac{-2}{\sqrt{17}}$,

e. $\frac{1}{\sqrt{17}}$.

Normals $\vec{a} (3, -4, 2)$ $\vec{b} (2, 3, 4)$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{6 - 12 + 8}{\sqrt{29} \sqrt{29}} = \frac{2}{29}$$

8. A particle starts at time $t = 0$ at position $(1, 2, 3)$ and velocity $\mathbf{v} = (0, 0, 0)$. It is then subjected to a constant acceleration $\mathbf{a} = (1, 0, -2)$. What is the position of the particle at time $t = 1$?

a. $(\frac{3}{2}, 0, -2)$,

b. $(0, \frac{2}{3}, 1)$,

c. $(\frac{3}{2}, 2, 2)$,

d. $(\frac{1}{2}, \frac{3}{2}, 0)$,

e. $(1, 1, -2)$.

$$\mathbf{a}(t) = (1, 0, -2)$$

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(r) dr$$

$$= (1, 0, -2) \cdot t + \vec{c}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(s) ds$$

$$= (1, 0, -2) \frac{t^2}{2} + \vec{c}_0 t + \vec{c}_1$$

$$\vec{c}_1 = \text{start position} = (1, 2, 3)$$

$$\vec{c}_0 = \text{start velocity} = (0, 0, 0)$$

$$t = 1 \quad \text{Ans} = (1, 0, -2) \cdot \frac{1}{2} + (1, 2, 3)$$

9. Set $\mathbf{r}(t) = (2t+1)\mathbf{i} + (\sin(t^2))\mathbf{j} - 3\mathbf{k}$, find $\mathbf{r}'(0)$.

- a. $\mathbf{r}'(0) = 2\mathbf{i}$,
 b. $\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j}$,
 c. $\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$,
 d. $\mathbf{r}'(0) = \mathbf{j} - 3\mathbf{k}$,
 e. $\mathbf{r}'(0) = -3\mathbf{k}$.

$$\begin{aligned}\mathbf{r}(t) &= (2t+1)\hat{\mathbf{i}} + \sin(t^2)\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \\ \mathbf{r}'(t) &= 2\hat{\mathbf{i}} + (\cos t^2)(2t)\hat{\mathbf{j}} \\ \mathbf{r}'(0) &= 2\hat{\mathbf{i}} + 0 = 2\hat{\mathbf{i}}\end{aligned}$$

10. Evaluate

$$\int_1^2 \left[\frac{1}{t}\mathbf{i} - 2t\mathbf{j} + \mathbf{k} \right] dt.$$

- a. $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$,
 b. $\ln 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$,
 c. $\ln 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$,
 d. $(\arctan 2 - \arctan 1)\mathbf{i} - 3\mathbf{j} + \mathbf{k}$,
 e. $2\mathbf{i} - \mathbf{j}$.

$$\int_1^2 \left[\frac{1}{t}\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + \hat{\mathbf{k}} \right] dt$$

$$\begin{aligned}&= \left(\int_1^2 \frac{1}{t} dt \right) \hat{\mathbf{i}} \\ &\quad + \left(\int_1^2 -2t dt \right) \hat{\mathbf{j}} \\ &\quad + \left(\int_1^2 1 dt \right) \hat{\mathbf{k}}\end{aligned}$$

11. Find the length of the curve

$$\mathbf{r}(t) = 2t \mathbf{i} + 4 \cos t \mathbf{j} - 4 \sin t \mathbf{k}; \quad 0 \leq t \leq 2\pi.$$

a. $2\sqrt{2}\pi$,

b. $2 + \pi$,

c. $2\sqrt{20}\pi$,

d. $2 + 20\pi$,

e. $1 + 8\pi$.

$$\begin{aligned} \mathbf{r}'(t) &= 2\mathbf{i} - 4 \sin t \mathbf{j} - 4 \cos t \mathbf{k} \\ |\mathbf{r}'(t)| &= \sqrt{2^2 + (-4 \sin t)^2 + (4 \cos t)^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \\ L &= \int_0^{2\pi} |\mathbf{r}'(t)| dt = 2\pi\sqrt{20} \end{aligned}$$

12. A particle A moves from $(0, 0)$ to $(1, 1)$ as t varies from 0 to 1. Its position at time t is given by $\mathbf{r}_A(t) = (t, \sqrt{t})$. Particle B moves along from the same start point to the same final point with its position at time t given by $\mathbf{r}_B(t) = (t, t^2)$. For which t between 0 and 1 is the particle A moving faster than particle B ?

a. All t ,

b. No t ,

c. $t > 16^{-1/3}$,

d. $t < 16^{-1/3}$,

e. Can't be determined from the given information.

$$\begin{aligned} \text{speed of } A &= |\mathbf{r}'_A(t)| = \left| \left(1, \frac{1}{2\sqrt{t}} \right) \right| \\ &= \sqrt{1 + \frac{1}{4t}} \end{aligned}$$

$$\text{speed of } B = |\mathbf{r}'_B(t)| = \sqrt{1 + 4t^2}$$

$$|\mathbf{r}'_A| > |\mathbf{r}'_B| \text{ exactly if } \sqrt{1 + \frac{1}{4t}} > \sqrt{1 + 4t^2}$$

$$\text{or } 1 + \frac{1}{4t} > 1 + 4t^2 \quad \text{or } \frac{1}{4t} > t^2 \quad \text{or } t < 16^{-\frac{1}{3}}$$

13. The motion of a particle in space is described by the formula

$$\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$$

Find the angle between the velocity vector of the particle and the acceleration vector at time $t = 2$.

a. $\cos^{-1}\left(\frac{-2}{\sqrt{15}}\right) = 2.113,$

b. $\cos^{-1}\left(\frac{12}{\sqrt{65}\sqrt{15}}\right) = 1.176,$

c. $\cos^{-1}\left(\frac{7}{6\sqrt{11}}\right) = 1.211,$

d. $\cos^{-1}\left(\frac{-4}{\sqrt{14}\sqrt{7}}\right) = 1.986,$

e. $\cos^{-1}\left(\frac{8}{2\sqrt{17}}\right) = 0.244.$

$$\vec{v}(t) = \mathbf{r}'(t) = \vec{\mathbf{j}} + 2t\vec{\mathbf{k}}$$

$$\vec{a}(t) = \vec{2\mathbf{k}}$$

at time $t = 2$

$$\vec{v} = \vec{\mathbf{j}} + 4\vec{\mathbf{k}}$$

$$\vec{a} = 2\vec{\mathbf{k}}$$

$$\gamma = \theta = \cos^{-1} \frac{\vec{v} \cdot \vec{a}}{|\vec{v}| |\vec{a}|}$$

$$\gamma = \cos^{-1} \frac{8}{\sqrt{17} \sqrt{4}}$$

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Part III, Hand Graded, 20 pointsShow enough work so that it is clear how you arrived at your answer.The position of a particle at time t is given by

$$\mathbf{r}(t) = (t^2 + 1) \mathbf{i} + 2t \mathbf{j} - 6 \mathbf{k}$$

1. Find the velocity \mathbf{v} .
2. Find the acceleration \mathbf{a} .
3. Find the unit tangent vector \mathbf{T} for the path of the particle.
4. Find the unit normal vector \mathbf{N} for the path of the particle.
5. Decompose the acceleration into its tangential and normal components. That is, find numbers a_T and a_N so that

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

Note: All of these quantities are functions of the variable t .

$$1. \mathbf{v} = \mathbf{r}'(t) = 2t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}$$

$$2. \mathbf{a} = \mathbf{v}'(t) = \mathbf{r}''(t) = 2 \hat{\mathbf{i}}$$

$$3. \mathbf{T} = \frac{1}{|\mathbf{v}(t)|} \mathbf{v}(t) = \frac{1}{\sqrt{4t^2+4}} (2t, 2, 0)$$

$$= \frac{1}{\sqrt{t^2+1}} (t, 1, 0)$$

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4. $N = T' / |T'|$

$$T = (1+t^2)^{-1/2} (t, 1, 0)$$

$$T' = (1+t^2)^{-1/2} (1, 0, 0) + (-\frac{1}{2})(1+t^2)^{-3/2} (2t) (t, 1, 0)$$

$$= (1+t^2)^{-1/2} (1, 0, 0) - (1+t^2)^{-3/2} t (t, 1, 0)$$

$$= \left((1+t^2)^{-1/2} - (1+t^2)^{-3/2} t^2, -t (1+t^2)^{-3/2}, 0 \right)$$

5. $a_T =$

$$N = \frac{\left((1+t^2)^{-1/2} - (1+t^2)^{-3/2} t^2, -t (1+t^2)^{-3/2}, 0 \right)}{\sqrt{\left[(1+t^2)^{-1/2} - (1+t^2)^{-3/2} t^2 \right]^2 + \left(-t (1+t^2)^{-3/2} \right)^2}}$$

$$a_T = \frac{d}{dt} |V| = \frac{d}{dt} \sqrt{4t^2 + 4} = \frac{1}{2} (4t^2 + 4)^{-1/2} (8t)$$

$$= 4t (4t^2 + 4)^{-1/2}$$

$|g|^2$

$$a_N = \sqrt{4 - 4^2 t^2 (4t^2 + 4)^{-1}}$$

$$= \sqrt{4 - \frac{16t^2}{4t^2 + 4}}$$

$$= \sqrt{\frac{16}{4t^2 + 4}}$$

$$= \frac{2}{\sqrt{t^2 + 1}}$$

However it wasn't necessary to do those last few simplifications.

In fact. $\frac{1}{\sqrt{1+t^2}} = \frac{t^2}{(1+t^2)^{3/2}}$

$$= \frac{1}{\sqrt{1+t^2}} \left(1 - \frac{t^2}{1+t^2} \right)$$

$$= \frac{1}{\sqrt{1+t^2}} \left(\frac{1}{1+t^2} \right) = (1+t^2)^{-3/2}$$

which leads to

$$N = (1+t^2)^{-1/2} (1-t, 0)$$