TABLE 13.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\qquad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \, \delta \, dV, \qquad M_{xz} = \iiint_D y \, \delta \, dV, \qquad M_{xy} = \iiint_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta(x, y) \, dA$ $\qquad \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x \, \delta(x, y) \, dA, \qquad M_x = \iint_R y \, \delta(x, y) \, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

TABLE 13.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID		
About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\,dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\,dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\,dV$	
About a line L:	$I_L = \iiint r^2\,\delta\,dV$	$r(x, y, z)$ = distance from the point (x, y, z) to line L
TWO-DIMENSIONAL PLATE		
About the x-axis:	$I_x = \iint y^2\,\delta\,dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\,dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\,dA,$	$r(x, y)$ = distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\,dA = I_x + I_y$	

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

TABLE 14.1 Mass and moment formulas for coil springs, thin rods, and wires lying along a smooth curve C in space

Mass: $M = \int_C \delta(x, y, z)\, ds \qquad \delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\,\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\,\delta\, ds,$$

$I_L = \int_C r^2\,\delta\, ds \qquad r(x, y, z) =$ distance from the point (x, y, z) to line L

Math 233 Exam III November 12, '07

Part I, True or False, 3 points each. Answer A for True, B for False.

1. Suppose D is a region in three dimensional space that is filled by an object whose density, $\delta(x,y,z)$, is constant. Then $\bar{y}$, the y –coordinate of the center of mass of the object, equals the average value of the function $f(x,y,z) = y$ on D.

2. The area inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$ is given by

$$\text{area} = \int_0^{\pi/3} \int_1^{2\cos\theta} r\,dr d\theta.$$

3. If C is a curve in three dimensional space then $\int_C ds$ is the length of C.

4. If $f(x,y,z)$ gives the temperature at the point (x,y,z) then at each point ∇f, the gradient vector field of f, points in a direction is which the temperature is not changing.

5. If C is a loop in the plane, a closed curve, and $\mathbf{F}$ is a force field then the work done by the force $\mathbf{F}$ in moving an object around C is zero.

Part II, Multiple Choice, 6 points each

6. Evaluate the integral

$$\iint_R x - y \, dA$$

where R is the triangle with vertices $(0,0)$, $(2,0)$ and $(2,1)$.

a. 0

b. 1

c. 2

d. 3

e. 4

7. Find $\bar{x}$, the x coordinate of the center of mass (center of gravity) of a triangular metal plate with corners at $(0,1)$, $(1,0)$, and $(1,1)$ and having constant density $\delta(x,y) = 3$.

a. 1/4

b. 1/2

c. 3/4

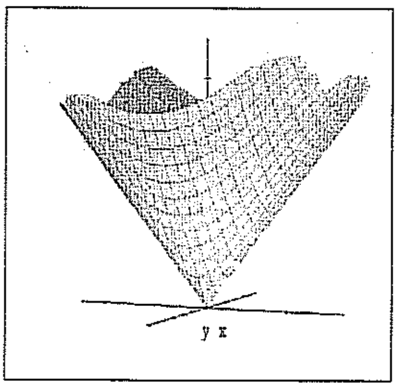
d. 1/3

e. 2/3

8. Suppose a metal plate with density $\delta(x,y) = 1$ occupies the region in which $1 \le x^2 + y^2 \le 4$ and $y \ge 0$. Find the second polar moment (the moment of inertia about the origin) of the plate.

a. $\frac{21\pi}{4}$
b. $\frac{15\pi}{4}$
c. $\frac{3\pi}{4}$
d. $\frac{7\pi}{2}$
e. $\frac{9\pi}{2}$

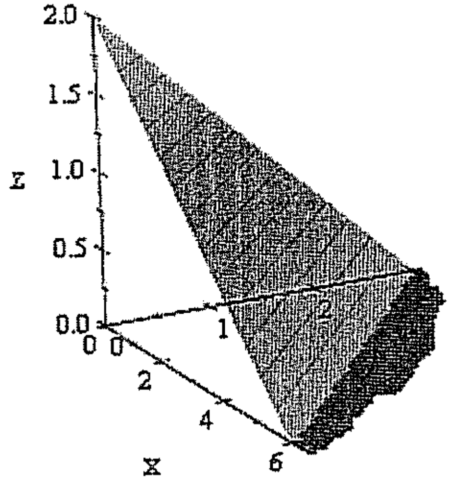
9. Find the volume of the region in three dimensions which is in the first octant, i.e., $x,y,z \ge 0$, inside the sphere centered at the origin of radius 2 and outside of the cone in the picture. That cone has vertex at the origin, the positive z-axis as its axis and boundary making an angle $\pi/4$ with the z-axis.



a. $\frac{2\pi}{3}$
b. $\frac{\pi\sqrt{2}}{3}$
c. $\frac{2\pi\sqrt{2}}{3}$
d. $\pi\sqrt{2}$
e. $2\pi\sqrt{2}$

10. Find the volume in the first octant (i.e. $x, y, z \geq 0$) below the plane $x + 2y + 3z = 6$.

a. 2
b. 4
c. 6
d. 8
e. 12



11. The region inside the sphere centered at the origin having radius 2 and in the first octant (i.e. x, y, and z are all positive) is filled with material of density z. Find the mass of this object.

a. $\pi/4$
b. $\pi/2$
c. π
d. 2π
e. 4π

12. Suppose R is the triangular region in the plane with vertices $(0,0)$, $(2,2)$, and $(3,1)$ and that you are going to evaluate

$$\int\int_R x + 3y \, dA$$

by making the substitutions $x = 3u + v$, $y = u + v$. This leads to the which integral?

a. $\displaystyle\int_0^1 \int_0^{2-2u} 12u + 8v \, dvdu$

b. $\displaystyle\int_0^1 \int_0^{1-u} 6u + 4v \, dvdu$

c. $\displaystyle\int_{-1}^1 \int_{u-1}^{u+1} 6u + 4v \, dvdu$

d. $\displaystyle\int_{-1}^0 \int_0^{1-u} -12u - 8v \, dvdu$

e. $\displaystyle\int_{-1}^0 \int_{-u}^{u} 3u + 2v \, dvdu$

13. To evaluate the integral

$$\int_C x + 2y\, ds$$

where C is the part of the curve $y = x^2$ starting at $(0,0)$ and finishing at $(1,1)$ you can compute which?

a. $\int_0^1 (t + 2t^2)\sqrt{t + t^2}\, dt$

b. $\int_0^1 (2t + 2t^2) dt$

c. $\int_0^1 (2t + t^2)\sqrt{t^2 + 4}\, dt$

d. $\int_0^1 (t + t^2)\, dt$

e. $\int_0^1 (t + 2t^2)\sqrt{1 + 4t^2}\, dt$

14. Suppose $\mathbf{F} = x\,\mathbf{i} + 2\,\mathbf{j}$ a force field. Suppose C is the part of the circle centered at the origin and with radius one in the first quadrant. C is oriented with starting point $(1,0)$ and finishing point $(0,1)$. Compute the work done by the force in moving an object along the curve.

a. 0

b. 1/2

c. 1

d. 3/2

e. 2

15. For the force field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ compute the flux of $\mathbf{F}$ across the circle of radius one centered at the origin traversed counterclockwise.

a. -2π

b. $-\pi$

c. 0

d. π

e. 2π

16. In order to have

$$\int_{-1}^{1}\left(\int_{0}^{\sqrt{1-y^2}} f(x,y)\,dx\right)dy = \int_{0}^{1}\left(\int_{a(x)}^{b(x)} f(x,y)\,dy\right)dx$$

we must have

a. $a(x) = \sqrt{1-x^2}$, $b(x) = 1-x^2$

b. $a(x) = 0$, $b(x) = \sqrt{1-x^2}$

c. $a(x) = -\sqrt{1-x^2}$, $b(x) = \sqrt{1-x^2}$

d. $a(x) = -\sqrt{1-x^2}$, $b(x) = 0$

e. $a(x) = -(1-x^2)$, $b(x) = 1-x^2$

Name	ID Number

Part III, Hand Graded, 20 points. **Show enough work so that it is clear how you arrived at your answer.**

1. Compute the volume of the region above the interior of the circle $(x-1)^2+y^2=1$ and below the plane $z=x$.

Name	ID Number

Part III, Hand Graded, 20 points. **Show enough work so that it is clear how you arrived at your answer.**

2. Suppose R is the triangle with vertices $(0,0)$, $(1,1)$ and $(1,-1)$. Evaluate the integral

$$\iint_R \sqrt{x^2-y^2}\, dA$$

using the substitution $x+y=2u$, $x-y=2v$ (or by some other method if you choose).