TABLE 13.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \, \delta \, dV, \qquad M_{xz} = \iiint_D y \, \delta \, dV, \qquad M_{xy} = \iiint_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta(x, y) \, dA$ $\delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x \, \delta(x, y) \, dA, \qquad M_x = \iint_R y \, \delta(x, y) \, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

TABLE 13.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID		
About the x-axis:	$I_x = \iiint (y^2 + z^2)\, \delta\, dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\, \delta\, dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\, \delta\, dV$	
About a line L:	$I_L = \iiint r^2\, \delta\, dV$	$r(x, y, z)$ = distance from the point (x, y, z) to line L
TWO-DIMENSIONAL PLATE		
About the x-axis:	$I_x = \iint y^2\, \delta\, dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\, \delta\, dA$	
About a line L:	$I_L = \iint r^2(x, y)\, \delta\, dA,$	$r(x, y)$ = distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\, \delta\, dA = I_x + I_y$	

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

TABLE 14.1 Mass and moment formulas for coil springs, thin rods, and wires lying along a smooth curve C in space

Mass: $M = \int_C \delta(x, y, z)\, ds \qquad \delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x\, \delta\, ds, \qquad M_{xz} = \int_C y\, \delta\, ds, \qquad M_{xy} = \int_C z\, \delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2)\, \delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\, \delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\, \delta\, ds,$$

$I_L = \int_C r^2\, \delta\, ds \qquad r(x, y, z) =$ distance from the point (x, y, z) to line L

Math 233 Exam III November 12, '07

Part I, True or False, 3 points each. Answer A for True, B for False.

1. Suppose D is a region in three dimensional space that is filled by an object whose density, $\delta(x,y,z)$, is constant. Then $\bar{y}$, the y-coordinate of the center of mass of the object, equals the average value of the function $f(x,y,z) = y$ on D.

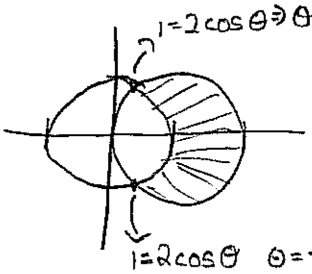
$\boxed{\text{True}}$ $M = \iiint_D \delta \, dV = \delta(\text{Volume of } D)$ since δ constant

$$\bar{y} = \frac{\iiint_D \delta y \, dV}{M} = \frac{\delta \iiint_D y \, dV}{\delta(\text{Volume of } D)} = \frac{\iiint_D y \, dV}{\text{Volume of } D} = \text{avg value of } f \text{ on } D.$$

2. The area inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$ is given by

$$\text{area} = \int_0^{\pi/3} \int_1^{2\cos\theta} r \, dr d\theta.$$

$r = 2\cos\theta$ $r = 1$



$\boxed{\text{False}}$

3. If C is a curve in three dimensional space then $\int_C ds$ is the length of C.

arclength

True

4. If $f(x,y,z)$ gives the temperature at the point (x,y,z) then at each point ∇f, the gradient vector field of f, points in a direction is which the temperature is not changing.

False: ∇f points in the direction of greatest positive change.

5. If C is a loop in the plane, a closed curve, and $\mathbf{F}$ is a force field then the work done by the force $\mathbf{F}$ in moving an object around C is zero.

False: Only true if $\vec{F}$ is a conservative force.

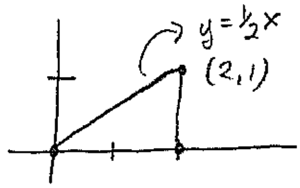
Part II, Multiple Choice, 6 points each

6. Evaluate the integral

$$\iint_R x-y\,dA$$

where R is the triangle with vertices $(0,0)$, $(2,0)$ and $(2,1)$.

- **a.** 0
- **b.** 1
- **c.** 2
- **d.** 3
- **e.** 4

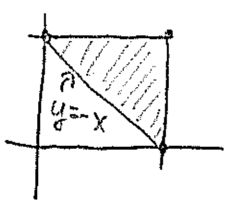


$$\int_0^2\int_0^{\frac12 x} x-y\;dy\,dx = \int_0^2 xy-\frac12 y^2\Big|_{y=0}^{\frac12 x} dx$$

$$=\int_0^2 \frac12 x^2-\frac18 x^2\,dx = \int_0^2 \frac{x^3}{8}\,dx = 1$$

7. Find $\bar{x}$, the x coordinate of the center of mass (center of gravity) of a triangular metal plate with corners at $(0,1)$, $(1,0)$, and $(1,1)$ and having constant density $\delta(x,y)=3$.

- **a.** 1/4
- **b.** 1/2
- **c.** 3/4
- **d.** ~~1/3~~
- **e.** 2/3



$$M=\int_0^1\int_{1-x}^1 3\,dy\,dx = 3\int_0^1 y\Big|_{y=1-x}^{y=1} dx$$

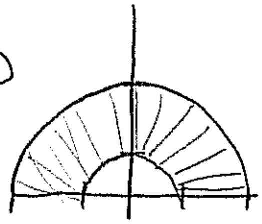
$$=3\int_0^1 1-(1-x)\,dx = 3\int_0^1 x\,dx = \frac{3}{2}$$

$$\bar{x}=\frac{2}{3}\int_0^1\int_{1-x}^1 3x\,dy\,dx = 2\int_0^1 xy\Big|_{y=1-x}^{1} dx = 2\int_0^1 x-(x-x^2)\,dx$$

$$=2\int_0^1 x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$

8. Suppose a metal plate with density $\delta(x,y) = 1$ occupies the region in which $1 \le x^2 + y^2 \le 4$ and $y \ge 0$. Find the second polar moment (the moment of inertia about the origin) of the plate.

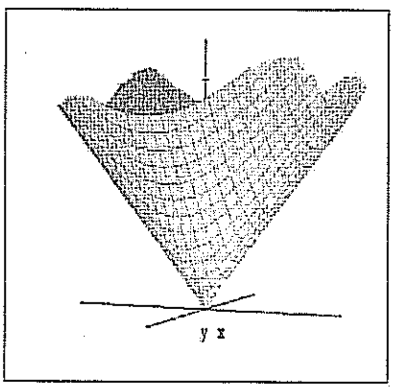
a. $\frac{21\pi}{4}$
b. $\frac{15\pi}{4}$
c. $\frac{3\pi}{4}$
d. $\frac{7\pi}{2}$
e. $\frac{9\pi}{2}$



$$\int_0^{\pi}\int_1^{2} r^2\, r\,dr\,d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} 4 - \tfrac{1}{4}\,d\theta = \frac{15}{4}\int_0^{\pi} d\theta = \frac{15\pi}{4}$$

9. Find the volume of the region in three dimensions which is in the first octant, i.e., $x,y,z \ge 0$, inside the sphere centered at the origin of radius 2 and outside of the cone in the picture. That cone has vertex at the origin, the positive z-axis as its axis and boundary making an angle $\pi/4$ with the z-axis.



$$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_{\pi/4}^{\pi/2} \frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{2} d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_{\pi/4}^{\pi/2} \frac{8}{3}\sin\phi\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi/2} -\cos\phi\Big|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} d\theta = \frac{8}{3}\int_0^{\pi/2} \frac{\sqrt{2}}{2}\, d\theta$$

$$= \frac{4\sqrt{2}}{3}\cdot\theta\Big|_0^{\pi/2} = \frac{4\pi\sqrt{2}}{6} = \frac{2\pi\sqrt{2}}{3}$$

a. $\frac{2\pi}{3}$
b. $\frac{\pi\sqrt{2}}{3}$
c. $\boxed{\frac{2\pi\sqrt{2}}{3}}$
d. $\pi\sqrt{2}$
e. $2\pi\sqrt{2}$

10. Find the volume in the first octant (i.e. $x,y,z \ge 0$) below the plane $x + 2y + 3z = 6$.

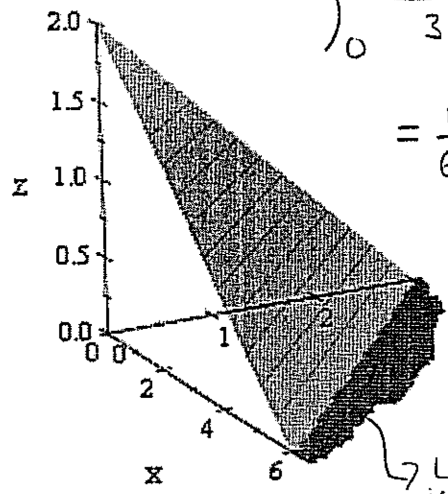
a. 2
b. 4
$\boxed{\textbf{c. } 6}$
d. 8
e. 12

$z=(6-x-2y)/3$

$$\int_0^3\int_0^{6-2y} \frac{6-x-2y}{3}\,dx\,dy = \int_0^3 \left(\frac{6-2y}{3}\right)x - \frac{x^2}{6}\Big|_{x=0}^{6-2y} dy$$

$$= \int_0^3 \frac{(6-2y)^2}{3} - \frac{(6-2y)^2}{6}\,dy = \frac{1}{6}\int_0^3 36-24y+4y^2\,dy = \frac{1}{6}\left(36y-12y^2+\frac{4}{3}y^3\right)\Big|_0^3$$

$$= \frac{1}{6}\left(36\cdot 3-12\cdot 9+\frac{4}{3}\cdot 27\right) = 6$$



Line is $x+2y=6$

11. The region inside the sphere centered at the origin having radius 2 and in the first octant (i.e. x,y, and z are all positive) is filled with material of density z. Find the mass of this object.

a. $\pi/4$
b. $\pi/2$
$\boxed{\textbf{c. } \pi}$
d. 2π
e. 4π

$z = \rho\cos\phi$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^4}{4}\cos\phi\sin\phi\Big|_{\rho=0}^{2} d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2} 4\cos\phi\sin\phi\,d\phi\,d\theta = 4\int_0^{\pi/2} \frac{\sin^2\phi}{2}\Big|_{\phi=0}^{\pi/2} d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\cdot\frac{\pi}{2} = \pi$$

12. Suppose R is the triangular region in the plane with vertices $(0,0)$, $(2,2)$, and $(3,1)$ and that you are going to evaluate

$$\iint_R x+3y\,dA$$

by making the substitutions $x = 3u+v$, $y = u+v$. This leads to the which integral?

a. $\displaystyle\int_0^1\int_0^{2-2u} 12u+8v\,dvdu$

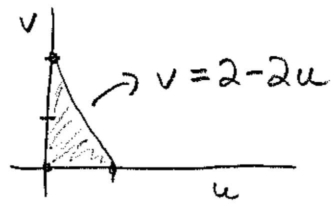
b. $\displaystyle\int_0^1\int_0^{1-u} 6u+4v\,dvdu$

c. $\displaystyle\int_{-1}^1\int_{u-1}^{u+1} 6u+4v\,dvdu$

d. $\displaystyle\int_{-1}^0\int_0^{1-u} -12u-8v\,dvdu$

e. $\displaystyle\int_{-1}^0\int_{-u}^{u} 3u+2v\,dvdu$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 3\cdot 1 - 1\cdot 1 = 2$$



$u = \frac{1}{2}(x-y)$

$v = \frac{1}{2}(3y-x)$

(x,y)	(u,v)
$(0,0)$	$(0,0)$
$(2,2)$	$(0,2)$
$(3,1)$	$(1,0)$

$$\int_0^1\int_0^{2-2u} [3u+v+3(u+v)]\cdot 2\,dvdu$$

$$\int_0^1\int_0^{2-2u} 12u+8v\,dvdu$$

13. To evaluate the integral

$$\int_C x + 2y\, ds$$

where C is the part of the curve $y = x^2$ starting at $(0,0)$ and finishing at $(1,1)$ you can compute which?

$r(t) = \langle t, t^2 \rangle$

$v(t) = \langle 1, 2t \rangle$

$|v(t)| = \sqrt{1+4t^2}$

$\int_0^1 (t+2t^2)(\sqrt{1+4t^2})\, dt$

a. $\int_0^1 (t+2t^2)\sqrt{t+t^2}\, dt$

b. $\int_0^1 (2t+2t^2)dt$

c. $\int_0^1 (2t+t^2)\sqrt{t^2+4}\, dt$

d. $\int_0^1 (t+t^2)\, dt$

e. $\int_0^1 (t+2t^2)\sqrt{1+4t^2}\, dt$

14. Suppose $\mathbf{F} = x\,\mathbf{i} + 2\,\mathbf{j}$ a force field. Suppose C is the part of the circle centered at the origin and with radius one in the first quadrant. C is oriented with starting point $(1,0)$ and finishing point $(0,1)$. Compute the work done by the force in moving an object along the curve.

a. 0
b. 1/2
c. 1
d. 3/2
e. 2

$r(t) = \langle \cos t, \sin t \rangle$ $\quad F(t) = \langle \cos t, 2 \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$

$$\text{Work} = \int_0^{\pi/2} F \cdot r'(t)\, dt = \int_0^{\pi/2} -\cos t \sin t + 2\cos t\, dt$$

$$= \frac{\cos^2 t}{2} + 2\sin t \Big|_0^{\pi/2} = -\tfrac{1}{2} + 2 = 1\tfrac{3}{2}$$

15. For the force field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ compute the flux of $\mathbf{F}$ across the circle of radius one centered at the origin traversed counterclockwise.

- **a.** -2π
- **b.** $-\pi$
- **c.** 0
- **d.** π
- **e.** $\boxed{2\pi}$

$r = \langle \cos t, \sin t \rangle$ $\quad dx = -\sin t\,dt$ $\quad dy = \cos t\,dt$

$M = x = \cos t$

$N = y = \sin t$

$$\int_0^{2\pi} \cos t \cos t\,dt - \sin t(-\sin t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

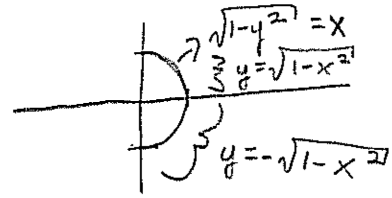
16. In order to have

$$\int_{-1}^{1}\left(\int_0^{\sqrt{1-y^2}} f(x,y)dx\right)dy = \int_0^1\left(\int_{a(x)}^{b(x)} f(x,y)dy\right)dx$$

$x^2 = 1 - y^2$

we must have

- **a.** $a(x) = \sqrt{1-x^2}$, $b(x) = 1-x^2$
- **b.** $a(x) = 0$, $b(x) = \sqrt{1-x^2}$
- **c.** $a(x) = -\sqrt{1-x^2}$, $b(x) = \sqrt{1-x^2}$
- **d.** $a(x) = -\sqrt{1-x^2}$, $b(x) = 0$
- **e.** $a(x) = -(1-x^2)$, $b(x) = 1-x^2$



Name	ID Number

Part III, Hand Graded, 20 points. **Show enough work so that it is clear how you arrived at your answer.**

1. Compute the volume of the region above the interior of the circle $(x-1)^2 + y^2 = 1$ and below the plane $z = x$.

In polar crds, $(x-1)^2+y^2=1$ becomes $r=2\cos\theta$ (see problem 2)

$x = r\cos\theta$

$$Vol = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\cos\theta\, r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\cos\theta \Big|_{r=0}^{2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\cos^4\theta}{3}\,d\theta$$

$$= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+2\cos 2\theta+\cos^2 2\theta\, d\theta$$

$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+2\cos 2\theta + \frac{1+\cos 2\theta}{2}\,d\theta = \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2}+\frac{5}{2}\cos 2\theta\, d\theta = \frac{2}{3}\left[\frac{3}{2}\theta + \frac{5}{4}\sin 2\theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\right]$$

$$\frac{2}{3}\left[\frac{3}{2}\frac{\pi}{2} + \frac{5}{4}\cdot 0 - \frac{3}{2}\left(-\frac{\pi}{2}+\frac{5}{4}\cdot 0\right)\right] = \pi$$

Name	ID Number

Part III, Hand Graded, 20 points. **Show enough work so that it is clear how you arrived at your answer.**

2. Suppose R is the triangle with vertices $(0,0)$, $(1,1)$ and ~~$(1,-1)$.~~ $(2,0)$ Evaluate the integral

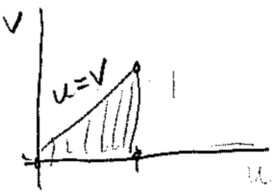
$$\iint_R \sqrt{x^2-y^2}\,dA$$

using the substitution $x+y=2u$, $x-y=2v$ (or by some other method if you choose).

$x = u+v$

$y = u-v$

(x,y)	(u,v)
$(0,0)$	$(0,0)$
$(1,1)$	$(1,0)$
$(2,0)$	$(1,1)$



$x^2-y^2=(x+y)(x-y)=2u(2v)=4uv$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2$$

$$\int_0^1\int_0^v \sqrt{4uv}\,|2|\,du\,dv = 4\int_0^1\left[\sqrt{v}\,\frac{2}{3}u^{3/2}\Big|_{u=0}^{u=v}\right]dv=\frac{8}{3}\int_0^1 v^2\,dv=\frac{8}{3}\left[\frac{1}{3}v^3\Big|_{v=0}^{v=1}\right]$$

$$=\frac{8}{9}$$