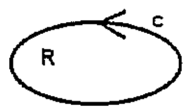
Math 233 Final Exam December 14, '07

Assume all functions are sufficiently well behaved (continuous, differentiable, etc.) so that out theories apply.

Part I, True or False, 3 points each. Answer A for True, B for False.

1. Suppose C is the positively oriented boundary curve of the region R, as in the figure, and that the area of R is A.

$$A = \oint_C -y\,dx.$$



2. If the vector field $\mathbf{F}(x,y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ is the flow field (= velocity field) of a fluid rotating steadily counterclockwise in the plane then the divergence of $\mathbf{F}$, $\operatorname{div}\mathbf{F}$, is positive.

3. If $M(x,y)$, $N(x,y)$ are functions defined in a region R in the plane and throughout that region

$$\frac{\partial}{\partial y}M = \frac{\partial}{\partial x}N,$$

then for any closed loop C in R we have

$$\oint_C M\,dx + N\,dy = 0.$$

4. If $f(x,y,z)$ has a local maximum at the point (x_0,y_0,z_0) in the interior (= inside) of its domain of definition then at (x_0,y_0,z_0) the gradient of f is zero, $\operatorname{grad} f(x_0,y_0,z_0) = \mathbf{0}$.

5. If $f(x,y)$ and $g(x,y)$ are two functions defined in a region R in the plane and if at every point in R $\operatorname{grad} f = \operatorname{grad} g$ then at all points of the region $f(x,y) = g(x,y)$.

Part II, Multiple Choice, 6 points each.

6. Suppose $f(x,y,z) = x^2 + 2y^3z$. Find the direction in which f is decreasing most rapidly at the point $(0,1,0)$.

   a. $(0,0,-1)$
   b. $(0,0,1)$
   c. $(0,1,0)$
   d. $(0,-1,0)$
   e. $(1,0,0)$
   f. $(-1,0,0)$

7. The tangent plane to the surface $x^2+y^3+z^4 = 18$ at the point $(3,0,1)$ cuts the z-axis at the point $(0,0,a)$. What is a?

   a. 9/2
   b. 11/2
   c. 13/2
   d. 15/2
   e. 17/2
   f. 19/2

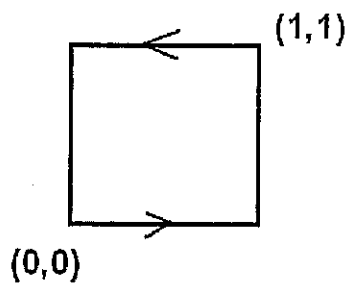
8. The function $f(x,y) = x^2 - 2xy + y^3 - y$ has two critical points, one, A, in the first quadrant (i.e. $x,y > 0$), and another one, B. Classify these critical points.

a. A loc. max; B loc min.
b. A loc. max; B loc max.
c. A loc. max; B saddle point.
d. A loc. min; B loc min.
e. A loc. min; B loc max.
f. A loc. min; B saddle point.
g. A saddle point; B loc min.
h. A saddle point; B loc max.
i. A saddle point; B saddle point.

9. Evaluate

$$\oint_C y^2\, dx + x\, dy$$

where C is the positively oriented boundary of the square bounded by the coordinate axes and the lines $x = 1$, and $y = 1$.



a. 0
b. 1
c. 2
d. 3
e. 4
f. 5

10. Let R be the rectangle $0 \le x \le 1,\ 1 \le y \le 3$. Evaluate

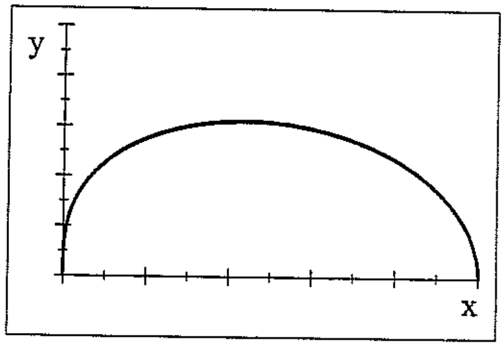
$$\iint_R \frac{x}{y}\,dA.$$

a. $\frac{1}{2}\ln 3$
b. $\frac{4}{3}$
c. $2\ln 3$
d. $\frac{3}{2}$
e. 3
f. $3+\ln 3$

11. Suppose R is the triangle with vertices $(0,0)$, $(0,2)$, $(1,2)$. Evaluate $\iint_R x\,dA$.

a. 1/2
b. 1/4
c. 1/3
d. 3/2
e. 2
f. 3

12. Find the area in the first quadrant (i.e., $x, y \geq 0$) between the curve $x = (x^2 + y^2)^{3/2}$ and the horizontal axis.



$x = (x^2 + y^2)^{3/2}$

a. $\frac{1}{4}$
b. $\frac{1}{2}$
c. $\frac{1}{4} + \frac{1}{2}\pi$
d. $1 - \pi$
e. $1 + \pi$
f. π

13. Find the volume of the region above the disk $x^2 + y^2 \leq 1$ in the $z = 0$ plane and below the surface $z = 1 + x^2 + y^2$.

a. $3\pi/2$
b. $5\pi/3$
c. $2\pi/3$
d. $\pi/2$
e. 2
f. $2\sqrt{2}$

14. Suppose $g(x,y,z)$ is the potential function $g(x,y,z) = x^2 + z^2$. Let C be the curve

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 2\pi.$$

Find the work done by the force field $\mathbf{F} = \nabla g$ in moving an object over this curve.

a. $1 + 4\pi^2$
b. 1
c. $4\pi^2$
d. $\pi^2 - 1$
e. 0
f. 2π

15. Find the average value of the function $f(x,y,z) = xz$ on the three dimensional region $0 \le x \le 1$, $0 \le y \le 3$, $1 \le z \le 3$.

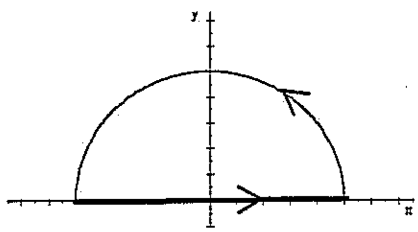
a. 1
b. 2
c. 3
d. 4
e. 5
f. 6

16. Suppose that S is the part of the plane $x + y + 3z = 6$ on which $x, y, z \geq 0$ and that $\mathbf{F}$ is the vector field $\mathbf{F}(x,y,z) = y\,\mathbf{k}$. Suppose that the normal vector for the surface is pointing toward you (rather than away). Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

a. 1
b. $9/(2\sqrt{7})$
c. $1/\sqrt{3}$
d. $\sqrt{11}/3$
e. $18/\sqrt{11}$
f. $2/(3\sqrt{7})$

17. Suppose $\mathbf{F}$ is the vector field $\mathbf{F}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$. Evaluate the circulation around (= flow integral along) C, the positively oriented boundary of the region $x^2 + y^2 \leq 1, y \geq 0$.



a. 0
b. 1
c. -1
d. π
e. $-\pi$
f. 2π

18. Suppose $\mathbf{F}$ is the vector field $\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2+1)\,\mathbf{j}$. Decide if $\mathbf{F}$ is a gradient vector field. If it is find its potential function $f(x,y)$ and evaluate $f(1,1) - f(0,0)$.

a. $f(1,1) - f(0,0) = -1$
b. $f(1,1) - f(0,0) = 0$
c. $f(1,1) - f(0,0) = 1$
d. $f(1,1) - f(0,0) = 2$
e. $f(1,1) - f(0,0) = 3$
f. $\mathbf{F}$ is not a gradient vector field.

19. Suppose B is the unit box, $B = \{(x,y,z) \text{ with } 0 \le x,y,z \le 1\}$. For the vector field $\mathbf{F}(x,y) = 2xy\,\mathbf{i} - x^2z\,\mathbf{j} - y\,\mathbf{k}$. Evaluate the divergence integral

$$\iiint_B \nabla \cdot \mathbf{F}\, dV.$$

a. 1
b. 2
c. 1/2
d. 3/4
e. 4
f. 6

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. M, N, and P are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$

Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$

Tangential form of Green's Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$

Stokes' Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Vector Identities

In the identities here, f and g are differentiable scalar functions, $\mathbf{F}$, $\mathbf{F}_1$, and $\mathbf{F}_2$ are differentiable vector fields, and a and b are real constants.

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$$

$$\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$$

$$\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$$

$$\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$$

$$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

$$(\nabla \times \mathbf{F}) \times \mathbf{F} = (\mathbf{F} \cdot \nabla)\mathbf{F} - \frac{1}{2}\nabla(\mathbf{F} \cdot \mathbf{F})$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{aligned}$$