4.2.2 a. $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$, so $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$.

b. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$, so $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

c. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -3 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & -2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 2 & -3 & 2 \end{array}\right]$,

so $A^{-1} = \begin{bmatrix} -1 & 3 & -2 \\ -1 & 2 & -1 \\ 2 & -3 & 2 \end{bmatrix}$.

4.2 d. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$, and so we see that A^{-1} does not exist.

e. $\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 0 & 5 & 8 & 1 & 0 & 2 \\ 0 & 3 & 5 & 0 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 0 & -1 & -2 & 1 & -2 & -2 \\ 0 & 3 & 5 & 0 & 1 & 2 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 3 & 5 & 0 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 0 & -1 \\ 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & -1 & 3 & -5 & -4 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -6 & 10 & 7 \\ 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \end{array}\right]$,

so $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$.

4.2.3 a. (i) $A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$; (ii) $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$; (iii) $\mathbf{b} = 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

b. (i) $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & -1 & -3 \\ -6 & 1 & 2 \end{bmatrix}$; (ii) $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$; (iii) $\mathbf{b} = 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

c. (i) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$; (ii) $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$; (iii) $\mathbf{b} = 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

d. (i) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$; (ii) $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$; (iii) $\mathbf{b} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

4.3.1 a. Correct: $2\mathbf{v}_1 - \mathbf{v}_3 = \mathbf{0}$.

b. Incorrect: $\mathbf{v}_2$ cannot be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$.

4.3.2 a. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \mathbf{0}$. Since $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we see that $c_1 = c_2 = 0$. Thus, $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \end{bmatrix} \right\}$ is linearly independent.

b. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} = \mathbf{0}$. From $\begin{bmatrix} 1 & 2 \\ 4 & 9 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, we see that $c_1 = c_2 = 0$. Thus, $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

c. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \mathbf{0}$. Since the reduced echelon form of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 31 \\ 0 & 1 & -14 \\ 0 & 0 & 0 \end{bmatrix}$, we see that this system has infinitely many solutions (e.g., $c_1 = -31$, $c_2 = 14$, $c_3 = 1$), and so the set $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \right\}$ is linearly dependent.

d. Suppose $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \mathbf{0}$. Since the echelon form of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has three pivots, we see that $c_1 = c_2 = c_3 = 0$ is the only solution, and so the vectors form a linearly independent set.

e. Suppose $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$. We find that

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -10 \end{bmatrix},$$

so the only solution is the trivial solution. Thus, the vectors form a linearly independent set.

f. These vectors form a linearly dependent set. For example, their sum is $\mathbf{0}$.

4.3.3 Suppose $c_1(\mathbf{v} - \mathbf{w}) + c_2(2\mathbf{v} + \mathbf{w}) = \mathbf{0}$. Then $(c_1 + 2c_2)\mathbf{v} + (c_2 - c_1)\mathbf{w} = \mathbf{0}$. Since $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, we infer that $c_1 + 2c_2 = -c_1 + c_2 = 0$. But $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so the only solution of this system is $c_1 = c_2 = 0$, as desired.

4.3.5 Suppose $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k = \mathbf{0}$. Then, for $i = 1, \dots, k$, $(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k)\cdot\mathbf{v}_i = 0$, so $c_1(\mathbf{v}_1\cdot\mathbf{v}_i)+\cdots+c_i(\mathbf{v}_i\cdot\mathbf{v}_i)+\cdots+c_k(\mathbf{v}_k\cdot\mathbf{v}_i) = c_i\|\mathbf{v}_i\|^2 = 0$. Since $\mathbf{v}_i \neq \mathbf{0}$, we must have $c_i = 0$ for $i = 1, \dots, k$. Hence $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

4.3.6 Suppose $k > n$, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$, and write $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$. Since $\text{rank}(A) \leq n < k$, the system $A\mathbf{x} = \mathbf{0}$ will have a nontrivial solution. Thus, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent. If we have k linearly independent vectors in $\mathbb{R}^n$, we conclude that $k \leq n$.

4.3.12 a. No: Since $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{0}$, the vectors form a linearly dependent set.

b. No: Any four vectors in $\mathbb{R}^3$ form a linearly dependent set.

c. No: Three vectors span a subspace whose dimension is at most 3.

4.3.14 To show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, it suffices by Proposition 3.9 to show that this is a linearly independent set of vectors.

a. From $\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & 5 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$, we see that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and that $\mathbf{b} = 3\mathbf{v}_1 - \mathbf{v}_2$, so the coordinates of $\mathbf{b}$ with respect to this basis are $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

4.3.26 The polynomials $p_j(x) = x^j$, $j = 0, 1, \dots, k$, evidently span the space in question. We must only argue that they form a linearly independent set. Now suppose $c_0p_0+c_1p_1+\cdots+c_kp_k = \mathbf{0}$, i.e., $c_0 + c_1x + c_2x^2 + \cdots + c_kx^k = 0$ for all $x \in \mathbb{R}$. Evaluating at $x = 0$, we see immediately that $c_0 = 0$. Since all the derivatives of the 0-function are again identically 0, differentiating repeatedly and evaluating at $x = 0$ gives $c_1 = 0$, $c_2 = 0$, $\dots$, $c_k = 0$, as required.

Alternatively, evaluating at $x = 0$ gives $c_0 = 0$. Now we have $c_1x + c_2x^2 + \cdots + c_kx^k = x(c_1 + c_2x + \cdots + c_kx^{k-1}) = 0$ for all x, so $c_1 + c_2x + \cdots + c_kx^{k-1} = 0$ for all x, from which we conclude that $c_1 = 0$. Continuing in this fashion, we have $c_0 = c_1 = \cdots = c_k = 0$, as we needed.