\[
\begin{align*}
\text{so } A^{-1} &= \\
\end{align*}
\]
d. \[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
4 & 5 & 6 & 0 & 1 & 0 \\
7 & 8 & 9 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & -3 & -6 & -4 & 1 & 0 \\
0 & -6 & -12 & -7 & 0 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & -3 & -6 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}
\]
and so we see that \(A^{-1}\) does not exist.

e. \[
\begin{bmatrix}
2 & 3 & 4 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 & 0 \\
-1 & 1 & 2 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & -2 & 0 & 0 & -1 \\
2 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 & 2 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & -1 & -2 & 0 & 0 & -1 \\
0 & 1 & 2 & -1 & 2 & 2 \\
0 & 0 & -1 & 3 & -5 & -4 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & -2 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
so \(A^{-1}\) is:

\[
\begin{bmatrix}
-1 & 2 & 1 \\
5 & -8 & -6 \\
-3 & 5 & 4 \\
\end{bmatrix}
\]

4.2.3

a. (i) \(A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\); (ii) \(x = \begin{bmatrix} -3 \\ -1 \end{bmatrix}\); (iii) \(b = 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \end{bmatrix}\).

b. (i) \(A^{-1} = \begin{bmatrix} 9 & -1 & -3 \\ -6 & 1 & 2 \end{bmatrix}\); (ii) \(x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\); (iii) \(b = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}\).

c. (i) \(A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\); (ii) \(x = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}\); (iii) \(b = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\).

d. (i) \(A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{bmatrix}\); (ii) \(x = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}\); (iii) \(b = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\).
4.3.1  a. Correct: $v_1 - v_3 = 0$.

b. Incorrect: $v_2$ cannot be written as a linear combination of $v_1$ and $v_3$.

4.3.2  a. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \end{bmatrix} = 0$. Since $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we see that $c_1 = c_2 = 0$. Thus, $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

b. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} = 0$. From $\begin{bmatrix} 1 & 2 \\ 4 & 9 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, we see that $c_1 = c_2 = 0$. Thus, $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

c. Suppose $c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 0$. Since the reduced echelon form of

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

is

$\begin{bmatrix} 1 & 0 & 31 \\ 0 & 1 & -14 \\ 0 & 0 & 0 \end{bmatrix}$,

we see that this system has infinitely many solutions (e.g., $c_1 = -31$, $c_2 = 14$, $c_3 = 1$), and so the set $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent.

d. Suppose $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$. Since the echelon form of the matrix

$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

has three pivots, we see that $c_1 = c_2 = c_3 = 0$ is the only solution, and so the vectors form a linearly independent set.

e. Suppose $c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 0$. We find that

$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & -10 \end{bmatrix}$,

so the only solution is the trivial solution. Thus, the vectors form a linearly independent set.

f. These vectors form a linearly dependent set. For example, their sum is 0.

4.3.3  Suppose $c_1(v - w) + c_2(2v + w) = 0$. Then $(c_1 + 2c_2)v + (c_2 - c_1)w = 0$. Since $\{v, w\}$ is linearly independent, we infer that $c_1 + 2c_2 = -c_2 + c_2 = 0$. But $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so the only solution of this system is $c_1 = c_2 = 0$, as desired.
4.3.5 Suppose $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$. Then, for $i = 1, \ldots, k$, $(c_1v_1 + c_2v_2 + \cdots + c_kv_k) \cdot v_i = 0$, so $c_1(v_1 \cdot v_i) + \cdots + c_k(v_k \cdot v_i) = c_i\|v_i\|^2 = 0$. Since $v_i \neq 0$, we must have $c_i = 0$ for $i = 1, \ldots, k$. Hence $\{v_1, \ldots, v_k\}$ is linearly independent.

4.3.6 Suppose $k > n$, let $v_1, \ldots, v_k \in \mathbb{R}^n$, and write $A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & \cdots & v_k \end{bmatrix}$. Since\text{rank}(A) \leq n < k$, the system $Ax = 0$ will have a nontrivial solution. Thus, $\{v_1, \ldots, v_k\}$ is linearly dependent. If we have $k$ linearly independent vectors in $\mathbb{R}^n$, we conclude that $k \leq n$.

4.3.12 a. No: Since $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$, the vectors form a linearly dependent set.

b. No: Any four vectors in $\mathbb{R}^3$ form a linearly dependent set.

c. No: Three vectors span a subspace whose dimension is at most 3.

4.3.14 To show that $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, it suffices by Proposition 3.9 to show that this is a linearly independent set of vectors.

a. From $\begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, we see that $\{v_1, v_2\}$ is linearly independent and that $b = 3v_1 - v_2$, so the coordinates of $b$ with respect to this basis are $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

4.3.26 The polynomials $p_j(x) = x^j$, $j = 0, 1, \ldots, k$, evidently span the space in question. We must only argue that they form a linearly independent set. Now suppose $c_0p_0 + c_1p_1 + \cdots + c_kp_k = 0$, i.e., $c_0 + c_1x + c_2x^2 + \cdots + c_kx^k = 0$ for all $x \in \mathbb{R}$. Evaluating at $x = 0$, we see immediately that $c_0 = 0$. Since all the derivatives of the $0$-function are again identically $0$, differentiating repeatedly and evaluating at $x = 0$ gives $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$, as required.

Alternatively, evaluating at $x = 0$ gives $c_0 = 0$. Now we have $c_1x + c_2x^2 + \cdots + c_kx^k = x(c_1 + c_2x + \cdots + c_kx^{k-1}) = 0$ for all $x$, so $c_1 + c_2x + \cdots + c_kx^{k-1} = 0$ for all $x$, from which we conclude that $c_1 = 0$. Continuing in this fashion, we have $c_0 = c_1 = \cdots = c_k = 0$, as we needed.