Math 318 Exam II March 28, '07

Name:____

ID Number:____

Do your work on this exam booklet. There is one page provided for each problem. Extra paper is available for scratch work. On questions 2, 3, and 4 give enough detail so that it is possible to follow your work.

1	2a	2b	2c	3a	3b	3c	4a	4b

1) Max # lin ind = dimension

2) if $\vec{x}$ in set so is $c\vec{x}$ for some $c > 1$

and $\|Tc\vec{x}\| = |c|\,\|T\vec{x}\| > \|T\vec{x}\|$

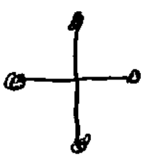
1. (20 points) True or False.

1a. It is impossible to find 4 linearly independent vectors in $\mathbb{R}^3$.

1b. If T is a linear map of $\mathbb{R}^3$ to $\mathbb{R}^3$ which is not the zero map then the function $f(\vec{x}) = \|T\vec{x}\|$ does not have a maximum on the set $\{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| < 1\}$.

1c. In the vector space of polynomials of degree at most 4 the polynomials $p(x) = x^2$, $q(x) = x$ and $r(x) = 2x^2 - 3x$ are linearly independent.

$r - 2p + 3q = 0$



1d. The set in of all vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ in $\mathbb{R}^2$ which satisfy the conditions $xy = 0$ and $-1 \leq x - y \leq 1$ is compact.

1e. If $f(\vec{x})$ is a continuous function on the set $S = \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] : x^2 + y^2 \leq 1\}$ then f has a maximum value on the set S and that maximum value is taken at a point $\vec{a}$ where $Df(\vec{a}) = \vec{0}$.

Question 1 answers:

a	b	c	d	e
F	T	F	T	F

no

$f\binom{x}{y} = x$

max at $\binom{1}{0}$

2. (35 points) Do 2 of 3.

2a. Perform a series of elementary operations on the rows of the matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 4 & 4 \end{pmatrix}$$

to obtain the reduced echelon form of the matrix. What is the rank of the matrix?

2b. For which numbers $a, b,$ and c does the system of equations

1. **2a.** Perform a series of elementary operations on the rows of the matrix

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row 2 → row 2 − row 1

row 3 → row 3 − 2 × row 1

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & -2 \\ 0 & 2 \end{pmatrix}$$

row 3 → row 3 + row 2

row 2 → $-\frac{1}{2}$ × row 2

$$\begin{pmatrix} 2 & 1 \\ 0 & -2 \\ 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

row 1 → row 1 − row 2

row 1 → $\frac{1}{2}$(row 1)

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

rank = 2

2b. For which numbers $a, b,$ and c does the system of equations

$$\begin{aligned} 2x + y &= a \\ 2x - y &= b \\ 4x + 4y &= c \end{aligned}$$

have a solution.

we look at the augmented matrix

$$\left(\begin{array}{cc|c} 2 & 1 & a \\ 2 & -1 & b \\ 4 & 4 & c \end{array}\right)$$

in fact the first 2 cols are the same as previous Q + we follow that path.

row 2 → row 2 − row 1
row 3 → row 3 − 2x row 1

$$(\ ) \to \left(\begin{array}{cc|c} 2 & 1 & a \\ 0 & -2 & b - a \\ 0 & 2 & c - 2a \end{array}\right)$$

now row 3 → row 3 + row 2.

$$(\ ) \to \left(\begin{array}{cc|c} 2 & 1 & a \\ 0 & -2 & b - a \\ \underline{0 \quad 0} & & \underline{c - 2a + b - a} \end{array}\right)$$

Because these are 0 we will only get a solution if this is zero

so $c + b - 3a = 0$ necessary.

rank is 2 insures that there is only one such

2c. Suppose $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are a basis of $\mathbb{R}^3$. Define $\vec{w}_1, \vec{w}_2$ and $\vec{w}_3$.

$$\vec{w}_1 = \vec{v}_2 + \vec{v}_3$$
$$\vec{w}_2 = \vec{v}_1 + \vec{v}_3$$
$$\vec{w}_3 = \vec{v}_1 + \vec{v}_2$$

Must it be true $\vec{w}_1, \vec{w}_2$ and $\vec{w}_3$ are a basis of $\mathbb{R}^3$? Give a complete justification for your answer.

3 vectors in a 3 dim space so they are a basis if independent or if the span.

~~we can~~

To show ind: if $c_1 w_1 + c_2 w_2 + c_3 w_3 = 0$ for scalar c_i's

then $(c_2 + c_3) v_1 + (c_1 + c_3) v_2 + (c_1 + c_2) v = 0$

so (v's ind) $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$

so, by a bit of algebra or some matrix analysis $c_1 = c_2 = c_3$ so w's ind.

To show span; By algebra.

$$\left.\begin{aligned} v_1 &= \tfrac{1}{2}(-w_1 + w_2 + w_3) \\ v_2 &= \tfrac{1}{2}(w_1 - w_2 + w_3) \\ v_3 &= \tfrac{1}{2}(w_1 + w_2 - w_3) \end{aligned}\right] *$$

The v's span $\longrightarrow$ any x can be written as a lin comb of v's $\longrightarrow$ sub using $*$ any x can be written as a lin comb of w's $\longrightarrow$ w's span

3a. Find the maximum and minimum value of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = x + 3y$ in the closed box $B = \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1\}$.

B is compact so there is a max + a min.

$Df = (1, 3)$ never 0 so no crit pts

so must have max min on boundary.

4 boundary segments

$x = 0$	$0 \le y \le 1$	$0 + 3y$
$x = 1$	$0 \le y \le 1$	$1 + 3y$
$y = 0$	$0 \le x \le 1$	x
$y = 1$	$0 \le x \le 1$	$1 + x$

$x + 3y$ on that segment

so by a quick analysis of the 4 cases.

min $= 0$ at ~~(0,0)~~ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

max $= 4$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3b. Find and classify the critical points of the function $f\binom{x}{y} = x^3 + y^2 - x - 2xy$.

$f_x = 3x^2 - 1 - 2y$

$f_y = 2y - 2x$

$f_y = 0 \rightarrow x = y \rightarrow f_x = 0$ can be ~~as~~ rewritten

as $3x^2 - 1 - 2x = 0$

so $(3x+1)(x-1) = 0$

$x = 1$ or $x = -1/3$

cp are $\binom{1}{1}$ $\binom{-1/3}{-1/3}$

$f_{xx} = 6x$ $f_{xy} = -2$ $f_{yy} = 2$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 6x & -2 \\ -2 & 2 \end{pmatrix}$$

$\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$ at first c.p. so loc min

$\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$ at second c.p so saddle pt.

3c. Use the method of Lagrange multipliers to maximize the function $f\begin{pmatrix} x \\ y \end{pmatrix} = xy^2$ subject to the constraint $x^2 + y^2 = 3$.

$f\begin{pmatrix} x \\ y \end{pmatrix} = xy^2 \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 3$

$f_x = \lambda g_x \rightarrow y^2 = 2\lambda x$ ①

$f_y = \lambda g_y \rightarrow 2xy = 2\lambda y$ ②

eqn ② $\rightarrow y = 0$ or $x = \lambda$

if $y = 0$ then from $g = 0$ we get $x = \pm\sqrt{3}$

$A = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} -\sqrt{3} \\ 0 \end{pmatrix}$

if $x = \lambda$ then from ① $y^2 = 2\lambda^2$.

and $g = 0$ becomes $\lambda^2 + 2\lambda^2 = 3$ so $\lambda = \pm 1$

$\lambda = 1 \rightarrow x = 1 \rightarrow y = \pm\sqrt{2} \quad C = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

$\lambda = -1 \rightarrow x = -1 \quad y = \pm\sqrt{2} \quad E = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} \quad F = \begin{pmatrix} -1 \\ -\sqrt{2} \end{pmatrix}$

$f(A) = f(B) = 0$

$f(C) = f(D) = 2 \leftarrow$ max

$f(E) = f(F) = -2 \leftarrow$ min

4a. Suppose $\vec{u}$ and $\vec{v}$ are linearly independent vectors in $\mathbb{R}^4$. For which numbers α, β are the two vectors $\vec{w} = \vec{u} + \vec{v}$ and $\vec{z} = \alpha\vec{u} + \beta\vec{v}$ linearly independent.

If $\vec{w}$ + $\vec{z}$ lin <u>dep</u> then there are scalars c_1, c_2 not both 0

so $c_1 w + c_2 z = 0$

so $(c_1 + \alpha c_2) u + (c_1 + \beta c_2) v = 0$

u + v lin ind so $c_1 + \alpha c_2 = 0$

$c_1 + \beta c_2 = 0$

so $(\alpha - \beta) c_2 = 0$

if $c_2 = 0$ then by $c_1 = 0$ and we have both equal 0, a ~~case not of interest~~ case not of interest in checking lin <u>dep</u>

otherwise $\alpha = \beta$

if $\alpha = \beta = 0$ then $\vec{z}$ is 0 + not lin ind from $\vec{w}$

otherwise $w = \vec{u} + \vec{v}$ $\quad z = \alpha(\vec{u} + \vec{v})$

+ $\alpha\vec{w} - \vec{z} = 0$ lin dep.

Conclusion: lin ind <u>unless</u> $\alpha = \beta$

4b. Recall that if T is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ then the norm of T, $\|T\|$ is defined to be

$$\|T\| = \max_{\|\vec{x}\|=1} \|T\vec{x}\|.$$

Show that is S is another linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ then

$$\|S-T\| \le \|S\| + \|T\|.$$

$$\|S-T\| \underset{\uparrow \text{ def}}{=} \max_{\|x\|=1} \|(S-T)(x)\|$$

$$= \max_{\|x\|=1} \|S(x) - T(x)\|$$

$$\underset{\uparrow \text{ triangle inequality}}{\le} \max_{\|x\|=1} \|S(x)\| + \|-T(x)\|$$

$$\underset{\uparrow \text{ because } \|\vec{a}\| = \|-\vec{a}\|}{=} \max \|S(x)\| + \|T(x)\|$$

$$\le \max \|S(x)\| + \max \|T(x)\|$$

$$= \|S\| + \|T\|$$