Math 318 Exam I February 21, '07

1. **(12 points) True or False.**

   (a) Suppose A and B are 2×2 matrices which have inverses A^{-1} and B^{-1} respectively; then the matrix AB has an inverse and it is given by the formula $(AB)^{-1} = A^{-1}B^{-1}$.

   (b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$ with $\mathbf{u}$ perpendicular to $\mathbf{v}$ and with $\mathbf{v}$ perpendicular to $\mathbf{w}$ then $\mathbf{u}$ is perpendicular to $\mathbf{w}$.

   (c) If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^n$ then $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a vector of length one and parallel to $\mathbf{x}$.

2. **(12 points) Which of the following subsets of the plane $\mathbb{R}^2$ are open, which are closed, and which are neither?**

   (a) $\mathbb{R}^2$ with the point $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ removed,

   (b) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : x = y\right\}$,

   (c) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : xy \neq 0\right\}$,

   (d) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : 0 < x,\ y = 0\right\}$.

3. **(12 points) Which of the following sets are subspaces of $\mathbb{R}^2$?**

   (a) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : x = 2y\right\}$,

   (b) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : xy = 0\right\}$,

   (c) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : y \leq 0\right\}$,

   (d) $\left\{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] : x = y + 1\right\}$.

4. **(20 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose T is a linear transformation which maps $\mathbb{R}^2$ to $\mathbb{R}^2$ and that

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \text{ and } T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}3\\2\end{bmatrix}.$$

Recall that T^2 is the linear transformation defined by $T^2(\mathbf{a}) = T(T(\mathbf{a}))$.

Find:

(a) A, the standard matrix for T,

(b) the determinant of that matrix A,

(c) $T\binom{2}{-1}$,

(d) the standard matrix for T^2.

5. **(15 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}^1$ is given by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + y^2 z$.

Suppose

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Compute

(a) The derivative $Df(\mathbf{a})$,

(b) the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$,

(c) the equation of the plane tangent to the surface $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 14$ at the point **a**.

6. **(15 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose $\mathbf{r} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Find

   (a) $\mathbf{r} \cdot \mathbf{s}$,

   (b) $\|\mathbf{r}\|$,

   (c) a vector perpendicular to both **r** and **s**.

7. **(14 points) Do 2 of 3. Show your work on this sheet using the other side if necessary.**

   (a) Show that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ with $\mathbf{x}$ perpendicular to $\mathbf{y}$ then for any scalar (i.e. number) c it is true that $\|\mathbf{x} + c\mathbf{y}\| \geq \|\mathbf{x}\|$.

   (b) Suppose $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that is not the zero vector. Show that if a vector $\mathbf{y}$ is both perpendicular to $\mathbf{x}$ and parallel to $\mathbf{x}$ then $\mathbf{y}$ is the zero vector.

   (c) Suppose f is the function from $\mathbb{R}^2$ to $\mathbb{R}$ given by $f\binom{x}{y} = 3x + 2y$. Suppose $\mathbf{a}$ is the vector $\mathbf{a} = \binom{4}{5}$ and $\mathbf{v}$ is the vector $\mathbf{v} = \binom{1}{2}$. Using only the definition of directional derivative compute the directional derivative $D_{\mathbf{v}}f(\mathbf{a})$.