Math 318 Exam I February 21, '07

1. **(12 points) True or False.**

NO: $(AB)^{-1} = B^{-1}A^{-1}$

(a) Suppose A and B are 2×2 matrices which have inverses A^{-1} and B^{-1} respectively; then the matrix AB has an inverse and it is given by the formula $(AB)^{-1} = A^{-1}B^{-1}$.

NO: example $\vec{u} = \vec{w} = \binom{1}{0}$, $v = \binom{0}{1}$

(b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$ with $\mathbf{u}$ perpendicular to $\mathbf{v}$ and with $\mathbf{v}$ perpendicular to $\mathbf{w}$ then $\mathbf{u}$ is perpendicular to $\mathbf{w}$.

(c) If $\mathbf{x}$ is a nonzero vector in $\mathbb{R}^n$ then $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a vector of length one and parallel to $\mathbf{x}$.

yes

2. **(12 points) Which of the following subsets of the plane $\mathbb{R}^2$ are open, which are closed, and which are neither?**

(a) $\mathbb{R}^2$ with the point $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ removed, open

(b) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = y \right\}$, closed

(c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \neq 0 \right\}$, open

(d) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 < x,\ y = 0 \right\}$. neither

3. **(12 points) Which of the following sets are subspaces of $\mathbb{R}^2$?**

(a) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 2y \right\}$, subspace

(b) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\}$, not

(c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \leq 0 \right\}$, not

(d) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = y + 1 \right\}$. not

4. **(20 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose T is a linear transformation which maps $\mathbb{R}^2$ to $\mathbb{R}^2$ and that

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \text{ and } T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}3\\2\end{bmatrix}.$$

Recall that T^2 is the linear transformation defined by $T^2(\mathbf{a}) = T(T(\mathbf{a}))$.

Find:

(a) A, the standard matrix for T,

(b) the determinant of that matrix A,

(c) $T\binom{2}{-1}$,

(d) the standard matrix for T^2.

$$A = \left(T\binom{1}{0}\ T\binom{0}{1}\right) = \begin{pmatrix}0 & 3\\2 & 2\end{pmatrix}$$

$$\det A = 0\cdot 2 - 2\cdot 3 = -6$$

$$T\binom{2}{-1} = \begin{pmatrix}0 & 3\\2 & 2\end{pmatrix}\binom{2}{1} = \begin{pmatrix}0\cdot 2 + 3\cdot 1\\2\cdot 2 + 2\cdot 1\end{pmatrix} = \binom{3}{5}$$

Matrix for T^2 is A^2

$$A^2 = \begin{pmatrix}0 & 3\\2 & 2\end{pmatrix}\begin{pmatrix}0 & 3\\2 & 2\end{pmatrix} = \begin{pmatrix}6 & 6\\4 & 10\end{pmatrix}$$

5. **(15 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}^1$ is given by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + y^2 z$.

Suppose

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Compute

(a) The derivative $Df(\mathbf{a})$,

(b) the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$,

(c) the equation of the plane tangent to the surface $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 14$ at the point $\mathbf{a}$.

$$Df = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right)$$

$$= \left(y \quad x + 2yz \quad y^2 \right)$$

at $a = (2 \quad 1 + 2 \cdot 2 \cdot 3 \quad 2^2)$

$$= (2 \quad 13 \quad 4)$$

$$D_V f = (Df)V = (2 \quad 13 \quad 4)\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$= 26 + 4 = 30$$

Tan plane $\left(\begin{pmatrix} x \\ Y \\ z \end{pmatrix} - \vec{a} \right) \cdot \nabla f(a) = 0$

$$\begin{pmatrix} x - 1 \\ Y - 2 \\ z - 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 13 \\ 4 \end{pmatrix} = 0$$

$$2(x - 1) + 13(Y - 2) + 4(z - 3) = 0$$

6. **(15 points) Show any appropriate work on this sheet. Put your answers on the answer sheet.** Suppose $\mathbf{r} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Find

   (a) $\mathbf{r} \cdot \mathbf{s}$,

   (b) $\|\mathbf{r}\|$,

   (c) a vector perpendicular to both $\mathbf{r}$ and $\mathbf{s}$.

a) $r \cdot s = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2 \cdot 2 + 0 \cdot 1 + 1(-1)$
$= 4 - 1 = 3$

b) $\|r\| = \sqrt{r \cdot r} = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$

c) $\vec{r} \times \vec{s}$ is $\perp$ to both

$\vec{r} \times \vec{s} = \begin{vmatrix} e_1 & x_1 & y_1 \\ e_2 & x_2 & y_2 \\ e_3 & x_3 & y_3 \end{vmatrix}$

$= \begin{vmatrix} e_1 & 2 & 2 \\ e_2 & 0 & 1 \\ e_3 & 1 & -1 \end{vmatrix} = 0 + 2e_3 + 2e_2$
$- 0 - (-2e_2) \cancel{+e_1} - e_1$
$= 4e_2 + 2e_3 - e_1 = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \vec{t}$

& check $\vec{t} \cdot \vec{r} = \ldots = 0$ ✓
$\vec{t} \cdot \vec{s} = \ldots = 0$ ✓

7. **(14 points) Do 2 of 3. Show your work on this sheet using the other side if necessary.**

   (a) Show that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$ with $\mathbf{x}$ perpendicular to $\mathbf{y}$ then for any scalar (i.e. number) c it is true that $\|\mathbf{x}+c\mathbf{y}\| \geq \|\mathbf{x}\|$.

   (b) Suppose $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that is not the zero vector. Show that if a vector $\mathbf{y}$ is both perpendicular to $\mathbf{x}$ and parallel to $\mathbf{x}$ then $\mathbf{y}$ is the zero vector.

   (c) Suppose f is the function from $\mathbb{R}^2$ to $\mathbb{R}$ given by $f\binom{x}{y} = 3x+2y$. Suppose $\mathbf{a}$ is the vector $\mathbf{a} = \binom{4}{5}$ and $\mathbf{v}$ is the vector $\mathbf{v} = \binom{1}{2}$. Using only the definition of directional derivative compute the directional derivative $D_{\mathbf{v}}f(\mathbf{a})$.

a) $\|x+cy\|^2 = (x+cy)\cdot(x+cy) = x\cdot x + 2c\, x\cdot y + c^2 y\cdot y$

$= \|x\|^2 + 0 + c^2\|y\|^2$

$\geq \|x\|^2$

now take $\sqrt{\ }$ of both sides

b) y parallel to $x \Leftrightarrow y = cx$ for some # c

$y \perp x \Longleftrightarrow y\cdot x = 0$

combine. $cx\cdot x = 0$

$c\|x\|^2 = 0$ but x not zero

so $\|x\|^2 > 0$ so $c = 0$ so $y = 0$

c) $(D_v f)(a) = \lim\limits_{t\to 0} \dfrac{f\left(\binom{4}{5}+t\binom{1}{2}\right) - f\binom{4}{5}}{t}$

$= \lim \dfrac{f\binom{4+t}{5+2t} - f\binom{4}{5}}{t}$

$= \lim \dfrac{3(4+t)+2(5+2t) - (3\cdot 4 + 2\cdot 5)}{t}$

$= \lim \dfrac{3t+4t}{t} = \lim 7 = 7$