Bayesian updating based on number of successes
in a series of Bernoulli trials

Suppose that we will do a series of N Bernoulli trials where the probability of success of each trial is a random variable P which has a Beta Distribution with parameters $\alpha, \beta > 0$. That is, the density for P is

$$f(p|\alpha,\beta) = f(p) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} & \text{if } 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose a series of N Bernoulli trials produces K successes. The conditional probability that $K = k$, given that $P = p$ is

$$\Pr(K = k|P = p) = \binom{N}{k} p^k (1-p)^{N-k}.$$

By Bayes' theorem we have the density for P given the value $K = k$ is

$$g(p|k) = \frac{\Pr(K = k|P = p) f(p)}{\text{marginal probability that } K = k}$$

and that marginal probability is

$$\int \Pr(K = k|P = p) f(p) dp.$$

(This is the law of total probability discussed on page 142.) Putting this all together, for $0 < p < 1$.

$$g(p|k) = \frac{\binom{N}{k} p^k (1-p)^{N-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}}{\int_0^1 \binom{N}{k} p^k (1-p)^{N-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\, dp}.$$

Now we cancel the binomial coefficients, cancel the gamma functions, and combine powers of p and of $(1-p)$. This gives

$$\begin{aligned} g(p|k) &= \frac{p^{\alpha+k-1}(1-p)^{\beta+(N-k)-1}}{\int_0^1 p^{\alpha+k-1}(1-p)^{\beta+(N-k)-1}\, dp} \\ &= cp^{\alpha+k-1}(1-p)^{\beta+(N-k)-1} \text{ for } 0 < p < 1. \end{aligned}$$

with $c^{-1} = \int ...dp$. So, the density of $g(p|k)$ is a constant multiple of the density of a Beta Distribution with parameters $\alpha + k$, $\beta + (N - k)$. Hence it must be the density of that distribution (because density functions have total integral one, so if one is a multiple of another they must be the same). Hence

$$g(p|k) = \begin{cases} \frac{\Gamma(\alpha+\beta+N)}{\Gamma(\alpha+k)\Gamma(\beta+N-k)} p^{\alpha+k-1}(1-p)^{\beta+(N-k)-1} & \text{if } 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}.$$