Bayesian updating based on number of successes in a series of Bernoulli trials

Suppose that we will do a series of \( N \) Bernoulli trials where the probability of success of each trial is a random variable \( P \) which has a Beta Distribution with parameters \( \alpha, \beta > 0 \). That is, the density for \( P \) is

\[
f(p|\alpha, \beta) = f(p) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1} & \text{if } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

Suppose a series of \( N \) Bernoulli trials produces \( K \) successes. The conditional probability that \( K = k \), given that \( P = p \) is

\[
Pr(K = k|P = p) = \binom{N}{k} p^k (1-p)^{N-k}.
\]

By Bayes’ theorem we have the density for \( P \) given the value \( K = k \) is

\[
g(p|k) = \frac{Pr(K = k|P = p)f(p)}{\text{marginal probability that } K = k}
\]

and that marginal probability is

\[
\int Pr(K = k|P = p)f(p)dp.
\]

(This is the law of total probability discussed on page 142.) Putting this all together, for \( 0 < p < 1 \).

\[
g(p|k) = \frac{\binom{N}{k} p^k (1-p)^{N-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}}{\int_0^1 \binom{N}{k} p^k (1-p)^{N-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1} dp}.
\]

Now we cancel the binomial coefficients, cancel the gamma functions, and combine powers of \( p \) and of \( (1-p) \). This gives

\[
g(p|k) = \frac{p^{\alpha+k-1} (1-p)^{\beta+(N-k)-1}}{\int_0^1 p^{\alpha+k-1} (1-p)^{\beta+(N-k)-1} dp} = cp^{\alpha+k-1} (1-p)^{\beta+(N-k)-1} \text{ for } 0 < p < 1.
\]

with \( c^{-1} = \int ... dp \). So, the density of \( g(p|k) \) is a constant multiple of the density of a Beta Distribution with parameters \( \alpha+k, \beta+(N-k) \). Hence it must be the density of that distribution (because density functions have total integral one, so if one is a multiple of another they must be the same). Hence

\[
g(p|k) = \begin{cases} \frac{\Gamma(\alpha+\beta+N)}{\Gamma(\alpha+k)\Gamma(\beta+N-k)}p^{\alpha+k-1} (1-p)^{\beta+(N-k)-1} & \text{if } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}
\]

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