Math 493

Exam II

November 1, '06

You don't need to reduce your answers to decimal form.

Part 1: True or False, no reasons need be given in this section, 4 points each. (T = always true; F = not always true.)

1. If $E(X)$ is the expectation of the continuous random variable $X$ then $\Pr(X < E(X)) = .5$.
2. If the joint distribution of the pair of random variables $X$ and $Y$ is uniform over the square with corners $(0, 0), (1, 0), (0, 1)$, and $(1, 1)$; then $X$ and $Y$ are independent.
3. If the continuous random variables $X$ and $Y$ are independent then the density function for the conditional density of $X$ given that $Y = y$, $g_1(x | y)$, is the same as $f_1(x)$, the marginal density of $X$.
4. If the random variable $X$ is uniformly distributed on the interval $[2, 3]$ then the random variable $X^2$ is uniformly distributed on the interval $[4, 9]$.
5. If $X$ is any continuous random variable and $F(x)$ is its distribution function then the random variable $Y = F(X)$ is uniformly distributed on the interval $[0, 1]$.

Part 2: Do 4 of 7, 20 points each.

Show enough work or explanation so that it is clear how you arrived at your answer.

1. Suppose the random variables $X_1$, $X_2$ and $X_3$ are independent and each uniformly distributed on the interval $[1, 3]$. Find:
   a. the probability $\Pr(X_1 < 2X_2)$,
   b. the expected value $E(X_1^2 - 2X_1X_2)$, and
   c. the probability that $W = \text{Max}(X_1, X_2, X_3) > 2.5$.

2. The density function $f(x, y)$ of the random variables $X, Y$ is

   $f(x) = \begin{cases} 
   2x + 2y & \text{if } (x, y) \text{ is in the shaded triangle} \\
   0 & \text{otherwise.} 
   \end{cases}$

   Find
   a. $\Pr(X < 1/2)$,
   b. $\Pr(X = 1/4)$,
   c. the expected value $E(Y)$. 
3. The joint p.f. of the discrete random variables $X$ and $Y$ are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>.1</td>
<td>.05</td>
<td>.1</td>
</tr>
<tr>
<td></td>
<td>.3</td>
<td>.2</td>
<td>.25</td>
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</tbody>
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- a. Are $X$ and $Y$ independent?
- b. Find the marginal probability function of $Y$.
- c. Find the conditional probability $\Pr(Y = 2 \mid X = 2)$.
- d. Find the probability $\Pr(X^2 + Y^2 > 5)$.

4. The pair of random variables $X_1$ and $X_2$ are uniformly distributed over the interior of the triangle with vertices $(x_1, x_2) = (0, 0)$, $(1, 0)$, $(1, 2)$. The variables $Y_1$ and $Y_2$ are functions of $X_1, X_2$

$$Y_1 = r_1(X_1, X_2) = 2X_1 + X_2,$$
$$Y_2 = r_2(X_1, X_2) = 2X_1 - X_2.$$

- a. Find the transformations $s_1$ and $s_2$ so that $s_1(Y_1, Y_2) = X_1, s_2(Y_1, Y_2) = X_2$.
- b. Where is the density function for $Y_1$ and $Y_2$ not equal to zero.
- c. On that region what is the formula for the density function.

5. Suppose the random variable $X$ is uniformly distributed on the interval $[-2, 2]$. Find
- a. the density function of the random variable $V = X^3$, and
- b. the density function of the random variable $W = X^4$.
- c. Suppose $Y$ has the same distribution as $X$ and is independent of $X$. Find the probability $\Pr(X^2 + Y^2 \leq 3)$.

6. The pair of random variables $X$ and $Y$ is uniformly distributed on the disk centered at the origin and of radius one; $\{(x, y) : x^2 + y^2 \leq 1\}$. Find the density function of the random variable $R = \text{distance of } (X, Y) \text{ to the origin } (0, 0); R = \sqrt{X^2 + Y^2}$.

7. The random variables $X_1$ and $X_2$ are independent and identically distributed with density functions

$$f(x) = \begin{cases} 
2e^{-2x} & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}$$

- a. Find the joint density function $g(x_1, x_2)$ of the pair $X_1, X_2$.
- b. Find the density function $h(z)$ of the random variable $Z = X_1 + X_2$. 

(102) binomial
\[ f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{if } x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \]

(129) marginal
\[ f_1(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \]
\[ f_2(y) = \int_{-\infty}^{\infty} f(x, y) \, dx \]

(130) conditional
\[ g_1(x \mid y) = \frac{f(x, y)}{f_2(y)} \]
\[ g_2(y \mid x) = \frac{f(x, y)}{f_1(x)} \]

(161) \( Y = r(X) \): if \( r(x) \) then for \( y \) in the range ...
\[ g(y) = f(s(y)) \left| \frac{d}{dy} s(y) \right| \]

(166) \( Y_n = \max(X_1, \ldots, X_n), Y_1 = \min(X_1, \ldots, X_n) \)
\[ g_n(y) = n [F(y)]^{n-1} f(y) \]
\[ g_1(y) = n [1 - F(y)]^{n-1} f(y) \]
\[ g(y_1, y_n) = n(n-1) [F(y_n) - F(y_1)] f(y_1) f(y_n) \]

(168) \( Y_1 = r_1(X_1, X_2), Y_2 = r_2(X_1, X_2) \)
\[ g(y_1, y_2) = f(s_1(y_1, y_2), s_2(y_1, y_2)) \left| \frac{\partial s_1}{\partial y_1} \frac{\partial s_2}{\partial y_2} - \frac{\partial s_1}{\partial y_2} \frac{\partial s_2}{\partial y_1} \right| \]

(172) convolution
\[ g(y) = \int_{-\infty}^{\infty} f_1(y-z) f_2(z) \, dz \]

(185, 190) expectation
\[ E(r(X)) = \int_{-\infty}^{\infty} r(x) f(x) \, dx \]
\[ E(aX + b) = aE(X) + b \]
\[ E(XY) = E(X) E(Y) \left( \prod_{i=1}^{n} \right) \]
<table>
<thead>
<tr>
<th>Section</th>
<th>Problems</th>
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<tbody>
<tr>
<td>4.3</td>
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<td>4.4</td>
<td>2, 4, 6, 8, 10, 12</td>
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