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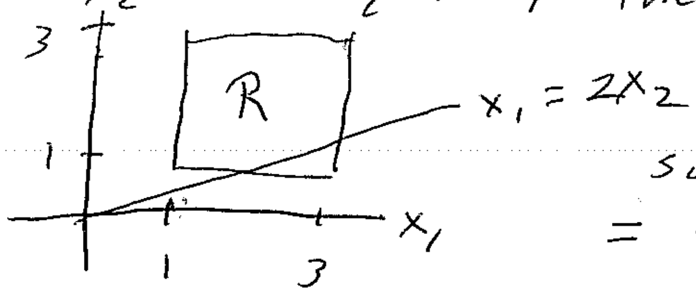
Math 493 Exam II Fall '06 Solutions

Part I
1) F the median has that property
2) T
3) T
4) F X^2 is distributed on $[4,9]$ but not uniformly
5) T

Part II

1) a) the pair X_1, X_2 are uniformly distributed on the square, The region in the square where $X_1 < 2X_2$ is everything except the small triangle so it's area is. $4 - \text{Area small } \Delta$



$$= 4 - \tfrac{1}{2}\cdot 1 \cdot \tfrac{1}{2} = 4 - \tfrac{1}{4}$$

so answer $= \dfrac{\text{Area of R}}{\text{Area of square}} = \dfrac{4-\frac{1}{4}}{4} = \boxed{\dfrac{15}{16}}$

b) $E(X_1^2 - 2X_1X_2) = E(X_1^2) - 2\underbrace{E(X_1)E(X_2)}_{\text{by independence.}}$

$$E(X_1^2) = \int_1^3 x^2 \cdot \tfrac{1}{2}\,dx$$

$$= \tfrac{1}{2}\tfrac{1}{3}x^3\Big|_1^3 = \tfrac{1}{6}\cdot 26 = \tfrac{13}{3}$$

$$E(X_1) = E(X_2) = \int_1^3 \tfrac{1}{2}x\,dx = \tfrac{1}{2}\tfrac{x^2}{2}\Big|_1^3 = 2$$

Ans $= \dfrac{26}{6} - 2\cdot 2\cdot 2 = \boxed{-11/3}$

c) Can use formula for density of Max or.

$$\begin{aligned}
Pr(\max > 2.5) &= 1 - Pr(\max < 2.5)\\
&= 1 - Pr(\text{all } 3 < 2.5)\\
&= 1 - (Pr(\text{any one} < 2.5))^3\\
&= 1 - \left(\tfrac{3}{4}\right)^3 = \tfrac{37}{64}
\end{aligned}$$

(2)

2 a) $\Pr(X < \frac{1}{2}) = \int_0^{1/2} \int_0^{x} 2x + 2y \, dy \, dx$

$$= \int_0^{1/2} (2xy + y^2) \Big|_{y=0}^{y=x} dx$$

$$= \int_0^{1/2} 3x^2 \, dx = x^3 \Big|_0^{1/2} = \boxed{\frac{1}{8}}$$

b) $\Pr(X = \frac{1}{4}) = 0$ ~~because~~ because the region $\{X = \frac{1}{4}\}$ has no area.

or. $\int_{1/4}^{1/4} \left(\int_0^{x} 2x + 2y \, dy \right) dx = 0$

c) $E(Y) = \int_0^1 \left(\int_0^x y(2x + 2y) \, dy \right) dx$

$$= \int_0^1 xy^2 + \frac{2y^3}{3} \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{5}{3} x^3 \, dx = \frac{5}{3} \cdot \frac{1}{4} = \boxed{\frac{5}{12}}$$

It is <u>not</u> enough to $P_{+i} P_{j+} = P_{ji}$ for a single cell.

3) a) no (for instance because 2^{nd} row not a multiple of first)

b) $\Pr$ $Y = 1, 2, 3$ is $.4, .25, .35$

c) $\Pr(Y = 2 \mid X = 2) = \frac{\Pr(Y = 2 \cancel{+} X = 2)}{\Pr(X = 2)} = \frac{.2}{.3 + .2 + .25}$

$$= \frac{20}{75} = \boxed{\frac{4}{15}}$$

d) x, y with $x^2 + y^2 > 5$

$(2, 2)$, $(2, 3)$, $\cancel{+}(1, 3)$

add their probabilities $.2 + .25 + .1 = \boxed{.55}$

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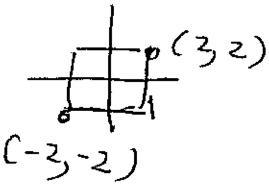
c) X, Y independent & uniform so pair is uniformly distributed on square



so $Pr(X^2+Y^2 \le 3) = \dfrac{\text{Area of Region } X^2+Y^2\le 3}{\text{Area of square}}$

$= \dfrac{3\pi}{16}$

6) The density of R is zero except for $0 < R < 1$

In $(0,1)$ the density is

$$\frac{d}{dr} Pr(R \le r) = \frac{d}{dr} Pr(X^2+Y^2 \le r)$$

$$= \frac{d}{dr} \frac{\text{Area of } X^2+Y^2\le r}{\text{Area of } X^2+Y^2 \le 1}$$

$$= \frac{d}{dr} \frac{\pi r^2}{\pi} = 2r$$

7) a) $f(x_1, x_2) = f(x_1) f(x_2) = \begin{cases} 2e^{-2x_1} \cdot 2e^{-2x_2} & x_1, x_2 \ge 0 \\ 0 & \text{otherwise} \end{cases}$

b) density of $X_1 + X_2 = 0$ $z < 0$ otherwise.

$$g(z) = \int_{-\infty}^{\infty} f(z-x) f(x)\, dx$$

$$= \int_0^z f(z-x) f(x)\, dx$$ because integrand is zero outside $0 \le x \le z$

$$= \int_0^z 2e^{-2(z-x)}\, 2e^{-2x}\, dx$$

$$= 4\int_0^z e^{-2z}\, dx = 4e^{-2z}\int_0^z dx = 4ze^{-2z}$$

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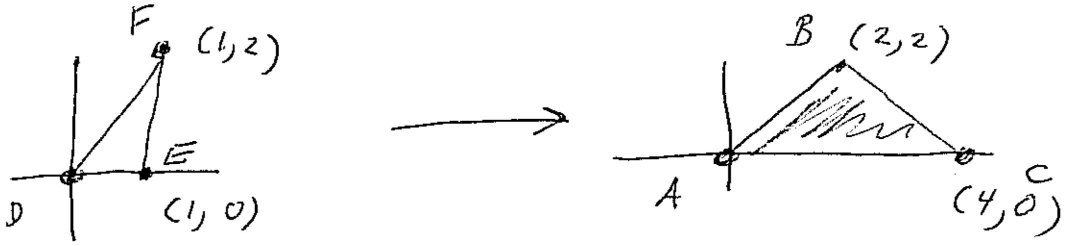
4) a) $Y_1 = 2X_1 + X_2$

$Y_2 = 2X_1 - X_2$

$$\left.\begin{aligned} Y_1 + Y_2 &= 4X_1 \\ Y_1 - Y_2 &= 2X_2 \end{aligned}\right\} \longrightarrow \begin{aligned} X_1 &= \tfrac{1}{4}Y_1 + \tfrac{1}{4}Y_2 \\ X_2 &= \tfrac{1}{2}Y_1 - \tfrac{1}{2}Y_2 \end{aligned}$$



b) density not zero ~~on~~ inside triangle ABC, zero outside

c) Density of X_1, X_2 is 1 ~~on~~ in Δ DEF

so where density of $Y_1\, Y_2$ inside ABC is

$$f(x_1, x_2)\,|J| = 1 \cdot \left|\tfrac{1}{4}\left(-\tfrac{1}{2}\right) - \tfrac{1}{4}\tfrac{1}{2}\right|$$
$$= \tfrac{1}{4}$$

5 a) $V = X^3$ so $X = V^{1/3}$

density of V is 0 outside $(-8, 8)$

in $(-8,8)$ it is $\frac{1}{4} \cdot \left|\frac{V^{-2/3}}{3}\right| = \frac{1}{12}|V|^{-2/3} = \frac{1}{12}V^{-2/3}$

b) density of w is 0 except on $(0, 16)$

there is no inverse function so we compute

$$g(w) = \frac{d}{dw} Pr(W \le w) = \frac{d}{dw} Pr(X^4 \le w)$$

$$= \frac{d}{dw} Pr\left(-w^{1/4} \le x \le w^{1/4}\right)$$

$$= \frac{d}{dw} \int_{-w^{1/4}}^{w^{1/4}} \frac{1}{4}\, dx = \frac{d}{dw}\, \frac{1}{4} \cdot 2w^{1/4} = \frac{1}{8} w^{-3/4}$$