Math 493 Exam II Fall '06 Solutions

Part I
1) F the median has that property
2) T
3) T
4) F $X^2$ is distributed on $[0, 9]$ but not uniformly
5) T

Part II
1) a) The pair $X_1, X_2$ are uniformly distributed on the square, the region in the square where $X_1 < 2X_2$ is everything except the small triangle, so its area is $4 - \text{Area Small } \Delta$

$$x_1 = 2x_2 \Rightarrow x_1 = 4 - \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = 4 - \frac{1}{4}$$

So answer = $\frac{\text{Area of } R}{\text{Area of square}} = \frac{4 - \frac{1}{4}}{4} = \frac{15}{16}$

b) $E(X_1^2 - 2X_1X_2) = E(X_1^2) - 2E(X_1)E(X_2)$

$E(X_1^2) = \int_{-3}^{3} x^2 \cdot \frac{1}{6} \ dx$

$= \frac{1}{6} \left[ \frac{1}{3} x^3 \right]_{-3}^{3} = \frac{1}{6} \cdot 26 = \frac{13}{3}$

$E(X_1) = E(X_2) = \int_{-3}^{3} \frac{1}{2} x \ dx = \frac{1}{2} \left[ \frac{x^2}{2} \right]_{-3}^{3} = 2$

Ans = $\frac{26}{6} - 2 \cdot 2 \cdot 2 = \frac{-11}{3}$

2) Can use formula for density of Max or.

$Pr(\text{Max} > 2.5) = 1 - Pr(\text{max} < 2.5)$

$= 1 - Pr(\text{all } 3 < 2.5)$

$= 1 - (Pr(\text{any one } < 2.5))^3$

$= 1 - (\frac{3}{4})^3 = \frac{37}{64}$
2. a) \[ P_r(x < \frac{1}{4}) = \int_0^{\frac{1}{4}} \int_0^{\frac{1}{2}} 2x + 2y \, dy \, dx \]
   \[ = \int_0^{\frac{1}{4}} \left( \frac{1}{2}x^2 + x^2 \right) \bigg|_{y=0}^{y=x} \, dx \]
   \[ = \int_0^{\frac{1}{4}} \frac{3}{2}x^2 \, dx = x^3 \bigg|_0^{\frac{1}{4}} = \frac{1}{8} \]

b) \[ P_r(x = \frac{1}{4}) = 0 \quad \text{because the region } \{ x = \frac{1}{4} \} \text{ has no area.} \]

Or, \[ \int_{\frac{1}{4}}^{1} \left( \int_0^{x} 2x + 2y \, dy \right) \, dx = 0 \]

3. a) no (for instance because 2nd row not a multiple of first)

b) \[ P_r \{ Y = \frac{5}{2} \} = \frac{1}{4}, \frac{3}{8}, \frac{35}{32} \]

c) \[ P_r \{ Y = 2 | X = 2 \} = \frac{P_r \{ Y = 2 + X = 2 \}}{P_r \{ X = 2 \}} = \frac{12}{0.3 + 2 + 2.5} = \frac{20}{25} = \frac{4}{5} \]

d) \[ Y \text{ with } X^2 + Y^2 > 5 \]
   \[ (2, 2), (2, 3), \text{ and (1, 3)} \]
   \[ \text{add their probabilities: } 2 + 2.5 + 1 = 5.5 \]
c) \( x, y \) independent and uniform so pair is uniformly distributed on square 

\[
\begin{array}{c}
\text{Area of Region } x^2 + y^2 \leq 3 \\
\text{Area of square}
\end{array}
\]

\[
= \frac{\pi \cdot 3}{16}
\]

6) The density of \( R \) is zero except for \( 0 < R < 1 \) within \((0,1)\) the density is:

\[
\frac{d}{dr} P_r (R \leq r) = \frac{d}{dr} \frac{\text{Area of } x^2 + y^2 \leq r}{\text{Area of } x^2 + y^2 \leq 1} = \frac{\pi r^2}{\pi} = r
\]

7) a) \( f(x_1,x_2) = f(x_1) f(x_2) = \begin{cases} 2e^{-2x_1} 2e^{-2x_2} & x_1 x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

b) density of \( x_1 + x_2 = 0 \) \( z < 0 \) otherwise.

\[
g(z) = \int_{-\infty}^{\infty} f(z-x) f(x) \, dx
\]

\[
= \int_{-\infty}^{0} f(z-x) f(x) \, dx \quad \text{because integrand is zero outside } 0 \leq x \leq z
\]

\[
= \int_{0}^{z} 2e^{-2(z-x)} 2e^{-2x} \, dx
\]

\[
= 4 \int_{0}^{z} e^{-2z} \, dx = 4e^{-2z} \int_{0}^{z} \, dx = 4ze^{-2z}
\]
4) \( a) \ \gamma_1 = 2x_1 + x_2 \\
\gamma_2 = 2x_1 - x_2 \\
\gamma_1 + \gamma_2 = 4x_1 \\
\gamma_1 - \gamma_2 = 2x_2 \)
\[
\begin{align*}
x_1 &= \frac{1}{4} \gamma_1 + \frac{1}{4} \gamma_2 \\
x_2 &= \frac{1}{2} \gamma_1 - \frac{1}{2} \gamma_2
\end{align*}
\]
b) density not zero inside triangle \( ABC \), zero outside

c) density of \( xy, xz \) is 1 in \( \Delta DEF \)

So where density of \( \gamma_1, \gamma_2 \) inside \( ABC \) is

\[ f(\gamma_1, \gamma_2) |J| = 1 \cdot |\frac{1}{4} (-\frac{1}{2}) - \frac{1}{4} \frac{1}{2}| \]
\[ = \frac{1}{4} \]

5) \( v = x^3 \) so \( x = v^{\frac{1}{3}} \)

density of \( v \) is 0 outside \((-8, 8)\)
in \([-8, 8]\) it is \( \frac{1}{4} \cdot |v^{\frac{2}{3}}| = \frac{1}{12} |v|^{\frac{-2}{3}} = \frac{1}{12} v^{-\frac{2}{3}} \)

b) density of \( w \) is 0 except on \((0, 16)\)

there is no inverse function so we compute

\[ g(w) = \frac{d}{dw} P_r(w \leq \omega) = \frac{d}{dw} P_r(X^\gamma \leq \omega) \]
\[ = \frac{d}{dw} P_r \left( w^{\frac{1}{4}} \leq x \leq w^{-\frac{1}{4}} \right) \]
\[ = \frac{d}{dw} \int_{w^{\frac{1}{4}}}^{w^{-\frac{1}{4}}} \frac{1}{4} dx = \frac{d}{dw} \left( \frac{1}{4} \cdot 2w^{\frac{1}{4}} \right) = \frac{1}{8} w^{-\frac{3}{4}} \]