Math 493 Final Exam
December '01

NAME:__________________________ ID NUMBER:__________________________

Return your blue book to my office or the Math Department office by Noon on Tuesday 11th.

On all parts after the first show enough work in your exam booklet to make it clear how you got your answer. Generally leave your answers in terms of fractions, factorials and binomial coefficients (i.e. rather than as decimals).

Part I (20 points): True or False (i.e., must be true or might be false); no reasons needed.

In this section A and B are events and B^c is the event complementary to B.

1. If A and B are independent then $P(AB^c) = P(A)(1 - P(B))$.
2. You roll two fair four sided dice. Let C be the event that the sum of the faces that show is even, D the event that at least one of the dice shows a 3. The events C and D are independent.
3. If the continuous random variable $\mathbf{W}$ is uniformly distributed on the interval $(0, 1)$ then the random variable $3\mathbf{W}+4$ is uniformly distributed on the interval $(4, 7)$.
4. If $\mathbf{X}$ is a normal random variable with mean 6 and $\mathbf{Y}$ is a normal random variable with mean 7 then $P(\mathbf{Y} > 8) > P(\mathbf{X} > 8)$.
5. If $\mathbf{X}$ and $\mathbf{Y}$ independent exponential random variables with the same value of the parameter θ then $\mathbf{X} + \mathbf{Y}$ is a gamma random variable.
6. If $\mathbf{X}$ is a Cauchy random variable then $1/\mathbf{X}$ is also a Cauchy random variable.
7. A jar has 30 balls in it; 10 red, 10 white, and 10 blue. Let p_1 be the probability that if you take 3 balls from the jar with replacement then they will all be the same color. Let p_2 be the probability that if you take 3 balls from the jar without replacement they will all be the same color; then $p_1 > p_2$.
8. With the same jar, let p_3 be the probability that if you take 3 balls from the jar with replacement then they will be three different colors and p_2 be the probability that if you take 3 balls from the jar without replacement they will be three different colors. Then $p_3 > p_4$.

9. If $\mathbf{Y}$ is a geometric random variable then $P(\mathbf{Y} > 10 \,|\, \mathbf{Y} > 7) = P(\mathbf{Y} > 3)$.

10. If the random variables $\mathbf{S}$ and $\mathbf{T}$ have the same moment generating function then $P(\mathbf{S} > 3) = P(\mathbf{T} > 3)$.

Part II (10 points): Do 1 of 2

1. You are dealt 5 cards from a regular deck of cards. What is the probability that you receive two pairs (e.g., two 3's and two 4's and a 7). (Note: four cards of the same denomination don't count as two pairs; neither does three of one denomination and two of another.)

2. A jar contains 10 coins, nine are fair and one gives heads with probability 2/3. You select a coin at random from the jar, toss it 3 times and obtain 3 heads. What is the probability that you have selected a fair coin ?

Part III (10 points): Do 1 of 2

1. You repeatedly throw a fair die and consider the throw a success if you throw a 5.

   (a) What is the probability that the sixth try produces the second success ?

   (b) What is the expected number of throws needed to obtain two successes ?

2. A fair coin is tossed 5 times. Let $\mathbf{A}$ be the event that you obtain an even number of heads, $\mathbf{B}$ be the event that you obtain 3 or more heads, and $\mathbf{C}$ the event that you obtain exactly 4 heads. Find:

   (a) $P(\mathbf{C} \mid \mathbf{A})$,

   (b) $P(\mathbf{C} \mid \mathbf{B})$,

   (c) $P(\mathbf{A} \mid \mathbf{B})$

Part IV (10 points): Do 1 of 2

1. $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$ are independent random variables each with the probability mass function:

$$f(x) = \begin{cases} .4 & \text{if} \quad x = 0 \\ .2 & \text{if} \quad x = 1 \\ .2 & \text{if} \quad x = 2 \\ .2 & \text{if} \quad x = 4 \\ 0 & \qquad \text{otherwise} \end{cases}$$

   What is the probability that $\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 = 4$?

2. The random variable $\mathbf{X}$ has the density function

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2) & \text{if} \quad 0 \leq x \leq 1 \\ \\ 0 & \qquad \text{otherwise} \end{cases}$$

(a) What is $P(\mathbf{X} > 1/3)$?

(b) What is $E(\mathbf{X})$?

(c) What is the density function of $\mathbf{Y} = \sqrt{\mathbf{X}}$?

Part V (20 points): Do 2 of 3

1. An experiment produces outcomes which are unit normal random variables. In 5 independent repetitions of the experiment the largest result obtained is $\mathbf{L}$. Find $P(\mathbf{L} > \mathbf{2})$.

2. The continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent and each is distributed uniformly on the interval (1,2). What is $P(\mathbf{XY} \geq 2)$?

3. The random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent, $\mathbf{X}$ is an exponential random variable with $\theta = 1$ and $\mathbf{Y}$ is exponential with $\theta = 2$.

   (a) What is the joint density function of the pair $(\mathbf{X}, \mathbf{Y})$?

   (b) What is the density function of $\mathbf{W} = \mathbf{X} + \mathbf{Y}$?

Part VI (15 points): Do 1 of 2

1. Let $\mathbf{X}$ be a random variable which has a binomial distribution with parameters $n = 200$ and $p = .01$.

   (a) What is $P(\mathbf{X} < 3)$?

   (b) Use the Poisson distribution as an approximation to the binomial distribution to estimate $P(\mathbf{X} < 3)$.

   (c) Use the Central Limit Theorem to estimate $P(\mathbf{X} < 3)$.

2. Suppose that during a particular thunderstorm lighting flashes at the rate of 2 flashes per minute.

   (a) What is the probability of exactly 3 flashes in a particular minute ?

   (b) Use the Central Limit Theorem to give a numerical estimate of the probability of more than 130 flashes in a single hour.

   (c) Let $\mathbf{W}$ be the time (in min., starting at a random time during the storm) you would need to wait in order to see 2 more flashes; What is $E(\mathbf{W})$?

Part VII (15 points): Do 1 of 2

1. Suppose $\overline{\mathbf{X}}_n$ is the fraction of heads in n tosses of a fair coin. The Law of Large Numbers says that $\overline{\mathbf{X}}_n$ will probably be close to .5 when n is large.

   (a) Use Chebyshev's inequality to obtain an estimate of how large n must be to insure

$$P(\left|\overline{\mathbf{X}}_n - .5\right| < .01) < .01.$$

(b) Use the Central Limit Theorem to obtain a more accurate estimate of that n.

2. At a carnival game you roll six six sided dices. You win if the total of the faces showing is less than 12 or more than 30. Use the Central Limit Theorem (or another method of your choosing) to estimate the probability of winning.