Math 128
Midterm Examination 1 – September 23, 2008

Name ______________________

6 problems, 100 points.

Instructions: Show all work – partial credit will be given, and "Answers without work are worth credit without points." You don't have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

1. *(18 points) Compute the following partial derivatives*

   (a) h_z, *where* $h(x,y,z) = \dfrac{x^2+y^2+z^2}{xyz}$

   Using the quotient rule,
$$h_z = \frac{2z \cdot xyz - (x^2+y^2+z^2)xy}{(xyz)^2}.$$

   (b) $\dfrac{\partial}{\partial x} \ln(e^{x^2y} + e^{y^2x})$

   Using the chain rule twice,
$$\frac{\partial f}{\partial x} = \frac{1}{e^{x^2y}+e^{y^2x}} \cdot (e^{x^2y} \cdot 2xy + e^{y^2x} \cdot y^2).$$

   (c) f_{xy}, *where* $f(x,y) = 3\sin xy$

   We first compute f_x:
$$f_x = 3y\cos xy.$$

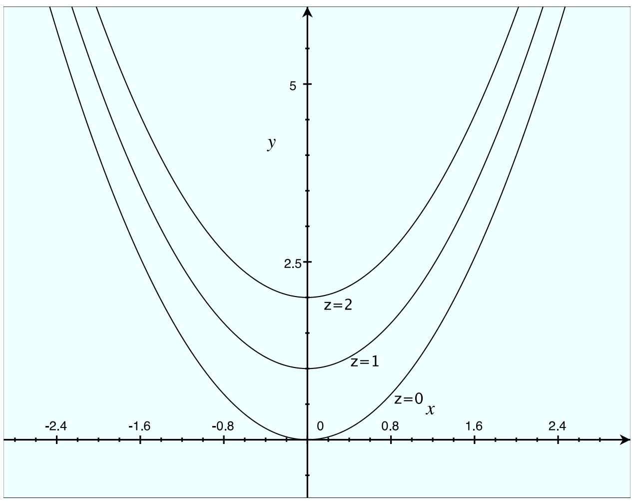
   And then $f_{xy} = f_{yx}$:
$$f_{xy} = 3\cos xy - 3xy\sin xy.$$

2. *(12 points) Sketch the level curves of $z = y - x^2$ at the levels $z = 0, 1, 2$. Make sure to label your graph.*

   We graph
$$0 = y - x^2, \quad 1 = y - x^2, \quad 2 = y - x^2,$$

which, after moving the y's to the right hand side and multiplying by -1 are

$$y = x^2, \quad y = x^2 + 1, \quad y = x^2 + 2.$$



3. *(20 points) Let* $f(x, y) = x^2 + 4y^3 - 6xy + 10$.

   (a) *Find the partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

   $\frac{\partial f}{\partial x} = 2x - 6y$, $\frac{\partial f}{\partial y} = 12y^2 - 6x$.

   (b) *Find the critical points for* f.

   We set $2x - 6y = 0$ and $12y^2 - 6x = 0$. Dividing the first equation by 2 and moving y to the rights gives us $x = 3y$. Substituting this into the other equation gives us $12y - 6x = 12y^2 - 18y = 0$. We factor this last equation as $6y(2y - 3) = 0$. Thus, $y = 0$ or $y = \frac{3}{2}$, and since $x = 3y$, we have the critical points $(0, 0)$ and $(\frac{9}{2}, \frac{3}{2})$.

   (c) *Calculate the 2nd derivatives* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, *and* $\frac{\partial^2 f}{\partial x \partial y}$.

   $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} 2x - 6y = 2$$
   $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} 12y^2 - 6x = -6$$
   $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} 12y^2 - 6x = 24y.$$

(d) *Using the 2nd derivative test, determine which points are relative maxima, relative minima, and saddle points.*

From the calculation of the 2nd derivative, we have the discriminant

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 48y - 36.$$

Plugging in the critical points, we get that
$D(0, 0) = 0 - 36 = -36 < 0$, hence $(0, 0)$ is a critical point.
$D(\frac{9}{2}, \frac{3}{2}) = 48 \cdot \frac{3}{2} - 36 = 36 > 0$, and since $f_{xx} = 2 > 0$, that $(\frac{9}{2}, \frac{3}{2})$ is a relative minimum.

4. *A small company has a local monopoly on two competing products: model houses, and model hotels. It costs them \$200 to build each model house, \$300 for each hotel. The total revenue from selling x houses and y hotels is $1000x + 1200y - 2xy - x^2 - 2y^2$.*

(a) *(5 points) What is the* profit *$P(x, y)$ from selling x houses and y hotels?*

Profit = Revenue - Cost, thus
$P(x, y) = 1000x + 1200y - 2xy - x^2 - 2y^2 - (200x + 300y) = 800x + 900y - 2xy - x^2 - 2y^2$.

(b) *(5 points) Calculate the partial derivatives P_x and P_y.*

$P_x = 800 - 2x - 2y$
$P_y = 900 - 2x - 4y$.

(c) *(5 points) Find the number of houses and hotels the company should build to maximize their profit.*

We set $P_x = P_y = 0$ and solve for x and y. First: $P_x = 0 = 800 - 2x - 2y$. When we divide by 2 and move the $2x$ and $2y$ to the other side, we get $x + y = 400$, or $x = 400 - y$. Then: $P_y = 0 = 900 - 2x - 4y$. But we plug in $x = 400 - y$ to give $0 = 900 - 800 + 2y - 4y = 100 - 2y$. Thus, $100 = 2y$, or $y = 50$ and $x = 400 - y - 350$.

A critical point occurs at 50 hotels and 350 houses.

(d) *(3 points) Explain briefly why your answer in (c) is a maximum.*

We didn't get the tools for a complete answer to this question. I gave full credit for a variety of answers. For example:

i. Finding the discriminant, we get $P_{xx} = -2$, $P_{yy} = -4$, and $P_{xy} = -2$, thus, that $D = 8 - 4 = 4$. Since $D > 0$ and $P_{xx} < 0$, this is a relative max.

ii. The function is an upside-down paraboloid based at the critical point, so has max there.

iii. Careful arguments about profit being unlikely to go to ∞, hence the given point must be a max.

5. *Consider the function* $z = 3x + y$ *on the ellipse* $4x^2 + y^2 = 25$.

(a) *(5 points) Set up a Lagrange multiplier function* $F(x, y, \lambda)$ *for* z *subject to this constraint.*

$F(x, y, \lambda) = 3x + y + \lambda(4x^2 + y^2 - 25)$.

(b) *(12 points) Find all critical points for* F.

We calculate the derivatives:

$F_x = 3 + 8\lambda x$

$F_y = 1 + 2\lambda y$

We then solve for λ. First, note that if x or y is zero, then $F_x = 3$ or $F_y = 1$, so it is not a critical point. Thus, we can safely divide by x and y.

Solving in $F_x = 0$, we get that $-3 = 8\lambda x$, so that $\lambda = -\dfrac{3}{8x}$. Similarly, from $F_y = 0$ we get that $\lambda = -\dfrac{1}{2y}$. Solving, we get that $-6y = -8x$, or that $y = \frac{4}{3}x$. Plugging into the constraint, we get that

$$4x^2 + \frac{16}{9}x^2 = \frac{36 + 16}{9}x^2 = \frac{52}{9}x^2 = 25,$$

so that $x^2 = \frac{9 \cdot 25}{52}$, or $x = \pm\frac{15}{2\sqrt{13}}$. Then $y = \frac{4}{3}x$. Thus, the critical points are

$$\left(\frac{15}{2\sqrt{13}}, \frac{10}{\sqrt{13}}, -\frac{3\sqrt{13}}{20}\right)$$

$$\left(-\frac{15}{2\sqrt{13}}, -\frac{10}{\sqrt{13}}, \frac{3\sqrt{13}}{20}\right)$$

(c) *(5 points) Determine the maximum and minimum value of z on the ellipse $4x^2 + y^2 = 25$.*

By Lagrange's Theorem, the maximum and minimum must both occur at the x and y values of the critical points from part (b). We evaluate the function:

$$\begin{aligned} z(\frac{15}{2\sqrt{13}}, \frac{10}{\sqrt{13}}) &= 3\frac{15}{2\sqrt{13}} + \frac{10}{\sqrt{13}} = \frac{65}{2\sqrt{13}} \\ z(-\frac{15}{2\sqrt{13}}, -\frac{10}{\sqrt{13}}) &= -3\frac{15}{2\sqrt{13}} - \frac{10}{\sqrt{13}} = -\frac{65}{2\sqrt{13}} \end{aligned}$$

so that $\frac{65}{2\sqrt{13}}$ is the maximum value and $-\frac{65}{2\sqrt{13}}$ is the minimum.

6. *(10 points) Consider the function $z = f(x, y) = x^2 - xy + y^2$.*

(a) *What is the slope of the tangent line to f in the cross section $x = 1$ at the point $(1, 2, 3)$?*

Since the partial derivative $\frac{\partial f}{\partial y}$ holds x fixed while differentiating with respect to y, it holds x fixed and finds the rate of change with respect to y. We plug in $y = 2$ to find the rate of change at this particular point. Thus,

$$\frac{\partial f}{\partial y}|_{x=1,y=2} = \frac{\partial}{\partial y}(x^2 - xy + y^2)|_{x=1,y=2} = -x + 2y|_{x=1,y=2} = 3$$

is the slope of the given tangent line.

(b) *The real number $f_x(2, 3) = \frac{\partial f}{\partial x}|_{x=2,y=3}$ is the slope of a tangent line to f at some point, in some cross sectional plane. What is the point, and what is the equation of the cross section plane?*

Working backwards from part (a), we are looking at f_x, which fixes y while differentiating with respect to x. Thus, we are looking in a plane $y =$something, and in particular, in the plane $y = 3$.

We are looking at the point $(2, 3, f(2, 3)) = (2, 3, 2^2 - 2 \cdot 3 + 3^2) = (2, 3, 7)$.