Math 128
Midterm Examination 2 – October 21, 2008

Name ___________________________

6 problems, 112 (oops) points.

Instructions: Show all work – partial credit will be given, and "Answers without work are worth credit without points." You don't have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

1. *(9 points each) Evaluate the following:*

   (a) $\int x \cos 3x \, dx$

   We apply integration by parts. Take $u = x$, so that $dv = \cos 3x \, dx$, $v = \frac{1}{3} \sin 3x$, $du = dx$. Thus

   $$\int x \cos 3x \, dx = \frac{1}{3} x \sin 3x - \int \frac{1}{3} \sin 3x \, dx = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C.$$

   (b) $\int_1^e \frac{\ln x}{x} \, dx$

   We apply integration by substitution. Let $u = \ln x$, so $du = \frac{1}{x} \, dx$. Then

   $$\int_1^e \frac{\ln x}{x} \, dx = \int_1^e \ln x \cdot \frac{1}{x} \, dx = \int_{u(1)}^{u(e)} u \, du = \int_0^1 u \, du = \left[\frac{u^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

   (c) $\int_1^\infty \frac{x}{(x^2+3)^5} \, dx$

   We are evaluating

   $$\lim_{b \to \infty} \int_1^b \frac{x}{(x^2+3)^5} \, dx.$$

   We substitute $u = x^2 + 3$, so that $du = 2x \, dx$. Then

   $$= \lim_{b \to \infty} \int_4^{b^2+3} \frac{1}{u^5} \frac{du}{2} = \lim_{b \to \infty} \frac{1}{2} \left[\frac{1}{-4} u^{-4} \right]_4^{b^2+3} = \lim_{b \to \infty} -\frac{1}{8} \frac{1}{(b^2+3)^4} + \frac{1}{2048}.$$

Since $\lim_{b\to\infty} b^2 + 3 = \infty$, $\lim_{b\to\infty} \frac{1}{(b^2+3)^5} = 0$, and the integral converges to $\frac{1}{2048}$.

2. Consider the function $f(x) = \cos x^2$.

   (a) *(4 points) Calculate M_3 for the integral $\int_{-2}^{1} \cos x^2\, dx$.*

   Plug into the formula, with $\Delta x = \frac{1-(-2)}{3} = 1$:

   $$M_3 = \left(\cos(-1.5)^2 + \cos(-0.5)^2 + \cos(0.5)^2\right) \cdot 1.$$

   (b) *(8 points) Find an upper bound A for $|f''(x)| = \left|\frac{d^2}{dx^2} \cos x^2\right|$ on the interval $[-2, 1]$.*

   First, we calculate $f'(x) = -2x \sin x^2$, so $f''(x) = -2\sin x^2 - 4x^2 \cos x^2$. Then

   $$\begin{aligned} |f''(x)| &= |-2\sin x^2 - 4x^2\cos x^2| \\ &\le |2\sin x^2| + |4x^2\cos x^2| \\ &= 2|\sin x^2| + 4|x^2| \cdot |\cos x^2|. \end{aligned}$$

   We bound each part separately. The trigonometric functions sin and cos are always bounded between -1 and 1; while $|x^2| \le 4$ on the interval $[-2, 1]$. Thus,

   $$|f''(x)| \le 2 \cdot 1 + 4 \cdot 4 \cdot 1 = 18.$$

   (c) *(4 points) Using your result from part (b), find an n so that the midpoint rule approximation M_n is accurate to 0.0025 for the integral $\int_{-2}^{1} \cos x^2\, dx$.*

   We plug the upper bound of 18 for the second derivative into the error formula:

   $$\text{error } M_n \le \frac{18(1-(-2))^3}{24 \cdot n^2} = \frac{18 \cdot 3^3}{24 \cdot n^2} = \frac{3^4}{4n^2}.$$

   We want the error to be less than $.0025 = \frac{1}{400}$, so we set

   $$\frac{3^4}{4n^2} \le \frac{1}{400}.$$

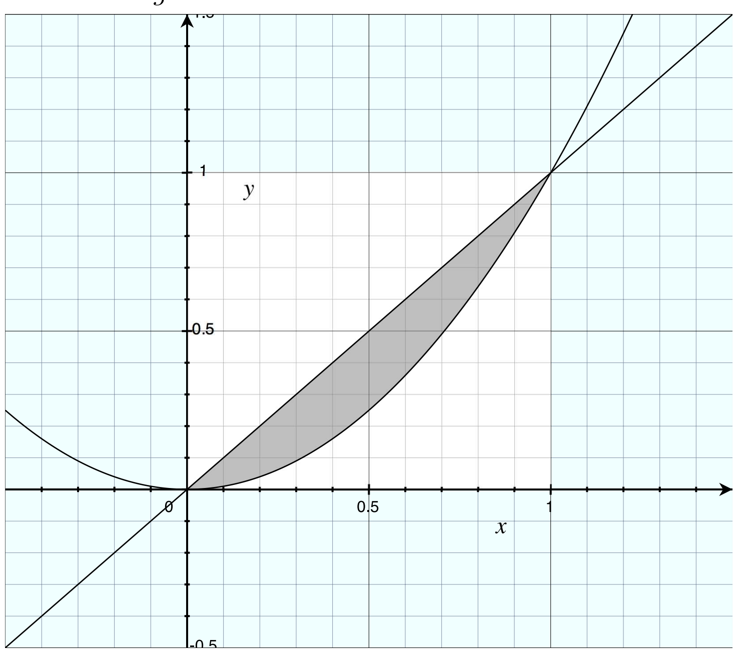
Multiplying through by the denominator of both sides, we get

$$3^4 \cdot 400 \leq 4n^2 \implies 3^4 \cdot 100 \leq n^2 \implies 90 \leq n.$$

(There are multiple possible correct answers to parts (b) and (c), depending how careful you are at each step of the bounding arguments.)

3. *Consider the integral* $\int_0^1 \int_{x^2}^x xe^y \, dydx$.

   (a) *(4 points) Draw the region in the xy-plane associated with the double integral.*



   (b) *(10 points) Evaluate the double integral. (Please leave your answer in terms of e.)*

   We first evaluate the inner integral. Since we are integrating dy, the variable x is treated as a constant, and we get

$$\int_0^1 x \int_{x^2}^x e^y \, dydx = \int_0^1 x \left[e^y\right]_{x^2}^x \, dx = \int_0^1 xe^x - xe^{x^2} \, dx.$$

We integrate xe^x by parts. Take $u = x$, so $dv = e^x\,dx$, $v = e^x$, and $u = dx$. We get

$$\int xe^x\,dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

We integrate xe^{x^2} by substitution. Take $u = x^2$, so $du = 2x\,dx$, and

$$\int xe^{x^2}\,dx = \int e^u \frac{du}{2} = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Combining, we get

$$\begin{aligned}
\int_0^1 xe^x - xe^{x^2}\,dx &= \left[xe^x - e^x - \frac{1}{2}e^{x^2}\right]_0^1 \\
&= (1 \cdot e^1 - e^1 - \frac{1}{2}e^1) - (0 - 1 - \frac{1}{2}) \\
&= -\frac{e}{2} + \frac{3}{2} = \frac{3-e}{2}.
\end{aligned}$$

4. *(10 points each) Solve the following differential equations:*

   (a) $y' = \dfrac{x}{y}, \quad y(1) = -3$

   The equation is not linear, so we need to apply the method of separation. Multiplying through by y we obtain $yy' = x$. Then

$$\begin{aligned}
y\,dy &= x\,dx \\
\int y\,dy &= \int x\,dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + C \\
y^2 &= x^2 + C.
\end{aligned}$$

   Since $y(1) = -3$, we have $(-3)^2 = 9 = 1^2 + C$, or $C = 8$. Furthermore, $y = \pm\sqrt{x^2 + 8}$, and since $y(1) = -3$ (as opposed to $+3$), our final answer is

$$y = -\sqrt{x^2 + 8}.$$

(b) $\dfrac{dy}{dx} = x^2 + \dfrac{y}{x}, \quad x > 0, y(2) = 6$

The equation is not separable, so we need to apply the integrating factor method. We put it in the required form

$$\frac{dy}{dx} - \frac{1}{x} \cdot y = x^2.$$

The integrating factor is

$$I(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = |x|^{-1} = \frac{1}{|x|} = \frac{1}{x},$$

where the last equality is because we are given $x > 0$. The solution is then

$$\begin{aligned} y(x) &= x \cdot \int x^2 \frac{1}{x}\,dx + C \cdot x = x \cdot \int x\,dx + C \cdot x \\ &= x \cdot \frac{x^2}{2} + C \cdot x = \frac{x^3}{2} + C \cdot x. \end{aligned}$$

Since $y(2) = 6$, we have the equality

$$6 = \frac{2^3}{2} + 2C = 4 + 2C.$$

Hence $C = 1$, and our final answer is

$$y = \frac{x^3}{2} + x.$$

5. *(5 points) Give an example of a differential equation which is neither separable nor linear. That is, give a differential equation that can't be solved with either of the 2 techniques we have discussed in Math 128. Explain briefly why each of the methods cannot be applied.*

The equation needs to be neither linear nor separable. The equation

$$y' = y^2 + x^2$$

is not linear because of the y^2, and is not separable since $y^2 + x^2$ does not factor.

Other correct examples include $y' = \sin xy$, $y' = x^y$, etc.

A particularly nice example (that a student in this class gave): $y'' = xy$. This is neither linear nor separable, since it is a second order differential equation!

6. *A 5000 liter tank starts with 500 liters of water in it, and 10 kilograms of salt dissolved in the water. At time 0, clean water starts pouring in at the rate of 4 liters per minute. Unfortunately, the drain on the tank only allows 3 liters per minute of solution to flow out.*

   (a) *(5 points) When does the tank overflow?*

   Since 1 liter/min is flowing into the tank in total, we have the volume of water $V(t) = 500 + t$. The tank starts to overflow when $V(t)$ reaches 5000, i.e., when $t = 4500$.

   (b) *(10 points) Setup a differential equation for the amount of salt in the tank (before overflow).*

   Since the water flowing in is clean water (with no salt), there is no input. To find the output, we first find the concentration. Let $S(t)$ be the amount of salt at time t. Then

$$\text{concentration} = \frac{S(t)}{V(t)} = \frac{S(t)}{500 + t}.$$

   Then the differential equation governing the amount of salt in the tank is

$$\frac{dS}{dt} = -\frac{S(t)}{500+t} \cdot 3 = -\frac{3S}{500+t}.$$

   (c) *(10 points) Solve your differential equation from (b).*

   We use separation of variables. (Note: the integrating factor method will also work.) We divide through by S to get

$$\frac{1}{S}\frac{dS}{dt} = -\frac{3}{500+t}.$$

   Integrating, we get

$$\begin{aligned} \int \frac{1}{S}\, dS &= \int -\frac{3}{500+t}\, dt \\ \ln|S| &= -3\ln|500+t| + C. \end{aligned}$$

   We now work a lot of algebra. Apply the function e^x to both sides:

$$\begin{aligned} |S| &= e^{-3\ln|500+t|+C} = e^C e^{-3\ln|500+t|} = e^C\left(e^{\ln|500+t|}\right)^{-3} \\ &= e^C|500+t|^{-3}. \end{aligned}$$

Remove the absolute value signs:

$$S = \pm e^C (500 + t)^{-3} = A(500 + t)^{-3}.$$

Finally, we use the initial condition $S(0) = 10$ to solve for A.

$$S(0) = 10 = \frac{A}{500^3} \qquad \Longrightarrow \qquad A = 10 \cdot 500^3.$$

Our final answer is

$$S(t) = \frac{10 \cdot 500^3}{(500 + t)^3}.$$

(d) *(5 points) How much salt is in the tank after 33 minutes?*

Plug in $t = 33$:

$$S(33) = \frac{10 \cdot 500^3}{(500 + 33)^3} \cong 8.26 \text{ kg}.$$