Math 128
Midterm Examination 3 – November 11, 2008

Name ______________________

6 problems, 100 points.

Instructions: Show all work – partial credit will be given, and "Answers without work are worth credit without points." You don't have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

1. *(16 points) Let $y' = x + y$, and suppose $y(0) = 0$. Use Euler's method with four steps ($n = 4$) to approximate $y(1)$.*

   Since we are going from $x = 0$ to $x = 1$ in 4 steps, the step size is $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. We calculate:

$$
\begin{aligned}
x_0 &= 0 & y_0 &= 0 = y(0) \\
x_1 = x_0 + \Delta x &= \tfrac{1}{4} & y_1 = y_0 + (x_0 + y_0) \cdot \Delta x &= 0 \\
x_2 = x_1 + \Delta x &= \tfrac{1}{2} & y_2 = y_1 + (x_1 + y_1) \cdot \Delta x &= \tfrac{1}{16} \\
x_3 = x_2 + \Delta x &= \tfrac{3}{4} & y_3 = y_2 + (x_2 + y_2) \cdot \Delta x &= \tfrac{1}{16} + \tfrac{9}{64} = \tfrac{13}{64} \\
x_4 = x_3 + \Delta x &= \tfrac{4}{4} = 1 & y_4 = y_3 + (x_3 + y_3) \cdot \Delta x &= \tfrac{13}{64} + \tfrac{61}{256} = \tfrac{113}{256}
\end{aligned}
$$

   The desired approximation is $y(1) \cong \dfrac{113}{256}$.

2. (a) *(8 points) Find the 3rd degree Taylor polynomial $T_3(x)$ centered at 0 of the function $f(x) = \ln(1 + x)$.*

   We take the first three derivatives:

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f^{(3)}(x) = \frac{2}{(1+x)^3}.$$

   So

$$
\begin{aligned}
T_3(x) =& \ln(1+0) + \frac{1}{1+0} \cdot \frac{1}{1!}x - \frac{1}{(1+0)^2} \cdot \frac{1}{2!}x^2 + \frac{2}{(1+0)^3} \cdot \frac{1}{3!}x^3 \\
=& 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3
\end{aligned}
$$

Note: Another way to approach this problem would be to integrate the Taylor series for $\frac{1}{1+x}$, and take the first three terms.

(b) *(4 points) Find the exact value of* $\sum_{k=0}^{\infty} x^{2k}$ *at* $x = \frac{2}{3}$.

We recognize this as a geometric series, with initial term $a = 1$ and ratio $(\frac{2}{3})^2 = \frac{4}{9}$. Thus, the series converges to $\frac{1}{1-\frac{4}{9}} = \frac{9}{5}$.

(c) *(4 points) Find the exact value of* $\sum_{k=0}^{\infty} x^{2k}$ *at* $x = \frac{3}{2}$.

We recognize this as a geometric series, with initial term $a = 1$ and ratio $(\frac{3}{2})^2 = \frac{9}{4}$. Thus, the series diverges to infinity.

3. *Consider the function* $f(x) = \sin x^2$.

(a) *(7 points) Find the Taylor series centered at* 0 *for* $\sin x^2$.

We recall the Taylor series for $\sin x$:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Plugging in x^2, we get

$$\sin x^2 = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}.$$

(b) *(7 points) Using part (a), evaluate the integral* $\int \sin x^2 \, dx$.

$$\begin{aligned}\int \sin x^2 \, dx &= \int \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} \, dx = \sum_{k=0}^{\infty} \int (-1)^k \frac{x^{4k+2}}{(2k+1)!} dx. \\ &= C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(4k+3) \cdot (2k+1)!}.\end{aligned}$$

4. *Let* $f(x) = \sin 2x$.

(a) *(4 points) Find the 3rd degree Taylor polynomial $T_3(x)$ centered at 0 approximating* $\sin 2x$.

The Taylor series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

thus, the series for $\sin 2x$ is

$$\sin 2x = \sum_{k=0}^{\infty}(-1)^k \frac{(2x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}.$$

The 3rd Taylor polynomial are the terms of the series up to the power x^3, i.e.

$$T_3(x) = 2x - \frac{8}{3!}x^3 = 2x - \frac{4}{3}x^3.$$

(b) *(4 points) What is the* exact *error of your $T_3(x)$ from part (a) at $x = \frac{\pi}{4}$?*

We plug in to find

$$|\sin(2 \cdot \frac{\pi}{4}) - T_3(\frac{\pi}{4})| = |1 - (\frac{\pi}{2} - \frac{4}{3} \cdot \frac{\pi^3}{4^3})| \cong 0.075.$$

(c) *(12 points) Let $T_n(x)$ be the degree n Taylor polynomial for* $\sin 2x$, *centered at 0. Find an upper bound for the error of $T_n(x)$ on the interval $[-1, 1]$. Your bound should depend on n.*

We start taking some derivatives:

$f'(x) = 2\cos 2x$

$f^2(x) = -4\sin 2x$

$f^{(3)}(x) = -8\cos 2x$

$f^{(4)}(x) = 16\sin 2x$

....

We see that the nth derivative for $\sin 2x$ is 2^n, multiplied by plus or minus, sin or cos of $2x$. Since $|\sin u|$ and $|\cos u|$ are both ≤ 1, we get that

$$|f^{(n)}(x)| \leq 2^n, \quad \text{so} \quad |f^{(n+1)}(x)| \leq 2^{n+1}.$$

We plug the bound on the $(n+1)$st derivative into the error bounding formula for Taylor polynomials to get

$$\text{error } T_n(x) \le \frac{2^{n+1}}{(n+1)!} \cdot 1^{n+1} = \frac{2^{n+1}}{(n+1)!}.$$

5. *(15 points) Let $f(x) = x \sin x^2$. Calculate $f^{(14)}(0)$ and $f^{(15)}(0)$.*
Hint: Use the coefficients of the Taylor series.

We first find the Taylor series for $x \sin x^2$. As in other problems, we use that

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Then

$$x \sin x^2 = x \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} = x \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(2k+1)!}.$$

We look at the coefficients of x^{14} and x^{15}. From the Taylor formula, they are

$$\frac{f^{(14)}(0)}{14!} \quad \text{and} \quad \frac{f^{(15)}(0)}{15!}.$$

On the other hand, we calculated the series from that for sin. The coefficient of x^{14} is 0, since $4k+3$ is always odd. Thus, $f^{(14)}(0) = 0$. The coefficient of x^{15} is $(-1)^3 \cdot \frac{1}{(2 \cdot 3 + 1)!} = -\frac{1}{7!}$. We solve to find that $f^{(15)}(0) = -\frac{15!}{7!}$.

6. *For each of the following series, show why it converges or diverges:*

   (a) *(7 points)* $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^{5/4}}$.

   We use direct comparison:

$$0 \le \sin^2 k \le 1 \implies 0 \le \frac{\sin^2 k}{k^{5/4}} \le \frac{1}{k^{5/4}}.$$

   From class, the series $\sum_{k=0}^{\infty} \frac{1}{k^{5/4}}$ converges, and so the given series does as well.

(b) *(7 points)* $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$.

You may use the fact that $\frac{\ln x}{x^{3/2}}$ *is decreasing on the interval* $[2, \infty)$.

We use the integral test, following the hint that $\ln x / x^{3/2}$ is decreasing on $[2, \infty)$. It is certainly continuous and positive on this interval. We take the integral:

$$\int_2^{\infty} \ln x \cdot x^{-3/2}\, dx.$$

We evaluate it via integration by parts. Take $u = \ln x$, so that $dv = x^{-3/2}$, and $du = 1/x$, $v = -2x^{-1/2}$. Then

$$\begin{aligned}
\int_2^{\infty} \ln x \cdot x^{-3/2}\, dx &= \left[-2 \ln x \cdot x^{-1/2}\right]_2^{\infty} + \int_2^{\infty} 2x^{-1/2} \cdot x^{-1}\, dx \\
&= \left[-2 \ln x \cdot x^{-1/2}\right]_2^{\infty} + \int_2^{\infty} 2x^{-3/2}\, dx \\
&= \left[-2 \ln x \cdot x^{-1/2} - 4x^{-1/2}\right]_2^{\infty} \\
&= \lim_{b \to \infty} -2\frac{\ln b}{\sqrt{b}} - \frac{4}{\sqrt{b}} + \frac{2 \ln 2}{\sqrt{2}} + \frac{4}{\sqrt{2}}.
\end{aligned}$$

By l'Hopital's rule (for example), $\ln b/\sqrt{b} \to 0$, and clearly $1/\sqrt{b} \to 0$. So the improper integral converges, and by the integral test the series does as well.

(c) *(5 points)* $\sum_{j=1}^{\infty} \frac{j+1}{j^2+1}$.

We do direct comparison:

$$\frac{j+1}{j^2+1} > \frac{j}{j^2+1}.$$

We then use the integral test to show that $\sum_{j=1}^{\infty} \frac{j}{j^2+1}$ diverges. The function $\frac{x}{x^2+1}$ is continuous and positive and decreasing on $[1,\infty)$, and the integral can be evaluated with the substition $u = x^2 + 1$:

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}\,dx &= \int_2^\infty \frac{1}{u}\,\frac{du}{2} \\ &= [\ln u]_2^\infty \\ &= \lim_{b\to\infty} \ln b - \ln 2 = \infty.\end{aligned}$$